Micro Problem Set #4

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1. $U(\pi) = \pi_2 z_2 + \sqrt{\pi_1 \pi_3}\,(z_1 + z_3)$

Let $v_i = u(z_i)$ for $i=1,2,3$

Suppose there is an expected utility representation

$$\tilde{U}(\pi) = \pi_1 v_1 + \pi_2 v_2 + \pi_3 v_3$$

$U(\frac{1}{2}, 0, \frac{1}{2}) = 0 + \frac{1}{2}(z_1 + z_3)$ $\qquad$ $\tilde{U}(\frac{1}{2}, 0, \frac{1}{2}) = \frac{1}{2}v_1 + \frac{1}{2}v_3$

$U(0,0,1) = 0$ $\qquad$ $\tilde{U}(0,0,1) = v_3$

$U(1,0,0) = 0$ $\qquad$ $\tilde{U}(1,0,0) = v_1$

$U(\frac{1}{2},0,\frac{1}{2}) > U(1,0,0) \Rightarrow (\frac{1}{2},0,\frac{1}{2}) \succ (1,0,0) \Rightarrow \tilde{U}(\frac{1}{2},0,\frac{1}{2}) > \tilde{U}(1,0,0) \Rightarrow \frac{v_1}{2} + \frac{v_3}{2} > v_1 \Rightarrow v_3 > v_1$

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CONTRADICTION, so $U(\pi)$ has no expected utility form.

2. (i) To satisfy the Allias Paradox given in MWG

$$L_1 \succ L_1' \quad V_0 F(0) + V_5 F(1) + V_{25}[1-F(1)] > V_0 F(.01) + V_5[F(.9)-F(.01)] + V_{25}[1-F(.9)]$$

$$V_5 > V_0 F(.01) + V_5[F(.9)-F(.01)] + V_{25}[1-F(.9)] \qquad (eq. 1)$$

$$\text{and } L_2' \succ L_2 \quad V_0 F(.9) + V_5[F(.9)-F(.9)] + V_{25}[1-F(.9)] > V_0 F(.89) + V_5[F(1)-F(.89)] + V_{25}[1-F(1)]$$

$$V_0 F(.9) + V_{25}[1-F(.9)] > V_0 F(.89) + V_5[1-F(.89)]$$

$$(eq. 1) \quad V_5 > V_0 F(.01) + V_5[F(.9)-F(.01)] + V_{25}[1-F(.9)]$$

$$V_5[1-F(.9)+F(.01)] - V_0 F(.01) > V_{25}[1-F(.9)]$$

$$(eq. 2) \quad V_5[1-F(.89)] - V_0[F(.9)-F(.89)] < V_{25}[1-F(.9)]$$

$$V_5[1-F(.89)] - V_0[F(.9)-F(.89)] < V_{25}[1-F(.9)] < V_5[1-F(.9)+F(.01)] - V_0 F(.01)$$

$$V_5 - V_0 F(.9) - (V_5-V_0) F(.89) < V_{25}[1-F(.9)] < V_5[1-F(.9)] + F(.01)(V_5-V_0) \qquad (eq. 3)$$

All algebra steps have been reversible, so eq. 3 is a sufficient and necessary condition for satisfying MWG's Allias Paradox.

Suppose $F(t)=t$. Then both end terms in eq. 3 equal each other, which is a contradiction. So F cannot be both pessimistic and optomistic.

Examples can be found for all other cases (using this intuition and a spreadsheet):

Example: Pessimistic

$V_0 = 1 \quad V_5 = 2 \quad V_{25} = 5$

$$F(t) = \begin{cases} 30t & \text{for } 0 \le t \le .01 \\ \frac{15t}{22} + \frac{129}{440} & .01 < t \le .89 \\ 5t - 71/20 & .89 < t \le .9 \\ \frac{t}{2} + \frac{1}{2} & .9 < t \le 1 \end{cases}$$

$U(L_1) = 2 > 1.85 = U(L_1')$

$U(L_2') = 1.2 > 1.1 = U(L_2)$

2(i)

Example: Optimistic

$V_0 = 1 \quad V_5 = 4 \quad V_{25} = 4.1$

$$f(t) = \begin{cases} \frac{11t}{10} & \text{for } 0 \le t \le .01 \\ t - \frac{1}{1000} & .01 < t \le .89 \\ \frac{t}{10} + \frac{4}{5} & .89 < t \le .9 \\ \frac{11t}{10} - \frac{1}{10} & .9 < t \le 1 \end{cases}$$

should not change the v's...

$u(L_1) = 4 > 3.984 = u(L_1')$

$u(L_2') = 1.341 > 1.333 = u(L_2)$

I am not sure f can be optimistic.

~~Are you sure your~~

Example: Neither optimistic nor pessimistic

$V_0 = 1 \quad V_5 = 2 \quad V_{25} = 5$

Let $F(0) = 0$

$F(.01) = .3$

$F(.89) = .88$

$F(.9) = .75$

$F(1) = 1$

$F(t)$ everywhere increasing

($F(t)$ linearly connecting those five dots would be a more specific example)

$u(L_1) = 2 > 1.85 = u(L_1')$

$u(L_2') = 1.2 > 1.12 = u(L_2)$

Depending on V, F can be optimistic, pessimistic, or neither

2. (ii)

CLAIM RDEU satisfies the von Neumann - Morgenstern independence axiom $\Leftrightarrow$

$F(t) = t \quad \forall t.$

Proof ($\Leftarrow$)

$F(t) = t \quad \forall t$ so $U(\pi) = \pi_1 v(z_1) + \pi_2 v(z_2) + (1-\pi_1-\pi_2) v(z_3)$

WTS: $\forall L, L', L'' \in \mathcal{L}$ and $\forall \alpha \in (0,1)$ $\quad L \succsim L' \Leftrightarrow \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$

$$U(L) \geq U(L') \Leftrightarrow l_1 v(z_1) + l_2 v(z_2) + (1-l_1-l_2) v(z_3) \geq l_1' v(z_1) + l_2' v(z_2) + (1-l_1'-l_2') v(z_3)$$

$$\Leftrightarrow \alpha l_1 v(z_1) + \alpha l_2 v(z_2) + \alpha(1-l_1-l_2) v(z_3) \geq \alpha l_1' v(z_1) + \alpha l_2' v(z_2) + \alpha(1-l_1'-l_2') v(z_3)$$

$$\Leftrightarrow (1-\alpha) l_1'' v(z_1) + (1-\alpha) l_2'' v(z_2) + (1-\alpha)(1-l_1''-l_2'') v(z_3) + \alpha l_1 v(z_1) + \alpha l_2 v(z_2) + \alpha(1-l_1-l_2) v(z_3)$$
$$\geq (1-\alpha) l_1'' v(z_1) + (1-\alpha) l_2'' v(z_2) + (1-\alpha)(1-l_1''-l_2'') v(z_3) + \alpha l_1' v(z_1) + \alpha l_2' v(z_2) + \alpha(1-l_1'-l_2') v_3$$

$$\Leftrightarrow [\alpha l_1 + (1-\alpha) l_1''] v(z_1) + [\alpha l_2 + (1-\alpha) l_2''] v(z_2) + [\alpha(1-l_1-l_2) + (1-\alpha)(1-l_1''-l_2'')] v(z_3) \geq$$
$$[\alpha l_1' + (1-\alpha) l_1''] v(z_1) + [\alpha l_2' + (1-\alpha) l_2''] v(z_2) + [\alpha(1-l_1'-l_2') + (1-\alpha)(1-l_1''-l_2'')] v(z_3)$$

$$\Leftrightarrow U(\alpha L + (1-\alpha)L'') \geq U(\alpha L' + (1-\alpha)L'').$$

Thus $U(\pi)$ satisfies the von Neumann - Morgenstern independence axiom.

$(\Rightarrow)$ RDEU satisfies the von Neumann - Morgenstern independence axiom.

Suppose $\exists\, \hat{t} \in [0,1]$ such that $F(\hat{t}) \neq \hat{t}$

Let $L, L' \in \mathcal{L}$ such that $L \sim L'$ and $L \neq L'$. Let $\alpha = \frac{1}{2}$

By independence axiom
$$v(z_1) F\left(\frac{l_1 + l_1''}{2}\right) + v(z_2)\left[F\left(\frac{l_1 + l_2 + l_1'' + l_2''}{2}\right) - F\left(\frac{l_1 + l_1''}{2}\right)\right] + v(z_3)\left[1 - F\left(\frac{l_1 + l_2 + l_1'' + l_2''}{2}\right)\right] =$$
$$v(z_1) F\left(\frac{l_1' + l_1''}{2}\right) + v(z_2)\left[F\left(\frac{l_1' + l_2' + l_1'' + l_2''}{2}\right) - F\left(\frac{l_1' + l_1''}{2}\right)\right] + v(z_3)\left[1 - F\left(\frac{l_1' + l_2' + l_1'' + l_2''}{2}\right)\right]$$

3. (a) $u(x) = \sum_{i=1}^{L} V_i(x_i)$ where $V_i : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and continuous is supermodular

Proof (WTS $u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$)

Fix $x, y \in \mathbb{R}^L_+$. Denote $A = \{i \in \{1,2,\dots,L\} \mid y_i > x_i\}$ and $B = \{i \in \{1,2,\dots,L\} \mid y_i \leq x_i\}$. Note $A \cap B = \emptyset$ and $A \cup B = \{1,2,\dots,L\}$

$$u(x \vee y) = \sum_A V_i(y_i) + \sum_B V_i(x_i)$$

$$u(x) = \sum_A V_i(x_i) + \sum_B V_i(x_i)$$

$$u(x \wedge y) = \sum_A V_i(x_i) + \sum_B V_i(y_i)$$

$$u(y) = \sum_A V_i(y_i) + \sum_B V_i(y_i)$$

$$u(x \vee y) - u(x) = \sum_A V_i(y_i) - V_i(x_i)$$

$$u(y) - u(x \wedge y) = \sum_A V_i(y_i) - V_i(x_i)$$

So, $u(x \vee y) - u(x) = u(y) - u(x \wedge y)$.

Therefore, $u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$. ∎

3 (b) $u(x) = \min x_i$ is supermodular

Proof (WTS: $u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$)

Fix $x, y \in \mathbb{R}^L_+$

$u(x \vee y) = \min_i (\max(x_i, y_i))$

$u(x) = \min_i x_i$

$u(x \wedge y) = \min_i (\min(x_i, y_i)) = \min(u(x), u(y))$

$u(y) = \min_i y_i$

CASE $u(y) \geq u(x)$

$u(x \vee y) = \min_i (\max(x_i, y_i)) \geq \min_i y_i = u(y)$

$u(x \vee y) - u(x) \geq u(y) - u(x)$

$u(y) - u(x \wedge y) = u(y) - u(x)$

Thus $u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$

CASE $u(y) < u(x)$

$u(x \vee y) = \min_i (\max(x_i, y_i)) \geq \min_i x_i = u(x)$

$u(x \vee y) - u(x) \geq u(x) - u(x) = 0$

$u(y) - u(x \wedge y) = u(y) - u(y) = 0$

$u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$ ∎

3. (c) $u(x_1, x_2) = g(x_1)h(x_2)$ for strictly increasing functions $g, h: \mathbb{R}_+ \to \mathbb{R}$ is supermodular.

Proof Fix $x, y \in \mathbb{R}^L_+$.

CASE $x_1 \geq y_1$, $x_2 \geq y_2$

$x \vee y = x$

$x \wedge y = y$

$u(x \vee y) - u(x) = u(x) - u(x) = 0$

$u(y) - u(x \wedge y) = u(y) - u(y) = 0$

$u(x \vee y) - u(x) = u(y) - u(x \wedge y)$

$u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$

CASE $x_1 < y_1$, $x_2 < y_2$

$x \vee y = y$

$x \wedge y = x$

$u(x \vee y) - u(x) = u(y) - u(x)$

$u(y) - u(x \wedge y) = u(y) - u(x)$

$u(x \vee y) - u(x) = u(y) - u(x \wedge y)$

$u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$

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3. (c) cont'd

CASE $x_1 \geq y_1, \quad x_2 < y_2$

$$u(x) = g(x_1)h(x_2)$$
$$u(y) = g(y_1)h(y_2)$$

$$u(x \vee y) = g(x_1)h(y_2)$$
$$u(x \wedge y) = g(y_1)h(x_2)$$

$h(y_2) > h(x_2)$ since h is strictly increasing

$$[g(x_1) - g(y_1)]h(y_2) \geq [g(x_1) - g(y_1)]h(x_2)$$ $g(x_1) - g(y_1) \geq 0$ since g increasing

$$g(x_1)h(y_2) - g(y_1)h(y_2) \geq g(x_1)h(x_2) - g(y_1)h(x_2)$$
$$u(x \vee y) - u(y) \geq u(x) - u(x \wedge y)$$
$$u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$$

CASE $x_1 < y_1 \quad x_2 \geq y_2$

$$u(x) = g(x_1)h(x_2)$$
$$u(y) = g(y_1)h(y_2)$$
$$u(x \vee y) = g(y_1)h(x_2)$$
$$u(x \wedge y) = g(x_1)h(y_2)$$

$h(y_2) \leq h(x_2)$ since h is increasing

$$[g(x_1) - g(y_1)]h(y_2) \geq [g(x_1) - g(y_1)]h(x_2)$$
$$g(x_1)h(y_2) - g(y_1)h(y_2) \geq g(x_1)h(x_2) - g(y_1)h(x_2)$$
$$u(x \wedge y) - u(y) \geq u(x) - u(x \vee y)$$
$$u(x \vee y) - u(x) \geq u(y) - u(x \wedge y)$$

For $\forall x, y \in \mathbb{R}^2_+$ $u(x \vee y) - u(x) \geq u(y) - u(x \wedge y) \Rightarrow u$ is supermodular. ∎

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