Fall 2005 Macro Prelim

I.3 First, find the implementability constraint

From the consumer's problem $\max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$ subject to $\sum_{t=0}^{\infty} p_t c_t(1+\tau_{ct}) + \sum_{t=0}^{\infty} p_t x_t \le \sum_{t=0}^{\infty} (F_{nt} n_t + F_{kt} k_t) p_t$

$$k_{t+1} = x_t + (1-\delta)k_t$$

$$n_t + l_t = 1$$

$$\mathcal{L}: \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) + \lambda \sum_{t=0}^{\infty} (F_{nt} n_t + F_{kt} k_t) p_t - \lambda \sum_{t=0}^{\infty} p_t c_t (1+\tau_{ct}) - \lambda \sum_{t=0}^{\infty} [p_t k_{t+1} - p_t (1-\delta) k_t]$$

FOCs

$\frac{\partial \mathcal{L}}{\partial c_t}$ $\beta^t u_{ct} = \lambda p_t (1+\tau_{ct})$ $\qquad$ $\frac{\partial \mathcal{L}}{\partial n_t}$ $\beta^t u_{lt} = \lambda F_{nt} p_t$

$\frac{\partial \mathcal{L}}{\partial k_{t+1}}$ $\lambda p_t F_{kt+1} = \lambda p_t - \lambda p_{t+1}(1-\delta)$

$$p_{t+1} F_{kt+1} + (1-\delta) p_{t+1} = p_t$$

From the budget constraint $\sum_{t=0}^{\infty} p_t c_t (1+\tau_{ct}) + p_t x_t = \sum_{t=0}^{\infty} (F_{nt} n_t + F_{kt} k_t) p_t$

$$\sum_{t=0}^{\infty} p_t c_t (1+\tau_{ct}) - F_{nt} n_t = \sum_{t=0}^{\infty} p_t F_{kt} k_t - p_t [k_{t+1} - (1-\delta) k_t]$$

Let $p_0 = 1$.

$$\sum_{t=0}^{\infty} p_t c_t (1+\tau_{ct}) - F_{nt} n_t = F_{k0} k_0 - (1-\delta) k_0 + \underbrace{\sum_{t=1}^{\infty} (p_t F_{kt} k_t + (1-\delta) p_t k_t - p_{t-1} k_t)}_{0 \text{ by } \frac{\partial \mathcal{L}}{\partial k_{t+1}} \text{ (arbitrage condition)}}$$

$$\sum_{t=0}^{\infty} (p_t c_t (1+\tau_{ct}) - F_{nt} n_t) = F_{k0} k_0 - (1-\delta) k_0$$

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by $\frac{\partial \mathcal{L}}{\partial c_0}$ $u_{c0} = \lambda (1+\tau_{c0})$

$$\lambda = \frac{u_{c0}}{(1+\tau_{c0})}$$

From $\frac{\partial \mathcal{L}}{\partial n_t}$ $\frac{\beta^t u_{lt}}{p_t F_{nt}} = \lambda$

$$\frac{\beta^t u_{lt}}{p_t F_{nt}} = \frac{u_{c0}}{1+\tau_{c0}}$$

by $\frac{\partial \mathcal{L}}{\partial c_t}$ $\beta^t u_{ct} = \lambda p_t (1+\tau_{ct})$

$$\beta^t \frac{u_{ct}}{u_{c0}} \frac{(1+\tau_{ct})}{(1+\tau_{c0})} = p_t$$

$$p_t F_{nt} = \frac{\beta^t u_{lt}}{u_{c0}} (1+\tau_{c0})$$

I.3 (cont'd)

$$\sum_{t=0}^{\infty}\left(p_t c_t (1+\tau_{ct}) - p_t F_{nt} n_t\right) = F_{k0} k_0 - (1-\delta) k_0$$

$$\sum_{t=0}^{\infty} p_t c_t (1+\tau_{ct}) - \frac{\beta_t u_{\ell t}}{u_{c0}} (1+\tau_{c0}) = F_{k0} k_0 - (1-\delta) k_0$$

$$\sum_{t=0}^{\infty} \beta^t \frac{(1+\tau_{c0})}{u_{c0}} \left[u_{ct} c_t - u_{\ell t}\right] = F_{k0} k_0 - (1-\delta) k_0 \qquad (IC)$$

Ramsey Problem $\quad \max \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t)$ subject to IC

Feasibility $\quad k_{t+1} - (1-\delta) k_t + c_t + g_t = F(k_t, n_t)$

given k_0

equivalently

$$\max_{c_t, n_t, k_{t+1}} \underbrace{\sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t) + \lambda F_{k0} k_0 - \lambda (1-\delta) k_0 - \lambda \sum_{t=0}^{\infty} \frac{\beta^t (1+\tau_{c0})}{u_{c0}} \left[u_{ct} c_t - u_{\ell t}\right]}_{V(c_t, \ell_t, \lambda)}$$

subject to $\quad k_{t+1} - (1-\delta) k_t + c_t + g_t = F(k_t, n_t)$

given k_0

FOCs of this

with c_t $\quad \beta^t V_c = \theta_t \qquad$ with n_t $\quad +\beta^t V_\ell = \theta F_n(k_t, n_t)$

with k_{t+1} $\quad \theta_t - (1-\delta)\theta_{t+1} = \theta_{t+1} F_k(k_{t+1}, n_{t+1})$

$$\theta_t = \theta_{t+1}\left[F_k(k_{t+1}, n_{t+1}) + 1 - \delta\right]$$

$$\frac{V_c(t)}{V_c(t+1)} = \beta \frac{\theta_t}{\theta_{t+1}} = \beta\left[F_k(k_{t+1}, n_{t+1}) + 1 - \delta\right]$$

$$V_c = u_c - \lambda \frac{1+\tau_{c0}}{u_{c0}}\left[u_{ctt} c_t + u_{ct}\right]$$

$$V_c = c_t^{-\sigma} - \lambda \frac{1+\tau_{c0}}{u_{c0}}\left[-\sigma c_t^{-\sigma} + c_t^{-\sigma}\right] = c_t^{-\sigma}\left[1 + \lambda \frac{1+\tau_{c0}}{u_{c0}} \sigma - \lambda \frac{1+\tau_{c0}}{u_{c0}}\right]$$

I.3 (cont'd)

$$V_c(t) = C_t^{-\sigma}\left[1+\lambda\frac{1+\tau_{co}}{u_{co}}\sigma-\lambda\frac{1+\tau_{co}}{U_{co}}\right]$$

$$\frac{V_c(t)}{V_c(t+1)} = \frac{C_t^{-\sigma}}{C_{t+1}^{-\sigma}}$$

Thus, $\frac{V_c(t)}{V_c(t+1)} = \left(\frac{C_t}{C_{t+1}}\right)^{-\sigma} = \beta\left[F_K(k_{t+1},n_{t+1})+1-\delta\right]$

From the household problem $\beta^t u_{ct} = \lambda p_t(1+\tau_{ct})$ $\quad u_{ct} = C_t^{-\sigma}$

$$\beta^{t+1}u_{ct+1} = \lambda p_{t+1}(1+\tau_{ct+1})$$

$$\frac{1}{\beta}\frac{C_t^{-\sigma}}{C_{t+1}^{-\sigma}} = \frac{p_t}{p_{t+1}}\frac{1+\tau_{ct}}{1+\tau_{ct+1}}$$

$$\left(\frac{C_t}{C_{t+1}}\right)^{-\sigma} = \beta\frac{p_t}{p_{t+1}}\frac{1+\tau_{ct}}{1+\tau_{ct+1}}$$

$$p_{t+1}F_{Kt+1}+(1-\delta)p_{t+1} = p_t$$

$$\frac{p_{t+1}}{p_t}(F_{kt+1}+1-\delta) = 1$$

$$F_{Kt+1}+1-\delta = \frac{p_t}{p_{t+1}}$$

$$\left(\frac{C_t}{C_{t+1}}\right)^{-\sigma} = \beta(F_{Kt+1}+1-\delta)\frac{1+\tau_{ct}}{1+\tau_{ct+1}}$$

$$\underbrace{\beta(F_{Kt+1}+1-\delta)\frac{1+\tau_{ct}}{1+\tau_{ct+1}}}_{\text{condition from consumer's problem}} = \underbrace{\beta(F_{Kt+1}+1-\delta)}_{\text{condition from Ramsey problem}} \Leftrightarrow \tau_{ct} = \tau_{ct+1}$$

in a solution to a Ramsey problem.

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