Problem Set 2
UMN, Macroeconomic Theory 8108, Spring 2007

- due April 5^{th} in lecture, each student submits his or her version of the homework

Problem 1

Consider the model of one sided no commitment as in LS section 19.3. In particular, focus on the recursive problem as in (19.3.4) - (19.3.8).

1. (thanks to Futoshi and Machiko for inspiration) Suppose that $v \geq v_{aut}$. Prove that the promise keeping constraint (19.3.5) is always binding. Hint: prove by contradiction. Suppose it is not binding, construct a new allocation s.t. (19.3.5) and (19.3.6) are satisfied and the new allocation yields higher value of the objective function.

2. Suppose now that we don't require that $w_s \geq v_{aut}$ and for some reason $v < v_{aut}$. Show that the requirement that $w_s \geq v_{aut}$ is WLOG. In particular, show that $\forall v \leq v_{aut} : w_s(v) = w_s(v_{aut}) \geq v_{aut}$, $c_s(v) = c_s(v_{aut}), P(v) = P(v_{aut})$.

Solution

1. The problem is

$$P(v) := \max_{w_s, c_s} \sum_{s=1}^{S} \Pi_s \left[\overline{y}_s - c_s + \beta P(w_s)\right] \;\; s.t.$$

$$\sum_{s=1}^{S} \Pi_s \left[u(c_s) + \beta w_s\right] \geq v \tag{1}$$

$$u(c_s) + \beta w_s \geq u(\overline{y}_s) + \beta v_{aut}, \;\; \forall s$$

Now suppose by contradiction that the PKC is not binding, i.e.

$$\sum_{s=1}^{S} \Pi_s \left[u(c_s) + \beta w_s\right] > v \tag{2}$$

An easy contradiction step done in Problem Set 1, 3 (iii) shows that at least one of the IC's is not binding, i.e. $\exists s^*$ s.t.

$$u(c_{s^*}) + \beta w_{s^*} > u(\overline{y}_{s^*}) + \beta v_{aut} \tag{3}$$

Clearly now by continuity of u $\exists \varepsilon > 0$ s.t.

$$u(c_{s^*} - \varepsilon) + \beta w_{s^*} > u(\overline{y}_{s^*}) + \beta v_{aut} \tag{4}$$

$$\sum_{s \neq s^*}^{S} \Pi_s \left[u(c_s) + \beta w_s\right] + u(c_{s^*} - \varepsilon) + \beta w_{s^*} > v \tag{5}$$

Thus this new allocation with $c_{s^*} - \varepsilon$ is in the constraint set. Moreover it delivers strictly higher value to the money lender, thus the original allocation was not optimal.

2. Note that we should prove our result without using what has been proved about P using this result (circular reasoning). Moreover, we haven't proved concavity and/or differentiability for $v < v_{aut}$. Now consider the problem (1) with $v < v_{aut}$. The PKC is:

$$\sum_{s=1}^{S} \Pi_s \left[u(c_s) + \beta w_s \right] \geq v \tag{6}$$

Summing over all IC's gives:

$$\sum_{s=1}^{S} \Pi_s \left[u(c_s) + \beta w_s \right] \geq v_{aut} > v \tag{7}$$

Thus the PKC is not binding as long as $v < v_{aut}$. Now what this says is that even though the explicit promise is that the agent will get $v < v_{aut}$, he knows that he will in fact get at least v_{aut}. Technically, as equation (7) shows, the the problem (1) with $v < v_{aut}$ is equivalent to (where the PKC is actually redundant):

$$\begin{aligned} \max_{w_s, c_s} \sum_{s=1}^{S} \Pi_s \left[\overline{y}_s - c_s + \beta P(w_s) \right] \ \ s.t. \\ \sum_{s=1}^{S} \Pi_s \left[u(c_s) + \beta w_s \right] \geq v_{aut} \\ u(c_s) + \beta w_s \geq u(\overline{y}_s) + \beta v_{aut}, \ \forall s \end{aligned} \tag{8}$$

But this is the definition of $P(v_{aut})$. We are done with

$$\forall v \leq v_{aut} : w_s(v) = w_s(v_{aut}), c_s(v) = c_s(v_{aut}), P(v) = P(v_{aut}) \tag{9}$$

To establish that $\forall v : w_s(v) \geq v_{aut}$, we can just refer to results established earlier. Or we can assume by contradiction that: $w_s(v) < v_{aut}$. Consider now an alternative allocation that sets $w_s(v) = v_{aut}$ and decreases $c_s(v)$ by the appropriate amount to keep the IC satisfied. $P(w_s)$ remains unchanged by the above argument. Thus the objective function is strictly higher. Thus $w_s(v) < v_{aut}$ is not optimal. That's it.

Problem 2

Consider the section 19.4 in LS on a Lagrangian method. Write the problem recursively as a Bellman equation in which the sum of the multipliers is the state variable. Try to derive as many properties of the optimal contract in section 19.3 as you can using this Bellman equation. (Before you start read Cooley, Marimon, Quadrini, *Journal of Political Economy* 2004. See especially the recursive formulation equations 12-14 on page 826 and the resulting analysis of the optimal contract.)

Solution:

The Lagrangean in LS is:

$$\begin{aligned} L = \quad & E_{-1}\sum_{t=0}^{\infty}\beta^t\left\{(y_t-c_t)+(\mu_t+\phi)u(c_t)-(\mu_t-\mu_{t-1})(u(y_t)-\beta v_{aut})\right\}-\phi v \\ & s.t. \\ & \mu_t \geq \mu_{t-1} \\ & \mu_{-1}=0 \end{aligned}$$

Using the Marcet, Marimon (1999) approach the maximization problem can be written as:

$$\begin{aligned} P(v) = \quad & \min_{\phi}\max_{c_t}\min_{\mu_t} E_{-1}\sum_{t=0}^{\infty}\beta^t\left\{(y_t-c_t)+(\mu_t+\phi)u(c_t)-(\mu_t-\mu_{t-1})(u(y_t)+\beta v_{aut})\right\}-\phi v \\ & s.t. \\ & \mu_t \geq \mu_{t-1} \\ & \mu_{-1}=0 \end{aligned}$$

Or alternatively:

$$\begin{aligned} P(v) = \quad & \min_{\phi}\max_{c_t}\min_{\mu_t} E_{-1}\sum_{t=0}^{\infty}\beta^t\left\{(y_t-c_t)+(\mu_t+\phi)u(c_t)-(\mu_t-\mu_{t-1})(u(y_t)+\beta v_{aut})-(1-\beta)\phi v\right\} \\ & s.t. \\ & \mu_t \geq \mu_{t-1} \\ & \mu_{-1}=0 \end{aligned}$$

This can be written as:

$$P(v) = \quad \min_{\phi} R(\phi)-\phi v$$

With $R(\phi)$ being defined as:

$$\begin{aligned} R(\phi) = \quad & \max_{c_t}\min_{\mu_t} E_{-1}\sum_{t=0}^{\infty}\beta^t\left\{(y_t-c_t)+(\mu_t+\phi)u(c_t)-(\mu_t-\mu_{t-1})(u(y_t)+\beta v_{aut})\right\} \\ & s.t. \\ & \mu_t \geq \mu_{t-1} \\ & \mu_{-1}=0 \end{aligned}$$

Now, we do the above to get rid of v and the choice variable ϕ. Assuming ϕ is fixed, taking the μ_t as state, we can rewrite the above recursively as:

$$
\begin{aligned}
Q(\mu) = \quad & \max_{(c_s)_{s=1}^S} \min_{(\mu_s)_{s=1}^S} \sum_{s=1}^S \Pi_s \left\{ (y_s - c_s) + (\mu_s' + \phi) u(c_s) - (\mu_s' - \mu)(u(y_s) + \beta v_{aut}) + \beta Q(\mu_s') \right\} \\
& s.t. \\
& \mu_s' \geq \mu
\end{aligned}
$$

Now, we can basically reproduce the analysis of LS. Let's see how it would go. First, we should establish whether Q is increasing or decreasing in μ. Intuitively, it should be decreasing. Next, since we are minimizing wrt the state μ, Q should be convex in μ - opposite of the value function on the original recursive set-up in LS, section 19.3. Proof of convexity is at the end of this section. Finally, we should also show that it is differentiable.

Now, we will proceed under these assumptions. Use $\Pi_s \lambda_s$ as the Lagrange multiplier on the constraint in state s. Then the FOC and ET imply:

$$
\begin{aligned}
u(c_s) + \beta Q'(\mu_s') + \lambda_s &= u(y_s) + \beta v_{aut} && (10) \\
(\mu_s' + \phi) u'(c_s) &= 1 && (11) \\
Q'(\mu) &= \sum_{s=1}^S \Pi_s [u(y_s) + \beta v_{aut} - \lambda_s] = \sum_{s=1}^S \Pi_s [u(c_s) + \beta Q'(\mu_s')] && (12)
\end{aligned}
$$

Now if $\mu = \mu_s'$, i.e. the incentive constraint is not binding, then c is as in the previous period. If, on the other hand, the incentive constraint is binding $\mu < \mu_s'$, consumption increases by equation (11). Thus consumption is non decreasing. Now, using convexity of Q it is straight forward (the proofs would resemble those of the max operators in Problem Set 1, problem 3(i)) to show that if ϕ is low enough (corresponds to $v = v_{aut}$) consumption is actually a function of the highest y realized so far. If on the other hand ϕ is high, consumption will be the maximum of the initial c and the one determined by the highest realization of y. Thus consumption is constant once y_S is realized. We get the same pattern as in the original set-up.

Claim: Q is convex.

"Proof": First I will present an informal proof. The idea here will be to find a mapping between the objects in this set-up and the original one in LS, section 19.3 and exploit the properties proved there. First note that FOC imply:

$$
\frac{1}{(\mu_s' + \phi)} = u'(c_s) = -\frac{1}{P'(w_s)} \tag{13}
$$

Assuming that the IC is binding (this is the only way to get to state w_s, μ_s' anyways), the IC's imply:

$$
Q'(\mu_s') = w_s \tag{14}
$$

Combining these 2 we will get:

$$
\begin{aligned}
P'(Q'(\mu_s')) &= -(\mu_s' + \phi) && (15) \\
Q'(\mu_s') &= P'^{-1}(-(\mu_s' + \phi)) && (16)
\end{aligned}
$$

Now, since P is concave, P' is decreasing and P'^{-1} is decreasing. Thus Q' is increasing. What is a little disturbing is that (14) suggests that whether Q is increasing or decreasing depends on w_s. Comments welcome. We can also proof convexity with brute force as in Problem Set 1, problem 3(ii).

Proof: Consider a convex function Q and define the operator T:

$$TQ(\mu) = \max_{(c_s)_{s=1}^S} \min_{(\mu_s)_{s=1}^S} \sum_{s=1}^S \Pi_s \left\{ (y_s - c_s) + (\mu_s' + \phi) u(c_s) - (\mu_s' - \mu)(u(y_s) + \beta v_{aut}) + \beta Q(\mu_s') \right\} \ s.t.$$
$$\mu_s' \geq \mu$$

Pick arbitrary $\mu_1, \mu_2 \in [0, \infty]$. Now pick arbitrary $\lambda \in [0,1]$ and define $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$. To prove TP is convex we wish to show:

$$TQ(\mu) \leq \lambda TQ(\mu_1) + (1-\lambda) TQ(\mu_2)$$

For any $\mu \in [0, \infty]$ define the solution to the right hand side of the above operator as $c_s(\mu), \mu_s'(\mu)$ for all s. Thus for μ_1, μ_2, μ we have $c_s(\mu_1), \mu_s'(\mu_1), c_s(\mu_2), \mu_s'(\mu_2)$ and $c_s(\mu), \mu_s'(\mu)$. Define:

$$\begin{aligned} \widehat{c}_s &= \lambda c_s(\mu_1) + (1-\lambda) c_s(\mu_2) \\ \widehat{\mu'}_s &= \lambda \mu_s'(\mu_1) + (1-\lambda)\mu_s'(\mu_2) \end{aligned}$$

Clearly $\widehat{\mu'}_s > \mu$, i.e. this allocation satisfies the constraint on the RHS of the operator T. Thus we get:

$$TQ(\mu) = \sum_{s=1}^S \Pi_s \left\{ (y_s - c_s(\mu)) + (\mu_s'(\mu) + \phi) u(c_s(\mu)) - (\mu_s'(\mu) - \mu)(u(y_s) + \beta v_{aut}) + \beta Q(\mu_s'(\mu)) \right\} \leq$$

since we are minimizing wrt μ_s'

$$\begin{aligned} &\leq \sum_{s=1}^S \Pi_s \left\{ (y_s - c_s(\mu)) + (\widehat{\mu'}_s + \phi) u(c_s(\mu)) - (\widehat{\mu'}_s - \mu)(u(y_s) + \beta v_{aut}) + \beta Q(\widehat{\mu'}_s) \right\} = \\ &= \sum_{s=1}^S \Pi_s \left\{ (y_s - c_s(\mu)) + (\widehat{\mu'}_s + \phi) u(c_s(\mu)) - (\widehat{\mu'}_s - \mu)(u(y_s) + \beta v_{aut}) + \beta Q(\lambda\mu_s'(\mu_1) + (1-\lambda)\mu_s'(\mu_2)) \right\} \leq \end{aligned}$$

by convexity of Q

$$\leq \sum_{s=1}^S \Pi_s \left\{ (y_s - c_s(\mu)) + (\mu_s'(\mu) + \phi) u(c_s(\mu)) - (\mu_s'(\mu) - \mu)(u(y_s) + \beta v_{aut}) + \beta[\lambda Q(\mu_s'(\mu_1)) + (1-\lambda) Q(\mu_s'(\mu_2))] \right\} =$$

by definition $\widehat{\mu'}_s$ above

$$\begin{aligned} &= \lambda \left[\sum_{s=1}^S \Pi_s \left\{ (y_s - c_s(\mu)) + (\mu_s'(\mu_1) + \phi) u(c_s(\mu)) - (\mu_s'(\mu_1) - \mu)(u(y_s) + \beta v_{aut}) + \beta Q(\mu_s'(\mu_1)) \right\} \right] + \\ &+ (1-\lambda) \left[\sum_{s=1}^S \Pi_s \left\{ (y_s - c_s(\mu)) + (\mu_s'(\mu_2) + \phi) u(c_s(\mu)) - (\mu_s'(\mu_2) - \mu)(u(y_s) + \beta v_{aut}) + \beta Q(\mu_s'(\mu_2)) \right\} \right] \leq \end{aligned}$$

since $c_s(\mu)$ is not optimal for μ_1, μ_2 and we are maximizing wrt c_s

$$\begin{aligned} &\leq \lambda \left[\sum_{s=1}^S \Pi_s \left\{ (y_s - c_s(\mu_1)) + (\mu_s'(\mu_1) + \phi) u(c_s(\mu_1)) - (\mu_s'(\mu_1) - \mu)(u(y_s) + \beta v_{aut}) + \beta Q(\mu_s'(\mu_1)) \right\} \right] + \\ &+ (1-\lambda) \left[\sum_{s=1}^S \Pi_s \left\{ (y_s - c_s(\mu_2)) + (\mu_s'(\mu_2) + \phi) u(c_s(\mu_2)) - (\mu_s'(\mu_2) - \mu)(u(y_s) + \beta v_{aut}) + \beta Q(\mu_s'(\mu_2)) \right\} \right] = \\ &= \lambda TQ(\mu_1) + (1-\lambda) TQ(\mu_2) \end{aligned}$$

Thus T maps convex functions into convex functions. Since it is a contraction, it has a unique convex fixed point.

Problem 3

1. Consider the model in section 19.5. Suppose now instead of being discrete that the set of states is a continuous variable $Y = [\underline{y}; \overline{y}]$ rather than a finite set of points. State and prove the analog of the Lemma stated in class that an allocation is globally incentive compatible if and only if the allocations are monotone (the b's are weakly decreasing and the w's are weakly increasing) and the local incentive constraints hold (in the discrete case these are one up and one down constraints). Before you start read Section 7.3.1 in Fudenberg and Tirole and mimic the proofs of Theorems 7.1-7.3.

2. Prove that P is concave in the above set-up with continuous states.

Solution sketch:

1. Let's denote the pdf on the state space μ. The money lender's problem is:

$$\max_{w(y),b(y)} \int_Y [-b(y) + \beta P(w(y))]\, d\mu \ s.t.$$
$$\int_Y [u(y + b(y)) + \beta w(y)]\, d\mu = v$$
$$u(y + b(y)) + \beta w(y) \geq u(y + b(\widehat{y})) + \beta w(\widehat{y}),\ \forall \widehat{y} \in Y$$

Note in this language an allocation are two functions $b, w : Y \longrightarrow \Re$.

Definition: an allocation $b(y), w(y)$ is *globally incentive compatible* if $\forall y \in Y$:

$$u(y + b(y)) + \beta w(y) \geq u(y + b(\widehat{y})) + \beta w(\widehat{y}),\ \forall \widehat{y} \in Y \tag{17}$$

Definition: an allocation $b(y), w(y)$ is *locally incentive compatible* if $\forall y \in Y$:

$$\frac{\partial w(y)}{\partial y} \geq 0 \tag{18}$$
$$\frac{\partial b(y)}{\partial y} \leq 0 \tag{19}$$
$$\frac{\partial u(y + b(y))}{\partial c}\frac{\partial b(y)}{\partial y} + \beta\frac{\partial w(y)}{\partial y} = 0 \tag{20}$$

Proposition: an allocation is GIC iff it is LIC.

Proof: I will be a little informal here in the sense that I assume that the policies are differentiable everywhere.

$\Longrightarrow$

This is sort of obvious, one just follows the steps in Fudenberg, Tirole. First note that

$$b(y), w(y) := \arg\max_{\widehat{y} \in Y} u(y + b(\widehat{y})) + \beta w(\widehat{y}) \tag{21}$$

Under differentiability assumptions, the first order condition, the derivative of the FOC as identity in y and the second order condition yield (18) - (20).

$\Longleftarrow$

Suppose by contradiction that (17) is violated, i.e. $\exists \widehat{y}$ s.t.

$$u(y + b(y)) + \beta w(y) - u(y + b(\widehat{y})) - \beta w(\widehat{y}) < 0 \tag{22}$$

Case 1: $\widehat{y} < y$. Then the above is equivalent to:

$$\int_{\widehat{y}}^{y} [u'(y + b(t))b'(t) + \beta w'(t)]dt < 0 \tag{23}$$

However $\forall t \in [\widehat{y}, y] : u'(t + b(t)) \geq u'(y + b(t))$ and since $b'(t) \leq 0 : u'(t + b(t))b'(t) \leq u'(y + b(t))b'(t)$. By monotonicity of integral then:

$$\int_{\widehat{y}}^{y} [u'(t + b(t))b'(t) + \beta w'(t)]dt \leq \int_{\widehat{y}}^{y} [u'(y + b(t))b'(t) + \beta w'(t)]dt \tag{24}$$

However, the LHS integral equals 0 by definition. This is a contradiction.

Case 2: $\widehat{y} > y$ is similar.

2. We want to show that the value function is concave. We define the operator T on the value function as:

$$\begin{aligned}
&TP(v) = \max_{w(y),b(y)} \int_Y [-b(y) + \beta P(w(y))] \, d\mu \ s.t. \\
&\int_Y [u(y + b(y)) + \beta w(y)] \, d\mu = v \\
&u(y + b(y)) + \beta w(y) \geq u(y + b(\widehat{y})) + \beta w(\widehat{y}), \ \forall \widehat{y} \in Y
\end{aligned}$$

Now we want to how it maps concave functions into concave functions. Pick arbitrary v_1, v_2. Now pick arbitrary $\lambda \in [0, 1]$ and define $v_\lambda = \lambda v_1 + (1 - \lambda)v_2$. To prove TP is concave we wish to show:

$$TP(v_\lambda) \geq \lambda TP(v_1) + (1 - \lambda)TP(v_2)$$

For any $v \in [v_{aut}, \overline{v}]$ define the solution to the right hand side of the above operator as $b_v(y), w_v(y)$ for all y. Thus for v_1, v_2, v_λ we have $b_{v_1}(y), w_{v_1}(y), b_{v_2}(y), w_{v_2}(y)$ and $w_{v_\lambda}(y), b_{v_\lambda}(y)$. Define $\forall y$:

$$\begin{aligned}
\widehat{b}_{v_\lambda}(y) : u(y + \widehat{b}_{v_\lambda}(y)) &= \lambda u(y + b_{v_1}(y)) + (1 - \lambda)u(y + b_{v_2}(y)) \\
\widehat{w}_{v_\lambda}(y) : &= \lambda w_{v_1}(y) + (1 - \lambda)w_{v_2}(y)
\end{aligned}$$

Now, we would like to show that at these $\widehat{}$ allocations:

1. payoff to the lender is higher at this allocation than $\lambda TP(v_1) + (1 - \lambda)TP(v_2)$,
2. the allocation is in the constraint set.

The first one is obvious by concavity of u and P. The second one is not. PKC is satisfied by construction. The IC is satisfied for downward deviations by the DARA Lemma we proved in Problem Set 1. To verify that the IC is satisfied upward, we need to construct a new allocation, that has PKC satisfied, delivers weakly higher payoff to the lender and has b monotone and downward IC satisfied. Then again by a Lemma proved in Problem Set 1, the upward IC will be satisfied as well. The construction is similar to Problem 4, but rather cumbersome so it will not be repeated here.

Problem 4

Start with the line on page 666 in the proof of the proposition that says: "But Thomas and Worrall construct a new contract ... that is incentive compatible and that offers both the borrower and the lender no less utility". Denote with a caret $\widehat{}$ these new contracts. Write out in detail the construction. Show, in particular, that the final contract (with the constant added to the b's and the w's) is incentive compatible.

The issue that you are supposed to address is on the 8^{th} line from the bottom, where it says "... adding a constant to each b_s to leave $\sum_s \Pi_s b_s$ constant cannot make the borrower worse off." That is true by concavity of u and the fact that the constructed allocation is a mean preserving decrease in spread. However LS claim that "So in this new contract, $C_{s,s-1} = 0$". This is not true if u is strictly concave and $\exists i, j : b_i \neq b_j$. Fix this, i.e. find allocations that are going to have $C_{s,s-1} = 0$.

Hint: what might work is to add the constant in the process of making $C_{s,s-1} = 0$ rather that afterwards.

Solution 1:

The hint points in the correct direction. It helped me realize that adding a constant to each b_s^* will actually not affect the way we change the b_s^* to $\widehat{b}_s$, since we only change them to get monotonicity. Anyways, the idea will be to reverse the process of adjusting b_s^* and w_s^*.

Step 1: Define recursively:

$$\begin{aligned} \widehat{b}_1 &= b_1^* \\ \widehat{b}_s &= \begin{matrix} \widehat{b}_{s-1} \; if \; b_s^* > \widehat{b}_{s-1} \\ b_s^* \; otherwise \end{matrix} \end{aligned}$$

We did this to get $\widehat{b}_s \leq \widehat{b}_{s-1}$. Recall that we have

$$\begin{aligned} C_{s,s-1}^* : &= u(y_s + b_s^*) + \beta w_s^* - u(y_s + b_{s-1}^*) - \beta w_{s-1}^* \geq 0 \\ w_s^* &\geq w_{s-1}^* \end{aligned}$$

Then:

$$\widehat{C}_{s,s-1} : = u(y_s + \widehat{b}_s) + \beta w_s^* - u(y_s + \widehat{b}_{s-1}) - \beta w_{s-1}^* \geq 0$$

Now define:

$$\widehat{\widehat{b}}_s : = \widehat{b}_s + \sum_{s=1}^{S} \Pi_s (b_s^* - \widehat{b}_s)$$

By concavity, we still have:

$$\widehat{\widehat{C}}_{s,s-1} := u(y_s + \widehat{\widehat{b}}_s) + \beta w_s^* - u(y_s + \widehat{\widehat{b}}_{s-1}) - \beta w_{s-1}^* \geq 0$$

Moreover, since this procedure is a mean preserving reduction in spread, this allocation delivers weakly higher utility to the borrower and the same utility to the lender.

Step 2: Now we decrease the w's to make $\widehat{\widehat{C}}_{s,s-1} = 0$ and add the constant to all w's so as to keep the average $\sum_{s=1}^{S} \Pi_s w_s$ constant. Let's denote these new allocations $\widehat{w}_s$. This procedure is a mean preserving reduction in spread thus delivers weakly higher utility to the lender (we assume P_{k-1} strictly concave) and the same utility to the borrower. Note that the monotonicity of w's is preserved. Thus we have created an allocation with:

$$\begin{aligned}\widehat{\widehat{C}}_{s,s-1} &= 0 \\ \widehat{\widehat{w}}_s &\geq \widehat{\widehat{w}}_{s-1} \\ \widehat{\widehat{b}}_s &\leq \widehat{\widehat{b}}_{s-1}\end{aligned}$$

This implies by a previously proved Lemma:

$$\widehat{\widehat{C}}_{s-1,s} \geq 0$$

Thus this allocation is in the constraint set. Moreover it delivers weakly higher value of the objective function than the * allocation. Thus:

$$\begin{aligned}TP_{k-1}(v^*) &= TP_{k-1}(\lambda v_0 + (1-\lambda)v_1) \geq \sum_{s=1}^{S} \Pi_s[-\widehat{\widehat{b}}_s + \beta P_{k-1}(\widehat{\widehat{w}}_s)] \geq \sum_{s=1}^{S} \Pi_s[-b_s^* + \beta P_{k-1}(w_s^*)] \\ &\geq \sum_{s=1}^{S} \Pi_s[-\lambda b_s^0 - (1-\lambda)b_s^1 + \beta[\lambda P_{k-1}(w_s^0) + (1-\lambda)P_{k-1}(w_s^1)]] = \lambda TP_{k-1}(v_0) + (1-\lambda)TP_{k-1}(v_1)\end{aligned}$$

Thus TP_{k-1} is concave. Note that the inequality in the second line uses the concavity of both u and P_{k-1}.

Solution 2:

Proceed as LS until "... adding a constant to each b_s to leave $\sum_s \Pi_s b_s$ constant cannot make the borrower worse off." Note that all they have done up to this point serves to restore monotonicity of b's. The problem is that in this new contract, it can happen that $C_{s,s-1} => 0$. Now we will construct new w's to keep $\forall s : C_{s,s-1} = 0$ and then add the average of what we subtracted to keep the PKC satisfied. So, let's define $\widehat{w}_1 = w_1^*$ and recursively each $\widehat{w}_s$ (note that the * allocation is not the original one anymore):

$$u(y_s + b_s^*) + \beta \widehat{w}_s - u(y_s + b_{s-1}^*) - \beta \widehat{w}_{s-1} = 0$$

Clearly $\widehat{w}_s \leq w_s^*$ and monotonicity is preserved. Define now:

$$\widehat{\widehat{w}}_s : = \widehat{w}_s + \sum_{s=1}^{S} \Pi_s (w_s^* - \widehat{w}_s)$$

Again, monotonicity is preserved, all downward constraints bind by construction and since this is a mean preserving decrease in spread, it weakly increases the payoff for the lender. Now proceed as before.

Problem 5

Explore a modified approach to prove that P in section 19.5 is concave. The approach consists of the following steps.

1. Consider a relaxed Bellman equation with just the downward constraints and monotonicity of the policies b and w. Apply SLP to prove the value function is concave. I.e. define the operator T, show it has a unique fixed point. Show that the constraint set is convex etc. and use this to prove that the fixed point is concave.

2. Prove that in the relaxed problem the downward constraint binds.

3. Prove that the solution to the relaxed problem is a solution to the original problem with both the downward and upward constraint. Try to do so by arguing, using the earlier lemma that in the relaxed problem that if the downward constraint binds, the current value function is concave, and the monotonicity of policies holds then the upward constraint is automatically satisfied.

Explore means you should try to prove it using this or similar approach. It might or it might not work.

Solution sketch:

This problem shares a lot with the previous one, so I will not provide a detailed solution.

1. The problem here is that it (probably) can't be shown that the constraint set is convex which implies we cannot use SLP directly, but rather some kind of proof as in previous problem. It might be a good exercise to actually construct an example in which the constraint set is not convex.

2. Here we can basically apply the same procedure as in the previous problem where we have shown that if $C_{s,s-1} > 0$ then $\exists$ some other allocation with $C_{s,s-1} = 0$ delivering (weakly) higher utility to the lender keeping the constraints of the borrower satisfied.

3. The solution is in the question - just write down (or cite) the Lemma mentioned there.