Problem Set 2
UMN, Macroeconomic Theory 8108, Spring 2007

- due April 5^{th} in lecture, each student submits his or her version of the homework

Problem 1

Consider the model of one sided no commitment as in LS section 19.3. In particular, focus on the recursive problem as in (19.3.4) - (19.3.8).

1. (thanks to Futoshi and Machiko for inspiration) Suppose that $v \geq v_{aut}$. Prove that the promise keeping constraint (19.3.5) is always binding. Hint: prove by contradiction. Suppose it is not binding, construct a new allocation s.t. (19.3.5) and (19.3.6) are satisfied and the new allocation yields higher value of the objective function.

2. Suppose now that we don't require that $w_s \geq v_{aut}$ and for some reason $v < v_{aut}$. Show that the requirement that $w_s \geq v_{aut}$ is WLOG. In particular, show that $\forall v \leq v_{aut} : w_s(v) = w_s(v_{aut}) \geq v_{aut}$, $c_s(v) = c_s(v_{aut}), P(v) = P(v_{aut})$.

Problem 2

Consider the section 19.4 in LS on a Lagrangian method. Write the problem recursively as a Bellman equation in which the sum of the multipliers is the state variable. Try to derive as many properties of the optimal contract in section 19.3 as you can using this Bellman equation. (Before you start read Cooley, Marimon, Quadrini, *Journal of Political Economy* 2004. See especially the recursive formulation equations 12-14 on page 826 and the resulting analysis of the optimal contract.)

Problem 3

1. Consider the model in section 19.5. Suppose now instead of being discrete that the set of states is a continuous variable $Y = [\underline{y}; \overline{y}]$ rather than a finite set of points. State and prove the analog of the Lemma stated in class that an allocation is globally incentive compatible if and only if the allocations are monotone (the b's are weakly decreasing and the w's are weakly increasing) and the local incentive constraints hold (in the discrete case these are one up and one down constraints). Before you start read Section 7.3.1 in Fudenberg and Tirole and mimic the proofs of Theorems 7.1-7.3.

2. Prove that P is concave in the above set-up with continuous states.

Problem 4

Start with the line on page 666 in the proof of the proposition that says: "But Thomas and Worrall construct a new contract ... that is incentive compatible and that offers both the borrower and the lender no less utility". Denote with a caret $\hat{}$ these new contracts. Write out in detail the construction. Show, in particular, that the final contract (with the constant added to the b's and the w's) is incentive compatible.

The issue that you are supposed to address is on the 8^{th} line from the bottom, where it says "... adding a constant to each b_s to leave $\sum_s \Pi_s b_s$ constant cannot make the borrower worse off." That is true by concavity of u and the fact that the constructed allocation is a mean preserving decrease in spread. However LS claim that "So in this new contract, $C_{s,s-1} = 0$". This is not true if u is strictly concave and $\exists i, j : b_i \neq b_j$. Fix this, i.e. find allocations that are going to have $C_{s,s-1} = 0$.

Hint: what might work is to add the constant in the process of making $C_{s,s-1} = 0$ rather that afterwards.

Problem 5

Explore a modified approach to prove that P in section 19.5 is concave. The approach consists of the following steps.

1. Consider a relaxed Bellman equation with just the downward constraints and monotonicity of the policies b and w. Apply SLP to prove the value function is concave. I.e. define the operator T, show it has a unique fixed point. Show that the constraint set is convex etc. and use this to prove that the fixed point is concave.

2. Prove that in the relaxed problem the downward constraint binds.

3. Prove that the solution to the relaxed problem is a solution to the original problem with both the downward and upward constraint. Try to do so by arguing, using the earlier lemma that in the relaxed problem that if the downward constraint binds, the current value function is concave, and the monotonicity of policies holds then the upward constraint is automatically satisfied.

Explore means you should try to prove it using this or similar approach. It might or it might not work.