Problem Set 2
UMN, Macroeconomic Theory 8105, Fall 2006

- due September 26^{th} in lecture

Problem 1 - equilibrium in an endowment economy and welfare theorems

Consider an endowment economy with 2 agents with the same preferences defined by $\sum_{t=0}^{\infty} \beta^t \log c_t$, $0 < \beta < 1$.

Agent 1 has the following stream of endowments: $(w_0, w_1, w_2, w_3, ...) = (5, 1, 5, 1, ...)$.
Agent 2 has the following stream of endowments: $(w_0, w_1, w_2, w_3, ...) = (1, 5, 1, 5, ...)$.

(i) (5 points) Define and calculate the Arrow-Debreu equilibrium. (Save your and my time, note that this is a special case of the equilibrium defined in (iv)).

(ii) (10 points) Define a Pareto efficient allocation for this economy. Consider the following problem:

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \alpha_1 \sum_{t=0}^{\infty} \beta^t \log c_t^1 + \alpha_2 \sum_{t=0}^{\infty} \beta^t \log c_t^2 \text{ s.t.}$$

$$c_t^1 + c_t^2 \leq w_t^1 + w_t^2, c_t^1 \geq 0, c_t^2 \geq 0, \forall t,$$

$$\alpha_1 \geq 0, \alpha_2 \geq 0, \exists i \text{ s.t. } \alpha_i > 0.$$

Prove that a solution to this problem is Pareto efficient. Show that $\forall$ Pareto efficient allocation $\exists (\alpha_1, \alpha_2)$ s.t. the Pareto efficient allocation solves the above problem. Thus we can express a Pareto efficient allocation as a function of (α_1, α_2).

(iii) (5 points) Prove that the Arrow-Debreu equilibrium is Pareto efficient (First Welfare Theorem).

(iv) (5 points) Define Arrow-Debreu equilibrium with transfers. I.e. we assume it is possible that the government takes some resources away from an agent or gives additional resources to an agent. We also assume that the government doesn't consume anything, thus in equilibrium, it must be true that resources taken away from one agent are given to the other agent. (Hint: the budget constraint of agent i will be: $\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i + \hat{t}_i$), where $\hat{t}_i$ stands for transfers.

(v) (5 points) Find the transfers necessary to implement a given Pareto efficient allocation from (ii) (i.e. any PE allocation as a function of (α_1, α_2)) as an equilibrium with transfers (Second Welfare Theorem). Show that transfers are homogenenous of degree k in (α_1, α_2). What is k? Show that transfers sum up to 0.

Problem 2 - a 2 sector production economy

Consider a 2 sector-economy defined in class with consumption-goods firms indexed by $j_c = 1, 2, ..., J_c$ and investment-goods firms indexed by $j_x = 1, 2, ..., J_x$. You may assume a CRS technology for each firm in each sector. There are I consumers indexed by $i = 1, 2, ..., I$ and preferences are $U^i(\underline{c}, \underline{l}) = \sum_{t=0}^{\infty} \beta^t u(c^i_t, l^i_t)$.

(i) (5 points) Define the Arrow-Debreu equilibrium.

(ii) (5 points) Suppose the solution to the consumer's problem is interior, that is, all quantities at all times are strictly positive. Derive a condition relating the price of investment goods and the rental rate of capital. Explain briefly in words what this means.

(iii) (5 points) Show that in equilibrium the constraints of the consumer's problem can be re-written so that only the initial endowment of capital enters the budget constraint. State clearly the condition that you need in order to make this claim (i.e. the transversality condition).

(iv) (5 points) Write down the necessary first-order conditions that a solution to each firm's problem must satisfy. Derive a condition relating the price of consumption goods and the price of investment goods.

(v) (5 points) Set up the Social Planner's problem for this economy. The social planner maximizes weighted sum of utilities (across agents) given feasibility constraints.

(vi) (5 points) Show that the Arrow-Debreu equilibrium (ADE) is "homogeneous of degree zero in population". That is, consider a new economy with 2I consumers, 2 exactly like each of the agents in the original economy. Assuming that all production functions are CRS, show that the original equilibrium prices and allocations are still an equilibrium.

(vii) (5 points) Suppose that consumers are allowed to trade their shares of firms' profits amongst themselves. Specifically, in period 0, consumer i can buy shares $\widetilde{\theta}^i_{j_x}$ of firm j_x's profits at price p_{j_x} per unit, and shares $\widetilde{\theta}^i_{j_c}$ of firm j_c's profits at price p_{j_c} per unit, for $j_x = 1, 2, ..., J_x$ and $j_c = 1, 2, ..., J_c$. Show that the equilibrium allocation is the same as in the original set-up.

Problem 3 - methods of solving the functional equation
This problem is on the growth model with inelastic labor supply, full depreciation, log utility and CRS technology (we consider labor fixed at 1). This is one of the few problems that have an analytical solution for the value function. Let the Bellman equation be defined as follows:

$$v(k) = \max_{k' \in \Gamma(k)} \log(\theta k^\alpha - k') + \beta v(k')$$

where $\Gamma(k) := \{k' \in \mathbf{R} : k' \geq 0, \theta k^\alpha - k' \geq 0\}$.

(i) **Guess and verify method.** (10 points) Solve for the value function $v(k)$ and the policy function $k'(k)$ using the "guess and verify" method.

Step 1: Guess that the value function has the form $v(k) = a_1 + a_2 \log k$. Plug the guess into the maximization problem:

$$\max_{k' \in \Gamma(k)} \log(\theta k^\alpha - k') + \beta(a_1 + a_2 \log k')$$

Solve this problem for k' as a function of k and the other constants. Argue that the solution will be interior and that the FOC are sufficient.

Step 2: Plug back into the very first equation for the optimal $k'(k)$ equation to get:

$$a_1 + a_2 \log k = \log(\theta k^\alpha - k'(k)) + \beta(a_1 + a_2 \log k'(k))$$

Find the a_1 and a_2 that satisfy the equation. This verifies the guess. Determine the policy function $k'(k)$.

(ii) **Analytical iterations on the value function.** (5 points) We will show that:

$$v = \lim_{n \to \infty} T^n v_0$$

for an initial guess v_0 where the operator T is defined as:

$$Tv(k) = \max_{k' \in \Gamma(k)} \log(\theta k^\alpha - k') + \beta v(k').$$

Step 1: Let the initial guess v_0 be $v_0(k) = 0, \forall k \geq 0$. Solve:

$$\max_{k' \in \Gamma(k)} \log(\theta k^\alpha - k') + \beta v_0(k') = \max_{k' \in \Gamma(k)} \log(\theta k^\alpha - k')$$

for the optimal $k'(k)$. Plug back in for the optimal $k'(k)$ to get:

$$v_1(k) = \log(\theta k^\alpha - k'(k)) + \beta v_0(k'(k)).$$

Step 2: solve:

$$\max_{k' \in \Gamma(k)} \log(\theta k^\alpha - k') + \beta v_1(k')$$

for the optimal $k'(k)$. Plug back in for the optimal $k'(k)$ to get:

$$v_2(k) = \log(\theta k^\alpha - k'(k)) + \beta v_1(k'(k)).$$

Step 3: do it a couple more times until you observe a pattern. Prove that the pattern you observe is true by induction, i.e. show that if v_t has some form then v_{t+1} will have some other form. Let $t \to \infty$ to get:

$$v = \lim_{n\to\infty} T^n v_0.$$

Check that v is equal to v in part (ii). Discuss briefly the relevance of Theorem 4.6 for this problem.

(iii) BONUS QUESTION - 10 points. Apply the guess and verify procedure to the following problem:

$$v(k) = \max_{k',l} \log(\theta k^\alpha l^{1-\alpha} - k') + \gamma \log(1-l) + \beta v(k')$$

s.t.

$$\theta k^\alpha l^{1-\alpha} - k' \geq 0$$

$$k' \geq 0, l \geq 0, 1 - l \geq 0.$$

Use the guess that $v(k) = a_0 + a_1 \log k$. Hint $l(k) = \bar{l}$. Solve for the policy functions as well as the value function. (Note: this problem has appeared on Prelims, e.g. in Fall 2006. In the exam, you have less than 1 hour to solve it.)

(iv) Numerical methods. Next time.

Problem 4 - 5.1 in SLP (20 points)