Final Exam

UMN, Macroeconomic Theory 8108, Spring 2007

Problem 1 - Insurance with Private Information in a 2-period, 2-shock Economy

Consider the following economy. There is a measure one of agents and 2 periods. The agents discount at rate 1. In each period, an agent has endowment y_L or y_H, with $y_L < y_H$. The shock process is iid over agents and time and the probabilities are π_L, π_H. The law of large numbers applies, so the aggregate endowment in every period is $y = \pi_L y_L + \pi_H y_H$. Suppose the planner can save and borrow at rate 1. Lending and borrowing is 0 at the end of the second period.

(i) (10 points) Characterize the ex-ante efficient allocation in a world where the endowment is public information. Is there any lending or borrowing by the planner?

(ii) (10 points) Assume that agents privately observe their own shocks. Set up the mechanism design problem.

(iii) (15 points) Characterize the transfers under the optimal contract. Interpret your result.

Hint: Think about the problem for a couple of minutes before you start solving it. What must be true of second period allocations? Which is the relevant incentive constraint?

Solution:
Let's first note that this is a simplified version of Problem 5 (2) in Problem Set 3.

(i) The problem is:

$$\max_{c_L, c_H, c_{LL}, c_{LH}, c_{HL}, c_{HH}} \Pi_L \{u(c_L) + \Pi_L u(c_{LL}) + \Pi_H u(c_{LH})\} + \Pi_H \{u(c_H) + \Pi_L u(c_{HL}) + \Pi_H u(c_{HH})\}$$

$$s.t.$$

$$\Pi_L c_L + \Pi_H c_H + B \leq y \quad (1)$$

$$\Pi_L (\Pi_L c_{LL} + \Pi_H c_{LH}) + \Pi_H (\Pi_L c_{HL} + \Pi_H c_{HH}) \leq B + y \quad (2)$$

Clearly $c = y$ in all periods and states. No borrowing and lending since the discount factor and the interest rate are the same.

(ii) First note that second period's transfer cannot depend on second period's report. Thus transfers are functions of first period's report only. Let's denote them b_L, b_H for the first period and e_L, e_H for the second. Then we can write the principal's problem as:

$$\max_{b_L, b_H, e_L, e_H} \Pi_L \{u(y_L + b_L) + E_{y_2}[u(y_2 + e_L)]\} + \Pi_H \{u(y_H + b_H) + E_{y_2}[u(y_2 + e_H)]\}$$

$$s.t.$$

$$u(y_L + b_L) + E_{y_2}[u(y_2 + e_L)] \geq u(y_L + b_H) + E_{y_2}[u(y_2 + e_H)] \quad (3)$$

$$u(y_H + b_H) + E_{y_2}[u(y_2 + e_H)] \geq u(y_H + b_L) + E_{y_2}[u(y_2 + e_L)] \quad (4)$$

$$\Pi_L b_L + \Pi_H b_H + \Pi_L e_L + \Pi_H e_H = 0 \quad (5)$$

(iii) The procedure is the same as in the homework.

(a) Guess that (3) is slack. Consider a relaxed problem without (3).

(b) Show that (4) binds in the solution to the relaxed problem, i.e. consider a completely relaxed problem without both (3) and (4) and show that the solution doesn't satisfy (4). Note that this is what we have seen before - the constraint that bites is the one of the high endowment guy pretending he has low endowment.

(c) Now solve the problem with (4) satisfied with equality and without (3). I.e. solve:

$$\max_{b_L,b_H,e_L,e_H} \Pi_L \left\{u(y_L+b_L)+\beta E_{y_2}[u(y_2+e_L)]\right\} + \Pi_H \left\{u(y_H+b_H)+\beta E_{y_2}[u(y_2+e_H)]\right\}$$

$$s.t.$$

$$u(y_H+b_H)+\beta E_{y_2}[u(y_2+e_H)] = u(y_H+b_L)+\beta E_{y_2}[u(y_2+e_L)] \tag{6}$$

$$R(\Pi_L b_L+\Pi_H b_H)+\Pi_L e_L+\Pi_H e_H = 0 \tag{7}$$

Combine the FOC w.r.t. e_L, e_H and use the positivity of the Lagrange multiplier on (6) to get that $e_L < e_H$. Now (6) implies that $b_L > b_H$.

(d) Verify that (3) is satisfied. $b_L > b_H$, (6) and concavity of u imply (3).

Problem 2 - Time Consistency and Debt

Consider a 2-period model of government debt with a representative household. The resource constraint in both periods is:

$$c_t + g_t = l_t, \ t = 1, 2$$

where c_t and g_t is household's consumption and government spending and l_t is labor. Assume that $g_1 > 0$ and $g_2 = 0$. The government can borrow in the first period and tax labor income in both periods. We denote the tax rates τ_1, τ_2, the holdings of one period risk free bond B and its price q. The government budget constraints are:

$$\begin{aligned} g_1 &= \tau_1 l_1 + qB \\ B &= \tau_2 l_2 \end{aligned}$$

The preferences of the household are:

$$\log c_1 + \alpha \log(1 - l_1) + \beta \left[\log c_2 + \alpha \log(1 - l_2)\right]$$

(i) (11 points) Define a competitive equilibrium.

(ii) (8 points) Define a Ramsey equilibrium.

(iii) (8 points) Consider an environment without commitment. Here, the government chooses the tax rates and whether to default in its debt at the beginning of the second period. Denoting δ the proportion of debt that the government will default on, its second period budget constraint becomes:

$$(1 - \delta)B = \tau_2 l_2$$

Define a sustainable equilibrium.

(iv) (8 points) Is there a time consistency problem? Justify your answer. If so, characterize the equilibrium outcomes without commitment.

Solution: Please see Chari, Kehoe: Sustainable Plans

Problem 3 - Regulating Road Congestion

Consider the following economy. A large number of people want to share a road. Each person differs from the others in their desire to use the road, as indexed by the privately observed θ with density $p(\theta)$ with support $[\underline{\theta}, \bar{\theta}]$. Let $\mu(\theta)$ denote the time that type θ is allowed to drive. Let x denote the average traffic on the road, as denoted by

$$x = \int \mu(\theta)p(\theta)\ d\theta,$$

Let $R(x, \mu, \theta)$ be the utility for type θ of driving on the road for μ units of time when the average traffic is x. Assume that R is decreasing in x, because people dislike higher average traffic and is increasing in μ. Let $w(\theta)$ denote the toll to drive $\mu(\theta)$. The budget constraint on tolls is

$$\int w(\theta)p(\theta)\ d\theta \leq \bar{w}$$

where $\bar{w}$ is the money needed to operate the road, possibly zero. The payoff to a consumer of type θ, that drives $\mu(\theta)$ when the average traffic is x and pays the toll $w(\theta)$ is

$$R(x, \mu(\theta), \theta) + w(\theta).$$

(i) (15 points) Set up the mechanism design problem in which the criterion is ex ante utility.
(ii) (15 points) Solve for the optimal mechanism.

Hint: This problem is best solved by putting down your pencil and thinking for 5 minutes–not by grinding the algebra. How would one choose $w(\theta)$ to make a given allocation $\mu(\theta)$ incentive compatible? What allocations $\mu(\theta)$ can be made incentive compatible?

Solution: This is based on Athey, Atkeson, Kehoe: The Optimal Degree of Discretion in Monetary Policy, Appendix E.

(i) Note that in the set-up, we assume $w(\theta) \leq 0$. The problem is:

$$\begin{aligned} \max \quad & \int [R(x, \mu(\theta), \theta) + w(\theta)]p(\theta)d\theta \\ s.t. \quad & \\ x \quad &= \int \mu(\theta)p(\theta)\ d\theta \\ R(x, \mu(\theta), \theta) + w(\theta) \quad &\geq R(x, \mu(\hat{\theta}), \theta) + w(\hat{\theta}) \\ \int w(\theta)p(\theta)\ d\theta \quad &\leq \bar{w} \end{aligned}$$

(ii) We want to show that first best is implementable as the optimal contract. So, consider the relaxed problem (information is public) with the "feasibility" constraint satisfied as equality:

$$\begin{aligned} \max \quad & \int R(x, \mu(\theta), \theta)p(\theta)d\theta + \bar{w} \\ s.t. \quad & \\ x \quad &= \int \mu(\theta)p(\theta)\ d\theta \end{aligned}$$

The FOC imply that:

$$R_\mu(x, \mu(\theta), \theta) = R_\mu(x, \mu(\hat{\theta}), \hat{\theta})$$
$$R_\mu(x, \mu(\theta), \theta) = -R_x(x, \mu(\theta), \theta)$$

Not surprisingly, we are equalizing marginal utilities across agents. Note that we are able to determine the function $\mu(\theta)$ from the FOC and the constraint. Start with $\underline{\theta}$ and pick a constant

$\mu(\underline{\theta})$. The first FOC determines $\mu(\theta), \forall\theta$. The constraint implies a value of x. Use the second FOC to find the correct constant $\mu(\underline{\theta})$. Note also that from the perspective of the planner, it doesn't matter who pays for the road - the disutility is linear. Now we assume that:

$$
\begin{aligned}
R_{\mu\theta}(x, \mu, \theta) &> 0 \\
R_{\mu\mu}(x, \mu, \theta) &\leq 0 \\
R_{\mu}(x, \mu, \theta) &> 0
\end{aligned}
$$

The first two and the first FOC imply that the optimal $\mu(\theta)$ is increasing in θ. This and the third inequality imply that we cannot implement first best by setting $w(\theta) = \bar{w}$ because low θ guys would have incentive to report high θ. Clearly then $w(\theta)$ has to be decreasing in θ, i.e. higher θ guys consume more of the road and pay more for that. Now, we will use the Lemma that relates global IC to local IC. Global IC is equivalent to $\mu(\theta)$ is non-decreasing in θ and

$$R_{\mu}(x, \mu(\theta), \theta) \cdot \frac{d\mu(\theta)}{d\theta} + \frac{dw(\theta)}{d\theta} = 0$$

Given that we have the function $\mu(\theta)$ we can determine the function $w(\theta)$ in a similar fashion by noting that:

$$w(\theta) = w(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} R_{\mu}(x, \mu(t), t) \cdot \frac{d\mu(t)}{dt} dt$$

So again we just need to find that correct $w(\underline{\theta})$, i.e. one that satisfies:

$$\int w(\theta) p(\theta) \; d\theta = \bar{w}$$