Spring 2007 Prelim - Micro (Majors)

Q1.1 [+] CLAIM Let $\succeq$ be a transitive and complete preference relationship on the consumption set $X = \mathbb{R}^L_+$. Then the following are equivalent:

C1 For every sequence $\{x^n\}$ and $\{y^n\}$ in X such that $\lim_{n\to\infty} x^n = x$, $\lim_{n\to\infty} y^n = y$ and $x^n \succeq y^n$, it holds $x \succeq y$.

C2 For every $x \in X$, the preferred-to-x set $\{y \in X \mid y \succeq x\}$ and the lower contour set $\{y \in X \mid x \succeq y\}$ are closed.

Proof

C1 ⇒ C2

Let $x \in X$ be given. Consider the sequence $\{x^n\} = x$ and any convergent sequence $\{y^n\}$ such that $y^n \succeq x \ \forall n$. $x^n = x \Rightarrow y^n \succeq x^n$. By C1, $\lim_{n\to\infty} y^n = y \succeq x$. Thus $\lim_{n\to\infty} y^n \in \{y \in X \mid y \succeq x\}$ for any convergent $\{y^n\} \subset \{y \in X \mid y \succeq x\}$, so $\{y \in X \mid y \succeq x\}$ is closed.

By a symmetric argument $\{y \in X \mid y \preceq x\}$ is closed.

C2 ⇒ C1 *

Let $\{x^n\}$ and $\{y^n\}$ in X be such that $\lim_{n\to\infty} x^n = x$, $\lim_{n\to\infty} y^n = y$ and $x^n \succeq y^n \ \forall n$. Let $\{p \in X \mid p \succeq q\}$ and $\{p \in X \mid p \preceq q\}$ be closed $\forall q \in X$. Suppose for contradiction that $x \prec y$.

$\{p \in X \mid p \succeq q\}$ closed implies that its complement, $\{p \in X \mid p \prec q\}$, is open
$\{p \in X \mid p \preceq q\}$ closed implies that its complement, $\{p \in X \mid p \succ q\}$, is open.

Lemma $\exists z \in X$ such that $x \prec z \prec y$

Suppose for contradiction $\nexists z$ such that $x \prec z \prec y$. Then $\forall z \in X$ $x \succeq z$ or $y \preceq z$
Thus $z \in \{p \in X \mid p \succeq y\} \cup \{p \in X \mid p \preceq x\} = X = \mathbb{R}^L_+$.
Also $x \prec y \Rightarrow \{p \in X \mid p \succeq y\} \cap \{p \in X \mid p \preceq x\} = \emptyset$
$\{p \in X \mid p \succeq y\}$ and $\{p \in X \mid p \preceq x\}$ closed $\mathbb{R}^L_+$ is contained in two closed, disjoint sets ⇒ $\mathbb{R}^L_+$ not connected CONTRADICTION, so $\exists z \in X$ such that $x \prec z \prec y$.

+ Also Econ 8101, Fall 2007 PS3 #2
* I thank Hitoshi Tsujiyama for the approach.

(cont'd)

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Q 1.1 (cont'd)

$y \succ z \Rightarrow y \in \{p \in X | p \succ z\}$. $\{p \in X | p \succ z\}$ open $\Rightarrow \exists \varepsilon_1 > 0$ such that $B(y, \varepsilon_1) \subset \{p \in X | p \succ z\}$

$x \prec z \Rightarrow x \in \{p \in X | p \prec z\}$ $\{p \in X | p \prec z\}$ open $\Rightarrow \exists \varepsilon_2 > 0$ such that $B(x, \varepsilon_2) \subset \{p \in X | p \prec z\}$

Since $\lim_{n \to \infty} y_n = y$, $\exists N_1$ such that $\forall m > N_1$ $y_m \in B(y, \varepsilon_1) \subset \{p \in X | p \succ z\}$

Since $\lim_{n \to \infty} x_n = x$, $\exists N_2$ such that $\forall m > N_2$ $x_m \in B(x, \varepsilon_2) \subset \{p \in X | p \prec z\}$

Let $k > \max(N_1, N_2)$ $y_k \in \{p \in X | p \succ z\} \Rightarrow y_k \succ z$
$x_k \in \{p \in X | p \prec z\} \Rightarrow x_k \prec z$

By transitivity $x_k \prec y_k$. CONTRADICTION, so $y \succ x$. ∎

A preference relation is <u>transitive</u> if $\forall a, b, c$ such that $a \succsim b$, $b \succsim c$, then $a \succsim c$

A preference relation is <u>complete</u> if $\forall x, y$, $x \succsim y$, $x \precsim y$, or both