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3.1 (a)

$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} U \\ D \end{array} & \begin{pmatrix} 1,1 & 0,0 \\ 0,0 & 0,0 \end{pmatrix} \end{array}$$

(D,R) is a Nash equilibrium but not a perfect equilibrium.

(b) i. $(a_{11} - a_{21}) \cdot (a_{12} - a_{22}) > 0$ or $(b_{11} - b_{12}) \cdot (b_{21} - b_{22}) > 0$

would guarantee a single pure equilibrium, because it insure that one of the two strategies will be strictly dominated for a player. After the dominated strategy is eliminated, the other player has a choice between two nonequal payoffs.

ii. $a_{11} \gtrless a_{21}$, $b_{12} \gtrless b_{11}$, $a_{22} \gtrless a_{12}$, $b_{21} \gtrless b_{22}$

$\Rightarrow$ no pure equilibria (see table on right)

$$\begin{array}{cccc} & & q & 1-q \\ & & L & R \\ p & U & (\underline{a_{11}}, b_{11}) & (a_{12}, \underline{b_{12}}) \\ 1-p & D & (a_{21}, \underline{b_{21}}) & (\underline{a_{22}}, b_{22}) \end{array}$$

using top inequalities

There exists a Nash equilibria for any mixed extension normal force game. To show that in this case the Nash Equilibrium is unique, suppose that $\{(p, 1-p), (q, 1-q)\}$ and $\{(\hat{p}, 1-\hat{p}), (\hat{q}, 1-\hat{q})\}$ are two Nash equilibria.

$$p \in \underset{p \in [0,1]}{\arg\max}\ a_{11}pq + a_{21}(1-p)q + a_{12}p(1-q) + a_{22}(1-p)(1-q)$$

$$\text{and } q \in \underset{q \in [0,1]}{\arg\max}\ b_{11}pq + b_{21}(1-p)q + b_{12}p(1-q) + b_{22}(1-p)(1-q)$$

$$\Rightarrow p = \frac{b_{12} - b_{22}}{b_{12} - b_{11} + b_{21} - b_{22}}, \quad q = \frac{a_{12} - a_{22}}{a_{21} - a_{11} + a_{22} - a_{12}}$$

$$\hat{p} \in \underset{\hat{p} \in [0,1]}{\arg\max}\ a_{11}\hat{p}\hat{q} + a_{21}(1-\hat{p})\hat{q} + a_{12}\hat{p}(1-\hat{q}) + a_{22}(1-\hat{p})(1-\hat{q})$$

$$\hat{q} \in \underset{\hat{q} \in [0,1]}{\arg\max}\ a_{11}\hat{p}\hat{q} + a_{21}(1-\hat{p})\hat{q} + a_{12}\hat{p}(1-\hat{q}) + a_{22}(1-\hat{p})(1-\hat{q})$$

$$\Rightarrow p = \hat{p} \quad q = \hat{q}$$

So the Nash equilibrium is unique.

(cont'd)

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3.1 (b) cont'd

iii. $a_{11} \gtrless a_{12}$, $b_{12} \lessgtr b_{11}$, $a_{22} \gtrless a_{12}$, $b_{21} \lessgtr b_{22}$

	L	R
U	$\underline{a_{11}}, \underline{b_{11}}$	a_{12}, b_{12}
D	a_{21}, b_{21}	$\underline{a_{22}}, \underline{b_{22}}$

using top inequalities

	L	R
U	a_{11}, b_{11}	$\underline{a_{12}}, \underline{b_{12}}$
D	$\underline{a_{21}}, \underline{b_{21}}$	a_{22}, b_{22}

using bottom inequalities

		$a_{11} > a_{21}$		$a_{11} < a_{21}$	
		$a_{12} > a_{22}$	$a_{12} < a_{22}$	$a_{12} > a_{22}$	$a_{12} < a_{22}$
$b_{11} > b_{12}$	$b_{21} > b_{22}$	Case i	i	i	i
	$b_{21} < b_{22}$	i	iii	ii	i
$b_{11} < b_{12}$	$b_{21} > b_{22}$	i	ii	iii	i
	$b_{21} < b_{22}$	i	i	i	i

3.1 (c) CLAIM In the game under consideration, all Nash equilibria are perfect.

Proof The three cases of part (b) are exhaustive. Consider the proposition in each case.

Case i. Without loss of generality, assume $a_{11} > a_{21}$, $a_{12} > a_{22}$, $b_{11} > b_{12}$. Let $\varepsilon_n = \{(\frac{1}{n}, \frac{1}{n}), (\frac{1}{n}, \frac{1}{n})\}$ for $n > 3$. In any perturbed game G^{ε_n}, $(1-\varepsilon_n, \varepsilon_n)$ is the unique best response of player 1 to any strategy of player 2. The problem $\max_{q \in [\frac{1}{n}, 1-\frac{1}{n}]} q\left(\frac{1}{n} b_{21} + (1-\frac{1}{n}) b_{11}\right) + (1-q)\left(\frac{1}{n} b_{22} + (1-\frac{1}{n}) b_{12}\right)$ has unique maximizer $q = 1 - \frac{1}{n}$ for $n > M \equiv \frac{b_{22} - b_{21} + b_{11} - b_{12}}{b_{11} - b_{12}}$. Thus $\{(1-\varepsilon_n, \varepsilon_n), (1-\varepsilon_n, \varepsilon_n)\} \in N(G^{\varepsilon_n})$ for $n > M$. The limit of the sequence is (U, L), so it is a perfect equilibrium.

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3.1 (c) cont'd

Case ii. The equilibrium is strictly mixed, so it is a perfect equilibrium.

Case iii. Assume without loss of generality that $a_{11} > a_{21}$, $a_{12} < a_{22}$, $b_{11} > b_{12}$, $b_{22} > b_{21}$. Then the Nash equilibria are (U,L), (D,R), and at least one mixed pair of stragies.

Let $\varepsilon_n = \{(\frac{1}{n}, \frac{1}{n}), (\frac{1}{n}, \frac{1}{n})\}$. Let $p = s^1(U)$, $q = s^2(L)$.

$$BR^1_{\varepsilon_n}(q, 1-q) = \underset{p \in [\frac{1}{n}, 1-\frac{1}{n}]}{\arg\max} \; pq\,a_{11} + (1-p)q\,a_{21} + p(1-q)\,a_{12} + (1-p)(1-q)\,a_{22}$$

$$BR^1_{\varepsilon_n}(q, 1-q) = \begin{cases} 1/n & \text{for } q < \frac{a_{22} - a_{12}}{a_{11} - a_{21} + a_{22} - a_{12}} \\ (\frac{1}{n}, 1-\frac{1}{n}) & q = \frac{a_{22} - a_{12}}{a_{11} - a_{21} + a_{22} - a_{12}} \\ 1 - 1/n & q > \frac{a_{22} - a_{12}}{a_{11} - a_{21} + a_{22} - a_{12}} \end{cases}$$

$$BR^2_{\varepsilon_n}(p, 1-p) = \underset{q \in [\frac{1}{n}, 1-\frac{1}{n}]}{\arg\max} \; pq\,b_{11} + (1-p)q\,b_{21} + p(1-q)b_{12} + (1-p)(1-q)\,b_{22}$$

$$BR^2_{\varepsilon_n}(p, 1-p) = \begin{cases} 1/n & \text{for } p < \frac{b_{22} - b_{21}}{b_{11} - b_{21} - b_{12} + b_{22}} \\ (1/n, 1-1/n) & p = \frac{b_{22} - b_{21}}{b_{11} - b_{21} - b_{12} + b_{22}} \\ 1 - 1/n & p > \frac{b_{22} - b_{21}}{b_{11} - b_{21} - b_{12} + b_{22}} \end{cases}$$

Thus $\{(\frac{1}{n}, 1-\frac{1}{n}), (\frac{1}{n}, 1-\frac{1}{n})\} \in NE(G^{\varepsilon_n})$. A sequence of these has limit (D,R).

$\{(1-\frac{1}{n}, \frac{1}{n}), (1-\frac{1}{n}, \frac{1}{n})\} \in NE(G^{\varepsilon_n})$ A sequence of these has limit (U,L).

$\Rightarrow$ (U,L), (D,R) perfect equilibria.

The mixed equilibria are strictly mixed, therefore perfect. ■