Question 1 (short)

Consider a one-period economy populated by a continuum of ex-ante identical agents. In each period agents receive idiosyncratic productivity shocks $\theta \in \{\theta_L, \theta_H\}$ which are i.i.d over time and across agents. Let p_H and p_L denote the probabilities that θ equals θ_H and θ_L respectively. An agent with an idiosyncratic productivity shock θ who works l hours produces $y = \theta l$ units of output. Agent's utility is given by $u(c) - v(l)$, which can be written as $u(c) - v(\frac{y}{\theta})$. Assume throughout that the planner maximizes agents' ex-ante utility.

1. Assume that all variables, including the shock θ are publicly observed. Write the planner's problem in this environment.

2. Assume that the productivity shock θ and the hours of work l of each agent are privately observed by that agent only. The output of each agent is publicly observed.

a. State the revelation principle for this environment and use it to write the planner's problem for this environment.

b. Is the solution to the problem in part 1 a solution to the problem in part 2a? Prove why or why not.

3. Which incentive compatibility constraint is binding in the problem in part 2a? State and prove your claim in details. (Hint: you need to assume $v(\cdot)$ is a convex function).

Now consider a new economy identical to the one described in part 2 except that now there are two and only two agents. Write the planner's problem for this environment. Hint: In defining the allocation functions think carefully about what defines that state of the economy.

Solution:

1.

$$\max_{(c_L,c_H,y_L,y_H)} P_L\left[u(c_L)-v(\frac{y_L}{\theta_L})\right] \quad + \quad P_H\left[u(c_H)-v(\frac{y_H}{\theta_H})\right]$$
$$s.t.$$
$$P_Lc_L+P_Hc_H \quad \leq \quad P_Ly_L+P_Hy_H$$

We will call this **Problem 1**.

2a. *Revelation principle:* The equilibrium in any mechanism can be achieved as truth telling equilibrium in a direct trading mechanism, i.e. in a mechanism in which agents report their types directly.

$$\max_{(c_L,c_H,y_L,y_H)} P_L\left[u(c_L)-v(\frac{y_L}{\theta_L})\right] \quad + \quad P_H\left[u(c_H)-v(\frac{y_H}{\theta_H})\right]$$
$$s.t.$$

(1) $$P_Lc_L+P_Hc_H \quad \leq \quad P_Ly_L+P_Hy_H$$

(2) $$u(c_L)-v(\frac{y_L}{\theta_L}) \quad \geq \quad u(c_H)-v(\frac{y_H}{\theta_L})$$

(3) $$u(c_H)-v(\frac{y_H}{\theta_H}) \quad \geq \quad u(c_L)-v(\frac{y_L}{\theta_H})$$

We will call this **Problem 2a**.

2b. The solution to 1. has:

$$\begin{aligned} c_L &= c_H \\ \frac{v'(\frac{y_L}{\theta_L})}{\theta_L} &= \frac{v'(\frac{y_H}{\theta_H})}{\theta_H} \Rightarrow \\ \frac{y_L}{\theta_L} &< \frac{y_H}{\theta_H} \Rightarrow y_L < y_H \end{aligned}$$

The implication follows from convexity of v. The second incentive constraint, i.e. condition (3) is violated.

3. Consider the relaxed problem, which will be called **Problem 3**:

$$\max_{(c_L,c_H,y_L,y_H)} P_L\left[u(c_L)-v(\frac{y_L}{\theta_L})\right] \quad + \quad P_H\left[u(c_H)-v(\frac{y_H}{\theta_H})\right]$$
$$s.t.$$
$$\begin{aligned} P_Lc_L+P_Hc_H &\leq P_Ly_L+P_Hy_H \\ u(c_H)-v(\frac{y_H}{\theta_H}) &\geq u(c_L)-v(\frac{y_L}{\theta_H}) \end{aligned}$$

Since the solution to **Problem 1** has the IC (3) violated, it is binding in the solution to this problem. Denoting μ the Lagrange multiplier on the IC, the solution to **Problem 3** has:

$$\frac{u'(c_H)}{u'(c_L)} = \frac{1-\frac{\mu}{p_L}}{1+\frac{\mu}{p_H}} \Rightarrow c_H > c_L$$

The IC being satisfied with equality and convexity of v imply: $y_H > y_L$.

We will show that the IC (2) is satisfied at the solution to **Problem 3**. I.e. WTS:

$$u(c_L)-v(\frac{y_L}{\theta_L}) = u(c_H)-v(\frac{y_H}{\theta_L}) \Rightarrow u(c_H)-v(\frac{y_H}{\theta_H}) \geq u(c_L)-v(\frac{y_L}{\theta_H})$$

This is equivalent to:

$$u(c_H)-u(c_L) = v(\frac{y_H}{\theta_L})-v(\frac{y_L}{\theta_L}) \Rightarrow u(c_H)-u(c_L) \geq v(\frac{y_H}{\theta_H})-v(\frac{y_L}{\theta_H})$$

Thus it is enough to show:

$$v(\frac{y_H}{\theta_L})-v(\frac{y_L}{\theta_L}) \geq v(\frac{y_H}{\theta_H})-v(\frac{y_L}{\theta_H})$$

First, note that by $\theta_H > \theta_L$ we have:

$$\frac{y_L}{\theta_L} < \frac{y_H}{\theta_L} - \left[\frac{y_H}{\theta_H} - \frac{y_L}{\theta_H}\right]$$

Convexity and $\theta_H > \theta_L$ imply:

$$v(\frac{y_H}{\theta_L}) - v(\frac{y_L}{\theta_L}) \geq v(\frac{y_H}{\theta_H}) - v\left(\frac{y_H}{\theta_H} - (\frac{y_H}{\theta_L} - \frac{y_L}{\theta_L})\right) >$$
$$> v(\frac{y_H}{\theta_H}) - v\left(\frac{y_H}{\theta_H} - \left[\frac{y_H}{\theta_H} - \frac{y_L}{\theta_H}\right]\right) = v(\frac{y_H}{\theta_H}) - v(\frac{y_L}{\theta_H})$$

We are done.

If there are only 2 agents, clearly, we have aggregate uncertainty. Using the fact, that we weight the 2 agents with equal weights, we can argue that there actually only 3 states $\{LL, LH, HH\}$. The planner is choosing consumption and output for the 2 agents when both report low or when both report low and consumption and output for the high and low type when one reports low and the other high. The mechanism design problem is:

$$\max_{c(LL),c(HH),c_L(LH),c_H(LH),y(LL),y(HH),y_L(LH),y_H(LH)}$$
$$2P_L^2[u(c(LL)) - v(\frac{y(LL)}{\theta_L})] + 2P_H^2[u(c(HH)) - v(\frac{y(HH)}{\theta_H})] +$$
$$+2P_LP_H[u(c_H(LH)) - v(\frac{y_H(LH)}{\theta_H}) + u(c_L(LH)) - v(\frac{y_L(LH)}{\theta_L})]$$
$$s.t.$$
$$2c_L(LL) \leq 2y_L(LL)$$
$$2c_H(HH) \leq 2y_H(HH)$$
$$c_L(LH) + c_H(LH) \leq y_L(LH) + y_H(LH)$$
$$P_L[u(c(LL)) - v(\frac{y(LL)}{\theta_L})] + P_H[u(c_L(LH)) - v(\frac{y_L(LH)}{\theta_L})] \geq$$
$$\geq P_L[u(c_H(LH)) - v(\frac{y_H(LH)}{\theta_L})] + P_H[u(c(HH)) - v(\frac{y(HH)}{\theta_L})]$$
$$P_L[u(c_H(LH)) - v(\frac{y_H(LH)}{\theta_H})] + P_H[u(c(HH)) - v(\frac{y(HH)}{\theta_H})] \geq$$
$$\geq P_L[u(c(LL)) - v(\frac{y(LL)}{\theta_H})] + P_H[u(c_L(LH)) - v(\frac{y_L(LH)}{\theta_H})]$$

Question 2 (long)

Consider an economy populated by a continuum of measure one of ex–ante identical households. The economy lasts for two periods. In each period, an agent has an stochastic endowment $y \in \{y_L, y_H\}$ with $y_L < y_H$. The shock process is i.i.d. over agents and time with p_L and p_H denoting the probabilies of y_L and y_H. Suppose the planner can save and borrow at rate R. Agent' utility is $u(c_1) + \beta u(c_2)$ where c_t denotes the consumption at time t.Assume $u(c)$ strictly concave and twice continuously differentiable with $u'(c) \to \infty$ as $c \to 0$ from above and that $u(c)$ has decreasing absolute risk aversion. Assume the endowment is privately observed by the agents. Assume throughout that the planner maximizes agents' ex-ante utility.

1. Assume that agents cannot save. Write the planner's problem in this environment. (Hint: How can the incentive constraints in the second period be used to simplify the definition of allocations and the relevant constraints?) Write down only the simplified problem but give an argument as to why you did.

2. Assume agents can save at rate R and that the amount saved is not observed by the planner. Write the planner's problem in this environment.

3. a. Prove that it suffices to restrict attention to allocations in which agents have zero savings.

b. If agents can save at rate $R^* < R$ does the claim in part (a.) hold? Why?.

c. If agents can save at rate $R^* > R$ does the claim in part (a.) hold ? Why?

4. Show that any allocation that solves the constrained efficient problem satisfies:

$$u'(c_1(y_s)) = \beta R \sum_{j=1}^{2} \pi_j u'(c_2(y_s, y_j)) \quad \forall s$$

To prove this claim you can use without proving the following lemma.

1. *Lemma.* Let u be strictly concave. Let $\varepsilon, \delta > 0$ satisfy $\delta > R\varepsilon$ and define

$$\begin{aligned} Z(m) &\equiv \max_{k \geq 0} u(m-k) + \beta E_y(y + Rk) \\ W(m) &\equiv \max_{k \geq 0} u(m-k+\varepsilon) + \beta E_y(y + Rk - \delta) \end{aligned}$$

If $Z(m_a) = W(m_a)$ and $m_b > m_a$ then $Z(m_b) > W(m_b)$.

Solution: inspired by Ljungquist, Sargent, chapter 19.6, where they do much of this stuff for T periods.

1. Transfers in the second period must be independent of the agents second period report, otherwise all agents would report the endowment that gives the higher transfer. Thus transfers in both periods will be functions of first period's report only. We will denote $b_i(j)$ a transfer in period $i \in \{1, 2\}$ for an agent who reports $j \in \{L, H\}$ in the first period. The planner's problem is:

$$\begin{aligned}\max_{b_1(L),b_1(H),b_2(L),b_2(H)} \quad & P_L\left[u(y_L + b_1(L)) + \beta\left(P_L u(y_L + b_2(L)) + P_H u(y_H + b_2(L))\right)\right] + \\ + \quad & P_H\left[u(y_H + b_1(H)) + \beta\left(P_L u(y_L + b_2(H)) + P_H u(y_H + b_2(H))\right)\right]\end{aligned}$$

s.t.

$$P_L b_1(L) + P_H b_1(H) + \frac{1}{R}\left[P_L b_2(L) + P_H b_2(H)\right] \leq 0$$

$$\begin{aligned}&u(y_H + b_1(H)) + \beta\left(P_L u(y_L + b_2(H)) + P_H u(y_H + b_2(H))\right) \geq \\ &\quad u(y_H + b_1(L)) + \beta\left(P_L u(y_L + b_2(L)) + P_H u(y_H + b_2(L))\right)\end{aligned}$$

$$\begin{aligned}&u(y_L + b_1(L)) + \beta\left(P_L u(y_L + b_2(L)) + P_H u(y_H + b_2(L))\right) \geq \\ &\quad u(y_L + b_1(H)) + \beta\left(P_L u(y_L + b_2(H)) + P_H u(y_H + b_2(H))\right)\end{aligned}$$

2. To simplify notation, let's first denote y the true realization of the endowment process in the first period - i.e. H or L and $\hat{y}(y)$ the report as a function of the true realization. Let $\hat{s}(y)$ be the individual savings decision by the agent as a function of the income realization. Then we can define the expected utility associated with a reporting strategy $\hat{y}$ and savings decision $\hat{s}$ given transfer scheme $b(\hat{y})$ - a function of reports, as follows:

$$\begin{aligned}&\Gamma(\hat{y}, \hat{s}, b) := \\ &P_L u(y_L + b_1(\hat{y}(L)) - \hat{s}(L)) + \\ &P_L\beta\left[P_L u(y_L + b_2(\hat{y}(L)) + R\hat{s}(L)) + P_H u(y_H + b_2(\hat{y}(L)) + R\hat{s}(L))\right] + \\ &P_H u(y_H + b_1(\hat{y}(H)) - \hat{s}(H)) + \\ &P_H\beta\left[P_L u(y_L + b_2(\hat{y}(H)) + R\hat{s}(H)) + P_H u(y_H + b_2(\hat{y}(H)) + R\hat{s}(H))\right]\end{aligned}$$

The planner's problem then is:

$$\max_{b_1(L),b_1(H),b_2(L),b_2(H),s(L),s(H)}$$
$$P_L\left[u(y_L+b_1(L)-s(L))+\beta\left(P_L u(y_L+b_2(L)+Rs(L))+P_H u(y_H+b_2(L)+Rs(L))\right)\right]+$$
$$+P_H\left[u(y_H+b_1(H)-s(H))+\beta\left(P_L u(y_L+b_2(H)+Rs(H))+P_H u(y_H+b_2(H)+Rs(H))\right)\right]$$

s.t.

$$P_L b_1(L)+P_H b_1(H)+\frac{1}{R}\left[P_L b_2(L)+P_H b_2(H)\right]\leq 0$$

$$\Gamma(y,s,b)=\max_{\hat{y}\in\{L,H\},\hat{s}(y)\leq y+b_1(\hat{y}(y))}\Gamma(\hat{y},\hat{s},b)$$

3a. **Claim:** if an allocation that has $s(L)>0$ or $s(H)>0$ is feasible (i.e. satisfies the first constraint) and incentive compatible then $\exists$ another feasible and incentive compatible allocation with $s(L)=s(H)=0$.

Proof: let's call the original transfer scheme b. Define the new transfer scheme as (by incentive compatibility we have $y=\hat{y}$):

$$\begin{aligned} b_1'(y) &:= b_1(y)-s(y) \\ b_2'(y) &:= b_2(y)+Rs(y) \end{aligned}$$

Since the original allocation transfer scheme was feasible, this one is feasible with truth telling as well. Thus we need to verify that $(y,0,b')$ is incentive compatible. First note that $\Gamma(y,s,b)=\Gamma(y,0,b')$. Now suppose by contradiction that $(y,0,b')$ is not IC. Then $\exists$ a feasible savings' and reporting strategy s',y' s.t. $\Gamma(y',s',b')>\Gamma(y,0,b')$. Now we define $s''(y)=s'(y)+s(y'(y))$. We get that $\Gamma(y',s'',b)=\Gamma(y,s',b')>\Gamma(y,0,b')=\Gamma(y,s,b)$. Thus $\Gamma(y',s'',b)>\Gamma(y,s,b)$ which is a contradiction.$\|$

3b. The argument still goes through, we will get that $\Gamma(y,s,b)\leq\Gamma(y,0,b')$. The rest is the same. Thus we can restrict attention to allocations with individual savings 0. Note that the optimal allocation in the case of $R^*<R$ could be different than if $R^*=R$. Consider the extreme case $R^*=0$, i.e. agents can't save at all. We get rid of the double deviation problem.

3c. The argument breaks down in that we cannot argue that $\Gamma(y, s, b) \leq \Gamma(y, 0, b')$. In fact at any allocation the government can borrow more resources from the outside world at rate R transfer it to the agent and have him/her save at rate R^*. The net gain is $(R^* - R)$ multiplied by the amount borrowed and saved. Thus consumption and utility of the agent strictly increases. This can be done in an incentive compatible way. In other words the problem doesn't have a solution.

4. This is done in Ljungqvist, Sargent, chapter 19.6.4 for the case of 2 periods, just like here. Please take a look there, or in the original Cole, Kocherlakota paper, proof of proposition 2.