Fall 2005 Micro (Majors) Prelim

3.1 (a) Kakutani's fixed point theorem

Let S be a non-empty, closed, and convex set. Let $F: S \to S$ be a nonempty, convex-valued upper hemi-continuous correspondence. Then $\exists x \in S$ such that $x \in F(x)$.

(b) A strategy profile s is a Nash equilibrium if $s^i \in BR^i(s)\ \forall i \in I$.

(c) $S_1, \dots, S_I$ non-empty, closed, convex $\Rightarrow$ S non-empty, closed, convex

Lemma BR is non-empty

$u^i(s^i, s^{-i})$ continuous in s^i, S^i compact $\Rightarrow \{t^i \in S^i \mid u(t^i, s^{-i}) = \max_{r^i \in S^i} u(r^i, s^{-i})\}$ is nonempty by Weierstrass $\Leftrightarrow BR^i(s)$ nonempty $\forall i \Leftrightarrow BR(s)$ nonempty. ∎

Lemma BR is convex-valued

Let $\hat{s}^i, \tilde{s}^i \in BR^i(s)$. Then $u^i(\hat{s}^i, s^{-i}) \geq u^i(t^i, s^{-i})\ \forall t^i \in S^i$. For $\alpha \geq 0$,
$\alpha u(\hat{s}^i, s^{-i}) \geq \alpha u(t^i, s^{-i}) \Rightarrow \alpha \sum_{a \in A}\left(\hat{s}^i(a^i) \prod_{j \neq i} s^j(a) \ u^i(a)\right) \geq \alpha \sum_{a \in A}\left[t^i(a^i) \prod_{j \neq i} s^j(a^j)\ u^i(a)\right]$
$\sum_{a \in A} \alpha \hat{s}^i(a^i) \prod_{j \neq i} s^j(a^j) u^i(a) \geq \sum_{a \in A} \alpha t^i(a^i) \prod_{j \neq i} s^j(a^j) u^i(a)\ \ \forall t^i \in S^i$.

Likewise $u^i(\tilde{s}^i, s^{-i}) \geq u^i(t^i, s^{-i})\ \forall t^i \in S^i \Rightarrow$ For $\alpha \leq 1$, $\sum_{a \in A} (1-\alpha) \tilde{s}^i(a^i) \prod_{j \neq i} s^j(a^j) u^i(a) \geq$ $\sum_{a \in A} (1-\alpha)\, t^i(a^i) \prod_{j \neq i} s^j(a^j) u^i(a)$. So $\sum_{a \in A} (\alpha \hat{s}^i(a) + (1-\alpha)\tilde{s}^i(a)) \prod_{j \neq i} s^j(a^j) u^i(a) \geq \sum_{a \in A} t^i(a^i) \prod_{j \neq i} s^j(a^j)\, u^i(a) \Rightarrow$
$u^i(\alpha \hat{s}^i + (1-\alpha)\tilde{s}^i, s^{-i}) \geq u^i(t^i, s^{-i})\ \forall t^i \in S^i \Rightarrow \alpha \hat{s}^i + (1-\alpha)\tilde{s}^i \in BR^i(s) \Rightarrow BR^i(s)$ is convex-valued. BR^i convex valued $\forall i \Leftrightarrow BR$ convex-valued

Lemma $BR: S \to S$ is upper hemi-continuous.

Let s_n be a sequence in S with $\lim_{n \to \infty} s_n = s$. Let r_n^i be a sequence in S^i such that $r_n^i \in BR^i(s_n)\ \forall n$ and $\lim_{n \to \infty} r_n^i = r^i \in S^i$.

$r_n^i \in BR^i(s_n) \Rightarrow u^i(r_n^i, s_n^{-i}) \geq u^i(t^i, s_n^{-i})\ \forall n\ \forall t^i \in S^i \Rightarrow u^i(r_n^i, s_n^{-i}) \geq u^i(t^i, s_n^{-i})\ \forall t^i \in S^i\ \forall n$.
Since u^i continuous, $u^i(r^i, s_n^{-i}) \geq u^i(t^i, s^{-i})\ \forall t^i \in S^i \Rightarrow r^i \in BR^i(s) \Rightarrow BR^i(s)$ is uhc $\forall i$ $\Leftrightarrow BR(s)$ is uhc

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3.1 (c) (cont'd)

THRM $NE(G) \neq \emptyset$

Pf The best response correspondence BR is non-empty, convex-valued, and upper hemi-continuous by the above lemmae. By Kakutani FTP $\exists s \in BR(s)$. Such an s is a Nash equilibrium.