Brian Adams

39

Economics 8101
Fall Semester 2007
Jan Werner

Problem Set 1
Due: Thursday, September 13

1. (10 points) Prove that the properties of constant, increasing, and decreasing returns to scale for a production function $f : \Re^n_+ \to \Re_+$ with single output and n inputs imply that the corresponding production set Y_f, as defined by (2) in Course Handouts, exhibits constant, nondecreasing, and nonincreasing returns to scale, respectively.

2. (10 points) Consider production set given by

$$Y = \{(y_1, y_2) : y_1 \leq 0, \quad y_2 \leq \max\{\ln(\frac{-y_1 + 1}{3}), 0)\}.$$

Find the profit function π^* and the supply function (or correspondence) s^* associated with Y. Clearly specify the sets of price vectors for which π^* and s^* are well defined.

3. (10 points) Consider production function of two inputs given by

$$f(x_1, x_2) = x_1 + \ln(x_2 + 1),$$

for $x_1 \geq 0$ and $x_2 \geq 0$.

   (a) Verify whether this production function exhibits decreasing returns to scale, increasing returns to scale, or neither one.

   (b) What is the range of prices $p > 0$, $w_1 > 0$, and $w_2 > 0$ for which there exist profit-maximizing input quantities? Find these quantities as functions of prices.

4. (10 points) Let Y be a production set in $\Re^L$. Assume that Y is closed and $0 \in Y$. Production set $Y \subset \Re^L$ is said to be *additive* if $y + y' \in Y$ for every $y, y' \in Y$.
For each of the following two statements A and B, if it is true, then prove it; if it false, then give a counterexample.

   (A) If Y is convex and additive, then Y exhibits constant returns to scale.

   (B) If Y exhibits nondecreasing returns to scale, then Y is convex.

Brian Adams
8 Sep 2007

Problem Set 1

1. CLAIM $F: \mathbb{R}^n_+ \to \mathbb{R}_+$ exhibiting constant returns to scale $\Rightarrow Y = \{(x,z) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n_-, \ 0 \le z \le F(-x)\}$ exhibits constant returns to scale.

Proof Let $(x,z) \in Y$ (WTS: $\forall \lambda \ge 0 \ (\lambda x, \lambda z) \in Y$). $0 \le z \le F(-x) \Rightarrow 0 \le \lambda z \le \lambda F(-x) \ \forall \lambda \ge 0$

F exhibits constant returns to scale $\Rightarrow \lambda F(-x) = F(-\lambda x) \overset{\forall \lambda \ge 0}{\Rightarrow} 0 \le \lambda z \le F(-\lambda x) \ \forall \lambda \ge 0 \Rightarrow$
$(\lambda x, \lambda z) \in Y \ \forall \lambda \ge 0$. ∎

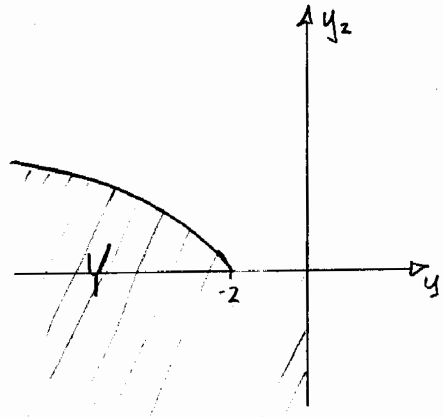
CLAIM $F: \mathbb{R}^n_+ \to \mathbb{R}_+$ exhibiting increasing returns to scale $\Rightarrow Y = \{(x,z) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n_-, \ 0 \le z \le F(-x)\}$ exhibits nondecreasing returns to scale.

Proof Let $(x,z) \in Y$ (WTS: $\forall \lambda \ge 1 \ (\lambda x, \lambda z) \in Y$) $0 \le z \le F(-x) \Rightarrow 0 \le \lambda z \le \lambda F(-x) \ \forall \lambda \ge 0$
F exhibits increasing returns to scale $\Rightarrow F(-\lambda x) \ge \lambda F(-x) \ \forall \lambda \ge 1 \Rightarrow 0 \le \lambda z \le F(-\lambda x) \ \forall \lambda \ge 1 \Rightarrow$
$(\lambda x, \lambda z) \in Y \ \forall \lambda \ge 1$. ∎

CLAIM $F: \mathbb{R}^n_+ \to \mathbb{R}_+$ exhibiting decreasing returns to scale $\Rightarrow Y = \{(x,z) \in \mathbb{R}^{n+1} \mid x \in \mathbb{R}^n_-, \ 0 \le z \le F(-x)\}$ exhibits nonincreasing returns to scale.

Proof Let $(x,z) \in Y$ (WTS: $\forall \lambda \in [0,1] \ (\lambda x, \lambda z) \in Y$) $0 \le z \le F(-x) \Rightarrow 0 \le \lambda z \le \lambda F(-x) \ \forall \lambda \ge 0$
F exhibits decreasing returns to scale $F(\mu t) \le \mu F(t) \ \forall \mu \ge 1$ Let $\mu = \frac{1}{\lambda} \ \forall \lambda \in (0,1]$ Let $t = \lambda x$
$F(-\frac{1}{\lambda} \lambda x) \le \frac{1}{\lambda} F(-\lambda x) \ \forall \lambda \in (0,1] \Rightarrow F(-x) \le \frac{1}{\lambda} F(-\lambda x) \Rightarrow \lambda F(-x) \le F(-\lambda x)$. Thus $0 < \lambda z \le F(-\lambda x) \ \forall \lambda \in (0,1] \Rightarrow$
$(\lambda x, \lambda z) \in Y \ \forall \lambda \in (0,1]$. $(0,0) \in Y$, so $(\lambda x, \lambda z) \in Y \ \forall \lambda \in [0,1]$. ∎

2. profit function $\pi^*(p) = \max_{y \in Y} p \cdot y$



$$Y_1 = \{(y_1, y_2) : y_1 \le 0,\ y_2 \le \ln\left(\frac{-y_1+1}{3}\right)\}$$

$$Y_2 = \{(y_1, y_2) : y_1 \le 0,\ y_2 \le 0\}$$

$$Y_1 \cup Y_2 = Y$$

CASE 1 $y^* = Y_1$ $\quad \max p_1 y_1 + p_2 y_2$ such that $-y_1 \ge 0$, $\ln\left(\frac{-y_1+1}{3}\right) - y_2 \ge 0$

$$\mathcal{L}: p_1 y_1 + p_2 y_2 - \lambda_1 y_1 + \lambda_2 \left(\ln\left(\frac{-y_1+1}{3}\right) - y_2\right)$$

$$0 = \frac{\partial \mathcal{L}}{\partial y_1} \quad p_1 = -\lambda_1 + \lambda_2 \frac{1}{y_1 - 1} \qquad p_1 = -\lambda_1 - p_2 \cdot \frac{1}{y_1 - 1}$$

$$0 = \frac{\partial \mathcal{L}}{\partial y_2} \quad p_2 = -\lambda_2$$

CASE 1a $y_1^* < 0 \Rightarrow \lambda_1 = 0$

$$p_1 = -p_2 \frac{1}{y_1 - 1}$$

$$y_1 - 1 = -\frac{p_2}{p_1}$$

$$y_1^* = -\frac{p_2}{p_1} + 1 \qquad y_2^* = \ln\left(\frac{p_2}{3p_1}\right)$$

$$\pi(p) = -p_2 + p_1 + p_2 \ln\left(\frac{p_2}{3p_1}\right)$$

CASE 1b $y_1^* = 0 \Rightarrow \lambda_1 > 0$

$$p_1 = -\lambda_1 - 3p_2$$

$$p_1 + 3p_2 = -\lambda_1$$

CASE 2 $y^* = Y_2$ $\quad \max p_1 y_1 + p_2 y_2$ such that $-y_1 \ge 0$ $\quad -y_2 \ge 0$

$p_1 > 0 \Rightarrow y_1^* = 0 \qquad p_1 < 0 \Rightarrow y_1^* \to \infty$

$p_2 > 0 \Rightarrow y_2^* = 0 \qquad p_2 < 0 \Rightarrow y_2^* \to \infty$

} no solution for $p_1 < 0,\ p_2 < 0$

$\pi(p) = 0$ for $p_1, p_2 > 0$

$$0 = -p_2 + p_1 + p_2 \ln\left(\frac{p_2}{3p_1}\right)$$

$$0 = -\frac{p_2}{p_1} + 1 + \frac{p_2}{p_1} \ln\left(\frac{p_2}{3p_1}\right)$$

$$1 = p - p \ln\left(\frac{p}{3}\right) \quad \text{where } p = \frac{p_2}{p_1}$$

$$p = 7.0808,\ .3040$$

$$\pi^*(p) = \begin{cases} -p_2 + p_1 + p_2 \ln\left(\frac{p_2}{p_1}\right) & \text{for } 0 < .3040p_1 < p_2 < 7.0808p_1 \\ 0 & \text{for } 0 < p_2 < .3040p_1 \text{ and } 0 < 7.0808p_1 < p_2 \end{cases}$$

$$y_1^*(p) = \begin{cases} 1 - \frac{p_2}{p_1} & \text{for } 0 < .3040p_1 < p_2 < 7.0808p_1 \\ 0 & \text{for } 0 < p_2 < .3040p_1 \text{ and } 0 < 7.0808p_1 < p_2 \end{cases}$$

$$y_2^*(p) = \begin{cases} \ln\left(\frac{p_2}{3p_1}\right) & \text{for } 0 < .3040p_1 < p_2 < 7.0808p_1 \\ 0 & \text{for } 0 < p_2 < .3040p_1 \text{ and } 0 < 7.0808p_1 < p_2 \end{cases}$$

3. (a) $F(x_1, x_2) = x_1 + \ln(x_2 + 1)$

For $(x_1, x_2) = (1, 0)$ $\quad F(\lambda x) = \lambda + \ln(1) = \lambda$

$\lambda F(x) = \lambda \cdot 1 + \ln(1) = \lambda$

For any $\lambda > 1$ $\quad \lambda F(x) \not> F(\lambda x) \Rightarrow$ returns to scale are not decreasing

$\lambda F(x) \not< F(\lambda x) \Rightarrow$ returns to scale are not increasing

For $(x_1, x_2) = (0, 1)$ $\quad F(\lambda x) = \ln(\lambda + 1)$

$\lambda F(x) = \lambda \ln 2$

/

For $\lambda = 2$ $\quad 2 \ln 2 \neq \ln 3 \Rightarrow$ returns to scale are not constant

(b) $F(x_1, x_2) = x_1 + \ln(x_2 + 1)$

$\max \; p \cdot F(x_1, x_2) - w_1 x_1 - w_2 x_2$, subject to $x_1 \geq 0, \; x_2 \geq 0$

$p \frac{1}{x^2 + 1} - w_2 = 0$

$\frac{p}{w_2} - 1 = x_2^*$ $\qquad x_2^* \Rightarrow w_2 < p$

$x_1^* = \begin{cases} 0 & \text{if } w_1 > p \\ \text{undefined} & w_1 < p \end{cases}$

Profit maximizing input quantities exist for $w_1 > p$ and $0 < w_2 < p$

Namely $(x_1^*, x_2^*) = \begin{cases} (0, \frac{p}{w_2} - 1) & \text{for } w_1 > p, \; 0 < w_2 < p \\ (0, 0) & \text{for } w_1 > p, \; w_2 > p \\ \text{undefined} & \text{for } w_1 < p \end{cases}$

-1

what if $w_1 = p$?

4. A) CLAIM If $Y \subset \mathbb{R}^L$ is closed, convex, additive and $0 \in Y$, then Y exhibits constant returns to scale.

Proof Let $y \in Y$. Want to show: $\lambda y \in Y$ for every $\lambda \geq 0$)

CASE 0 Let $\lambda = 0$. $0y = 0 \in Y$ by assumption.

CASE 1 Let $0 \leq \lambda < 1$. Y convex $\Rightarrow \forall x, z \in Y$ and $\forall \theta \in (0,1)$ $\theta x + (1-\theta) z \in Y$.

$y \in Y$ and $0 \in Y$ and $\lambda \in (0,1)$ so $\lambda y + (1-\lambda) 0 = \lambda y \in y$.

CASE 2 Let $\lambda = 1$. $\lambda y = y$. $y \in Y$.

CASE 3 Let $\lambda > 1$. Then $\lambda = a + b$ where $a < 1$ and $b \in \mathbb{Z}$. Y convex, $0, y \in Y \Rightarrow$ $ay \in Y$. Y additive $\Rightarrow \left(\sum_{i=1}^{b} y\right) + ay \in y$.

Thus $\lambda y \in Y$ $\forall \lambda \geq 0 \Rightarrow$ constant returns to scale.

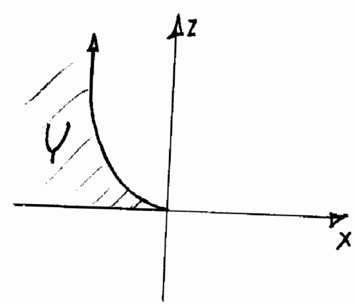
B) CLAIM If Y exhibits nondecreasing returns to scale, then Y is convex.

COUNTEREXAMPLE - Let $Y = \{(x,z) \in \mathbb{R}^2 \mid x \leq 0,\ 0 \leq z \leq x^2\}$

Let $\lambda \geq 1$

$(x,z) \in Y \Rightarrow 0 < z \leq x^2 \Rightarrow 0 \leq \lambda z \leq \lambda x^2$;

$0 < \lambda z \leq \lambda x^2 \leq (\lambda x)^2 \Rightarrow (\lambda x, \lambda z) \in Y \Rightarrow Y$ exhibits nondecreasing returns to scale.



$(0,0) \in Y$, $(1,1) \in Y$ $\quad \frac{1}{2}(0,0) + (1-\frac{1}{2})(1,1) = (\frac{1}{2}, \frac{1}{2}) \notin Y$ since $\frac{1}{2} \nleq \left(\frac{1}{2}\right)^2$

/ Therefore y is not convex.