Final exam
UMN, Macroeconomic Theory 8105, Fall 2006

Part 1. 35 points

Problem 1a - Dynamic programming, Thm 4.6 in SLP

Consider the following deterministic dynamic programming problem:

$$v(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta v(y)\} \tag{1}$$

Assume $\Gamma(x) \subset \Re^n$.
Hint: throughout you will need assumptions on $X, \Gamma : X \longrightarrow 2^X, \beta, F$.

1. (5 points) State the Maximum Theorem (you will need it later).
2. (5 points) State the appropriate assumptions that guarantee that the operator defined in (1) is mapping the space of continuous, bounded functions endowed with the supnorm $C(X)$ into itself. Prove you claim.
3. (5 points) Define a contraction with modulus β on $C(X)$.
4. (10 points) State Blackwell's Theorem and use it to prove that T is a contraction (provide the appropriate assumptions).
5. (5 points) State the (Banach's) Contraction Mapping Theorem. Use it to prove that T has a unique fixed point.
6. (5 points) Prove that the associated optimal policy correspondence is compact-valued and u.h.c.

Solution: see SLP

Problem 1b - Capital adjustment costs (SLP p. 95)

Consider the following problem. The revenues for a given firm at time t are given by:

$$R(x_t) = ax_t - \frac{bx_t^2}{2}; \ a, b > 0 \tag{2}$$

where x_t denotes its stock of capital. The cost of adjusting the capital level from x_t to x_{t+1} are given by:

$$C(x_t, x_{t+1}) = \frac{c(x_{t+1} - x_t)^2}{2}; \ c > 0 \tag{3}$$

The firm chooses a sequence $\{x_t\}_{t=0}^{\infty} \geq 0$ to maximize the present value of its net revenues (i.e. revenues - costs). The firm is discounting by $\delta = \frac{1}{1+r} \in (0,1)$.

1. (5 points) Write the problem that the firm is facing as a sequential problem in canonical form.

   Solution:

$$v^*(x_0) := \max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t [ax_t - \frac{bx_t^2}{2} - \frac{c(x_{t+1} - x_t)^2}{2}], \ a, b, c > 0 \ , \delta \in (0,1) \text{ s.t.}$$

$$x_{t+1} \in \Gamma(x_t) := \Re_+$$

$$x_0 \text{ given}$$

2. (5 points) Write down the corresponding functional equation.

   Solution:

$$v(x) = \max_{x' \in \Gamma(x)} [ax_t - \frac{bx_t^2}{2} - \frac{c(x_{t+1} - x_t)^2}{2}] + v(x') \tag{4}$$

3. (5 points) Can we apply the standard argument that $\exists!$ a bounded continuous function v^* that solves the functional equation defined in part 2.? Why or why not? If not, is there a way to fix the problem?

   Solution: $\forall x_t, x_{t+1}$: the return function

$$F(x_t, x_{t+1}) := ax_t - \frac{bx_t^2}{2} - \frac{c(x_{t+1} - x_t)^2}{2} \leq ax_t - \frac{bx_t^2}{2} \leq \frac{a^2}{2b}, \tag{5}$$

   where the maximizer for $ax_t - \frac{bx_t^2}{2}$ is $\frac{a}{b}$. Thus $x \leq \frac{a}{b} \Rightarrow x' \leq \frac{a}{b}$. So it makes sense to restrict the state space to $[0, \frac{a}{b}]$. Another way to proceed is to use Theorem 4.14 as SLP suggest. Note that (5) gives us the following upper bound on the value function:

$$v^*(x_0) := \max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t F(x_t, x_{t+1}) \leq \sum_{t=0}^{\infty} \delta^t \frac{a^2}{2b} = \frac{a^2}{2b(1-\delta)} < \infty, \forall x_0 \in \Re_+ \tag{6}$$

4. (20 points) Use the guess and verify method to obtain the value function and the policy function. (Hint: guess that $v(x) = Ex + \frac{Fx^2}{2} + G$ and derive the constants E, F, G.)

Solution:

$$v(x) = \max_{x' \in \Gamma(x)} [ax - \frac{bx^2}{2} - \frac{c(x'-x)^2}{2}] + \delta(Ex' + \frac{Fx'^2}{2} + G) \tag{7}$$

$$FOC : -c(x'-x) + \delta(E + Fx') = 0 \tag{8}$$

$$-cx' + cx + \delta E + \delta F x' = 0 \tag{9}$$

$$(c - \delta F)x' = cx + \delta E \tag{10}$$

$$x' = \frac{cx + \delta E}{c - \delta F} \tag{11}$$

$$\tag{12}$$

The policy function is correct by SLP.

$$SOC : -c + \delta F < 0 \Longrightarrow \tag{13}$$

$$F < \frac{c}{\delta} \tag{14}$$

$$v(x) = [ax - \frac{bx^2}{2} - \frac{c(x'(x)-x)^2}{2}] + \delta(Ex'(x) + \frac{Fx'(x)^2}{2} + G) \tag{15}$$

$$v(x) = [ax - \frac{bx^2}{2} - \frac{c(\frac{cx+\delta E}{c-\delta F} - x)^2}{2}] + \delta(E\frac{cx+\delta E}{c-\delta F} + \frac{F(\frac{cx+\delta E}{c-\delta F})^2}{2} + G) \Longrightarrow \tag{16}$$

$$F = \frac{(c - b\delta - c\delta) \pm \sqrt{(c - b\delta - c\delta)^2 + 4bc\delta}}{2\delta} \tag{17}$$

$$E = \frac{a}{1 - \frac{c\delta}{c-\delta F}} \tag{18}$$

$$G = \frac{(\delta E)^2}{2(c - \delta F)(1 - \delta)} \tag{19}$$

To verify what we have computed, let us use the Envelope Theorem. Applying it to (4) in the first step and plugging in for the optimal policy from (11) in the second step yields:

$$v'(x) = a - bx + c(x' - x) \tag{20}$$

$$v'(x) = a - bx + c(\frac{cx + \delta E}{c - \delta F} - x) = a + \frac{c\delta E}{c - \delta F} + x(\frac{c\delta F}{c - \delta F} - b) \tag{21}$$

Using our guess on the RHS of the last equation:

$$E + Fx = a - bx + c(\frac{cx + \delta E}{c - \delta F} - x) = a + \frac{c\delta E}{c - \delta F} + x(\frac{c\delta F}{c - \delta F} - b) \Longrightarrow \tag{22}$$

$$E = \frac{a}{1 - \frac{c\delta}{c-\delta F}} \tag{23}$$

$$F = \frac{c\delta F}{c - \delta F} - b \Longrightarrow \tag{24}$$

$$-\delta F^2 + F(c - c\delta - b\delta) + bc = 0 \Longrightarrow \tag{25}$$

$$F = \frac{(c - b\delta - c\delta) \pm \sqrt{(c - b\delta - c\delta)^2 + 4bc\delta}}{2\delta} \tag{26}$$

It seems like what we have computed above might be actually correct! :-) For those not yet convinced, we got the same result as SLP.

Finally, we need to worry now about F - which solution to the quadratic equation (25) is the one, we are looking for? Define $g(F)$ as:

$$g(F) := -\delta F^2 + F(c - c\delta - b\delta) + bc \Rightarrow \tag{27}$$
$$g(0) = bc > 0 \Rightarrow F_1 < 0, F_2 > 0 \tag{28}$$

where F_1, F_2 are the roots of the quadratic equation defined by (25). F_2 cannot be a solution to our problem since then obviously $v^*(x) > \frac{a^2}{2b(1-\delta)}$ for x sufficiently large, contradicting (6).

Part 2. 55 points

Problem 2a - Workers vs. capitalists and the $\tau_{kt} \longrightarrow 0$ proposition

Consider an economy with 2 representative consumers with the following endowments of capital and time $\overline{n}$ that can be used as leisure or labor: $k_0^1 = k_0 > 0, k_0^2 = 0, \overline{n}_t^1 = 0, \overline{n}_t^2 = 1$. Moreover, agent 2 is not allowed to accumulate capital, i.e. $\forall t : x_t^2, k_t^2 = 0$. There is also one representative firm facing a time stationary CRS technology, and a government that finances a given stream of expenditures $\{g_t\}_{t=0}^{\infty}$ by proportional taxes on capital and labor income, i.e. the budget constraint for the agents are:

$$\sum_{t=0}^{\infty} p_t(c_t^1 + x_t^1) \leq \sum_{t=0}^{\infty}(1 - \tau_{kt})r_t k_t^1. \tag{29}$$

$$\sum_{t=0}^{\infty} p_t c_t^2 \leq \sum_{t=0}^{\infty}(1 - \tau_{nt})w_t n_t^2. \tag{30}$$

The government is not required to balance its budget in every period. Preferences are the same for the two agents and are given by: $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$

1. (5 points) Define the TDCE for this economy.

   Solution:

   First note that $l_t^1 = n_t^1 = 0$.

   TDCE is a collection of sequences of allocations and prices $\{c_t^1, k_t^1, x_t^1, c_t^2, l_t^2, n_t^2, k_t^f, n_t^f, y_t^f, p_t, r_t, w_t\}_{t=0}^{\infty}$ s.t. given prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$ and policies $\{g_t, \tau_{kt}, \tau_{nt}\}_{t=0}^{\infty}$:

   (a) $\{c_t^1, k_t^1, x_t^1\}_{t=0}^{\infty}$ solve:

$$\max_{\{c_t^1, k_t^1, x_t^1\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^1, 0) \text{ s.t.}$$

$$\begin{aligned} \sum_{t=0}^{\infty} p_t(c_t^1 + x_t^1) &\leq \sum_{t=0}^{\infty}(1 - \tau_{kt})r_t k_t^1 \\ k_{t+1}^1 &\leq k_t^1(1 - \delta_k) + x_t^1 \\ k_0 &\text{ given} \\ \text{non} &- \text{negativities} \end{aligned}$$

   (b) $\{c_t^2, l_t^2, n_t^2\}_{t=0}^{\infty}$ solve:

$$\max_{\{c_t^2, l_t^2, n_t^2\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^2, l_t^2) \text{ s.t.}$$

$$\sum_{t=0}^{\infty} p_t c_t^2 \leq \sum_{t=0}^{\infty} (1-\tau_{nt}) w_t n_t^2$$
$$n_t^2 + l_t^2 \leq 1$$
$$\text{non}-\text{negativities}$$

(c) k_t^f, n_t^f, y_t^f solve $\forall t$:

$$\max_{k_t, n_t, y_t} p_t y_t - r_t k_t - w_t n_t$$

$$y_t \leq F(k_t, n_t)$$
$$\text{non}-\text{negativities}$$

(d) Markets clear $\forall t : k_t^1 = k_t^f, n_t^2 = n_t^f, c_t^1 + c_t^2 + x_t^1 + g_t = y_t^f$.

(e) Government BC is OK: $\sum_{t=0}^{\infty} p_t g_t = \sum_{t=0}^{\infty} \tau_{kt} r_t k_t^1 + \tau_{nt} w_t l_t^2$.

2. (5 points) What are the 2 implementability conditions.

Solution:

$$\sum_{t=0}^{\infty} \beta^t u_{c_t^1} \cdot c_t^1 = u_{c_0^1}[(1-\delta) + F_{k_0^1}(1-\tau_{k0})]k_0 \tag{31}$$

$$\sum_{t=0}^{\infty} \beta^t (u_{c_t^2} \cdot c_t^2 - u_{l_t^2} \cdot n_t^2) = 0 \tag{32}$$

3. (5 points) Suppose $\tau_{k0} \leq 1$ is given. Set up the Ramsey problem in which the government maximizes the utility of the agent with no capital.

$$\max_{\{c_t^1, k_t^1, x_t^1, c_t^2, l_t^2, n_t^2\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^2, l_t^2) \text{ s.t.}$$

$$(\eta) : \sum_{t=0}^{\infty} \beta^t u_{c_t^1} \cdot c_t^1 = u_{c_0^1}[(1-\delta) + F_{k_0^1}(1-\tau_{k0})]k_0$$
$$(\lambda) : \sum_{t=0}^{\infty} \beta^t (u_{c_t^2} \cdot c_t^2 - u_{l_t^2} \cdot n_t^2) = 0$$
$$(\mu\beta^t) : c_t^1 + c_t^2 + x_t^1 + g_t \leq F(k_t^1, n_t^2)$$
$$k_{t+1}^1 \leq k_t^1(1-\delta_k) + x_t^1$$
$$n_t^2 + l_t^2 \leq 1$$
$$\text{non}-\text{negativities}$$

4. (15 points) Suppose there is a steady state. Show that the solution to the Ramsey problem will have $\tau_{kt} \longrightarrow 0$.

Solution:

In the solution to the Ramsey problem we will have the following FOC (changing the notation for the derivatives slightly):

$$\begin{aligned}
(c_t^2) : \beta^t\{u_c(c_t^2, l_t^2) - \lambda[u_{cc}(c_t^2, l_t^2)\cdot c_t^2 + u_c(c_t^2, l_t^2) - u_{lc}(c_t^2, l_t^2)\cdot n_t^2] - \mu_t\} &= 0 \Longrightarrow \\
u_c(c_t^2, l_t^2) - \lambda[u_{cc}(c_t^2, l_t^2)\cdot c_t^2 + u_c(c_t^2, l_t^2) - u_{lc}(c_t^2, l_t^2)\cdot n_t^2] &= \mu_t \\
u_c(c_{t+1}^2, l_{t+1}^2) - \lambda[u_{cc}(c_{t+1}^2, l_{t+1}^2)\cdot c_{t+1}^2 + u_c(c_{t+1}^2, l_{t+1}^2) - u_{lc}(c_{t+1}^2, l_{t+1}^2)\cdot n_{t+1}^2] &= \mu_{t+1} \\
(k_{t+1}^1) : \beta^{t+1}[F_k(k_t^1, n_t^2) + 1 - \delta_k]\mu_{t+1} &= \beta^t\mu_t
\end{aligned}$$

The 2 FOC for c show clearly that in SS $\mu_{t+1} = \mu_t$. Combining with the last one, we get:

$$\beta[F_k(k_t^1, n_t^2) + 1 - \delta_k] = 1$$

However, in the TDCE, we will get:

$$\beta[(1 - \tau_{kt+1})F_k(k_t^1, n_t^2) + 1 - \delta_k] = 1$$

These two conditions give the desired conclusion.

5. (15 points) Show that if one of the implementability constraints is not binding (determine which one), i.e. the Lagrange multiplier $= 0$, then $\forall t > 0 : \tau_{nt} = 0$. Interpret this result.

Solution:

FOC w.r.t. l_t^2 in the Ramsey problem are $(l_t^2), \forall t > 0$:

$$\beta^t\{u_l(c_t^2, l_t^2) + \lambda[u_{ll}(c_t^2, l_t^2)\cdot n_t^2 - u_l(c_t^2, l_t^2) - u_{cl}(c_t^2, l_t^2)\cdot c_t^2] - \mu_t F_n(k_t^1, n_t^2)\} = 0$$

Combining with the FOC w.r.t. c_t^2 above and assuming steady state we will get:

$$F_n(k_t^1, n_t^2) = \frac{u_l(c_t^2, l_t^2) + \lambda[u_{ll}(c_t^2, l_t^2)\cdot n_t^2 - u_l(c_t^2, l_t^2) - u_{cl}(c_t^2, l_t^2)\cdot c_t^2]}{u_c(c_t^2, l_t^2) - \lambda[u_{cc}(c_t^2, l_t^2)\cdot c_t^2 + u_c(c_t^2, l_t^2) - u_{lc}(c_t^2, l_t^2)\cdot n_t^2]}$$

In the TDCE however:

$$F_n(k_t^1, n_t^2)(1 - \tau_{nt}) = \frac{u_l(c_t^2, l_t^2)}{u_c(c_t^2, l_t^2)}$$

Now the claim that if $\lambda = 0$ then $\forall t > 0 : \tau_{nt} = 0$ in the solution to the Ramsey problem is obvious. It means that all government revenue is raised in period 0 by a non-distortionary taxes on capital.

6. (10 points) Suppose now that the government is required to balance its budget in every period. Is it still true that the solution to the Ramsey problem will have $\tau_{kt} \longrightarrow 0$? How about τ_{nt}?

   Solution:
   This is left as an exercise.

Problem 2b - Government purchases entering the production function (Prelim, Fall 2006)

Consider a version of a Ramsey problem in a world with one representative consumer and one representative firm, but where government purchases increase the productivity of the firm and there is full depreciation of both k and g, i.e. feasibility is given by:

$$c_t + k_{t+1} + g_{t+1} \leq Ak_t^{\alpha} g_t^{1-\alpha}.$$

The initial stock of government goods and capital, g_0, k_0, are given. Assume that preferences of the representative consumer are given by:

$$\sum_t \beta^t u(c_t)$$

where $u(c) = c^{1-\sigma}/(1-\sigma)$.

1. (10 points) State the Social Planner's Problem for this economy and characterize its solution, i.e. determine the growth rate of the economy.
2. (5 points) If the planner wanted to implement this using a lump sum tax in each period keeping budget balance in each period, what would he do?
3. (10 points) Assume that the firm takes g_t as given when making its decisions. Define the TDCE when only capital income taxes, τ_{kt} can be used to finance the purchase of a given stream g_t. The government is NOT required to balance its budget in every period. Characterize the TDCE, i.e. determine the growth rate of the economy.
4. (5 points) What is the Implementability constraint for the Ramsey problem?
5. (5 points) Write down the Ramsey problem.
6. (15 points) Is it true that $\tau_{kt} \to 0$?
7. (5 points) Is it true that $\tau_{kt} \to 0$ if the government is required to balance its budget every period?

Outline of solution: by Larry and Anderson

1. ans. rewrite problem:

   max $\quad \sum_t \beta^t u(Ak_t^{\alpha} g_t^{1-\alpha} - k_{t+1} - g_{t+1})$

   $FOCk_{t+1}$: $\beta^t u'(c_t) = \beta^{t+1} u'(c_{t+1}) \frac{\alpha y_{t+1}}{k_{t+1}}$;

   $FOCg_{t+1}$: $\beta^t u'(c_t) = \beta^{t+1} u'(c_{t+1}) \frac{(1-\alpha) y_{t+1}}{g_{t+1}}$;

   $\frac{g_t}{k_t} = \frac{1-\alpha}{\alpha}$ all t.

   $\frac{u'(c_t)}{u'(c_{t+1})} = \beta \frac{\alpha y_{t+1}}{k_{t+1}} = \beta \frac{\alpha A k_{t+1}^{\alpha} g_{t+1}^{1-\alpha}}{k_{t+1}} = \beta \frac{\alpha A g_{t+1}^{1-\alpha}}{k_{t+1}^{1-\alpha}} = \beta \frac{\alpha A g_{t+1}^{1-\alpha}}{k_{t+1}^{1-\alpha}} = \beta \alpha A \left[\frac{1-\alpha}{a}\right]^{1-\alpha}$

Note that this does not depend on t.

Assume $u(c) = c^{1-\sigma}/(1-\sigma)$. Then, $\frac{c_{t+1}}{c_t} = \gamma_c$ is constant and is given by:

$$\gamma_c^\sigma = \beta\alpha A\left[\frac{1-\alpha}{a}\right]^{1-\alpha}.$$

2. straightforward

3. Now go to CE.

Firm takes g_t as given and solves:

$$Max_{k_t} \qquad p_t A k_t^\alpha g_t^{1-\alpha} - r_t k_t$$

FOC: $\qquad \alpha\frac{y_t}{k_t} = \frac{r_t}{p_t}$

$$\pi_t = (1-\alpha)y_t$$

i.e., the firm pays out the return to g in the form of profits.

Household problem:

$$Max_{\{c_t,k_t\}} \qquad \sum_t \beta^t u(c_t)$$

$$\sum_t p_t(c_t + k_{t+1}) \leq \sum_t \left[(1-\tau_{kt})r_t k_t + \pi_t\right]$$

k_0 given.

$FOCc_t: \qquad \beta^t u'(c_t) = \lambda p_t;$

$FOCk_{t+1}: \qquad \lambda p_t = \lambda r_{t+1}(1-\tau_{kt+1})$

$$\beta^t u'(c_t) = \beta^{t+1}u'(c_{t+1})\alpha\frac{y_{t+1}}{k_{t+1}}(1-\tau_{kt+1})$$

$$\gamma_t^\sigma = \left[\frac{c_{t+1}}{c_t}\right]^\sigma = \beta\alpha\frac{y_{t+1}}{k_{t+1}}(1-\tau_{kt+1}) = \beta\alpha\frac{Ak_{t+1}^\alpha g_{t+1}^{1-\alpha}}{k_{t+1}}(1-\tau_{kt+1}) = \beta\alpha A\frac{g_{t+1}^{1-\alpha}}{k_{t+1}^{1-\alpha}}(1-\tau_{kt+1})$$

$$= \beta\alpha A\left[\frac{1-\alpha}{\alpha}\right]^{1-\alpha}(1-\tau_{kt+1}).$$

Or, for use below:

$$1 = \beta\frac{u'(c_{t+1})}{u'(c_t)}F_k(t+1)(1-\tau_{kt+1}).$$

4. Substitution and simplification imply that the Implementability Constraint becomes:

$$\sum_t \beta^t u'(c_t)c_t = crap_0 + \sum_t \pi_t = crap_0 + (1-\alpha)\sum_t y_t$$

$$= crap_0 + (1-\alpha)\sum_t Ak_t^{\alpha} g_t^{1-\alpha}.$$

5. Thus, the RP is:

$$Max_{\{c_t,k_t,g_t\}} \qquad \sum_t \beta^t u(c_t)$$

$$\sum_t \beta^t u'(c_t)c_t = crap_0 + (1-\alpha)\sum_t Ak_t^{\alpha} g_t^{1-\alpha}.$$

$$c_t + k_{t+1} + g_{t+1} = Ak_t^{\alpha} g_t^{1-\alpha}$$

k_0, g_0 given.

6. Rewrite this as:

$$Max_{\{c_t,k_t,g_t\}} \qquad \sum_t \beta^t W(c_t, k_t, g_t; \lambda) + \lambda crap_0$$

$$c_t + k_{t+1} + g_{t+1} = Ak_t^{\alpha} g_t^{1-\alpha} \qquad (\beta^t \mu_t)$$

k_0, g_0 given.

Where

$$W(c_t, k_t, g_t; \lambda) = u(c_t) + \lambda \left[u'(c_t)c_t - (1-\alpha)Ak_t^{\alpha} g_t^{1-\alpha} \right]$$

and λ is the mulitiplier on the IC.

Note that in this case,

$$u(c_t) + \lambda u'(c_t)c_t = c_t^{1-\sigma}/(1-\sigma) + \lambda c_t^{1-\sigma} = [\lambda + 1/(1-\sigma)]\, c_t^{1-\sigma}.$$

FOC's for this problem are:

$$FOCk_{t+1}: \qquad \beta^t \mu_t = \beta^{t+1}\mu_{t+1}F_k(t+1) + \beta^{t+1}W_k(t+1)$$

$$FOCc_t: \qquad \beta^t W_c(t) = \beta^t \mu_t.$$

$$W_c(t) = \beta \left[W_c(t+1)F_k(t+1) + W_k(t+1) \right].$$

Hence,

$$1 = \beta \frac{W_c(t+1)}{W_c(t)} F_k(t+1) + \beta \frac{W_k(t+1)}{W_c(t)}.$$

Recall from above that we have in the TDCE:

$$1 = \beta \frac{u'(c_{t+1})}{u'(c_t)} F_k(t+1)(1 - \tau_{kt+1}).$$

But from the form of the utility function it follows that

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{W_c(t+1)}{W_c(t)}.$$

Thus, in the TDCE, we have:

$$1 = \beta \frac{W_c(t+1)}{W_c(t)} F_k(t+1)(1 - \tau_{kt+1}).$$

Thus, it follows that $\tau_{kt+1} = 0$ (resp. converges to 0) if and only if $\beta \frac{W_k(t+1)}{W_c(t)} = 0$ (resp. converges to 0). Since this doesn't hold in general (check this!!!), it will typically NOT be true that $\tau_{kt} \to 0$.

7. obvious