Homework #2
Math 5616

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February 7, 2004

Problem #1: Chapter 5, Problem #9 of Rudin.
Let f be a continuous real function on $\mathbb{R}$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and the $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?

Proof: We show $f'(0) = 3$. Consider the functions $f(x) - f(0)$ and $g(x) := x$, both go to zero as x goes to zero. The ratio of their derivatives is equal to $f'(x)$ which tends to 3 as x tends to zero, therefore by L'Hospital's rule we have $\frac{f(x)-f(0)}{g(x)} \to 3$ as $x \to 0$, i.e. $f'(0) \equiv lim_{x\to 0}\frac{f(x)-f(0)}{x} = 3$. $\square$

Problem #2: Chapter 5, Problem #10 of Rudin.
Let f and g be complex differentiable function on $(0,1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where A and B are complex numbers $B \neq 0$. Prove that:

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Compare with Example 5.18. *Hint:*

$$\frac{f(x)}{g(x)} = \left\{ \frac{f(x)}{x} - A \right\} \cdot \frac{x}{g(x)} + A \cdot \frac{x}{g(x)}. \tag{1}$$

Apply Theorem 5.13 to the real and imaginary parts of $\frac{f(x)}{x}$ and $\frac{g(x)}{x}$.

Proof: Decompose f, g into their real and imaginary parts: $f = f_1 + if_2$, $g = g_1 + ig_2$. Our assumptions tell us that: $(f_1, f_2) \to (0,0)$, $(g_1, g_2) \to (0,0)$, $(f_1', f_2') \to (ReA, ImA)$, and $(g_1', g_2') \to (ReB, ImB)$ as $x \to 0$.

Observe from this that: $\frac{f_1'}{x'} = f_1' \to ReA$, $\frac{f_2'}{x'} = f_2' \to ImA$, $\frac{g_1'}{x'} = g_1' \to ReB$, $\frac{g_2'}{x'} = g_2' \to ImB$ as $x \to 0$.

Thus we may apply Theorem 5.13 directly to the real and imaginary parts of $\frac{f(x)}{x}$ and $\frac{g(x)}{x}$ to get: $\frac{f_1}{x} \to ReA$, $\frac{f_2}{x} \to ImA$, $\frac{g_1}{x} \to ReB$, $\frac{g_2}{x} \to ImB$ as $x \to 0$. From this we may conclude that $\frac{f(x)}{x} \to A$, $\frac{g(x)}{x} \to B$ as $x \to 0$. Noting that $B \neq 0$ allows us to use (4.4 Theorem) and substitute these values directly into (1) to yield $\frac{f(x)}{g(x)} \to \frac{A}{B}$ as $x \to 0$ as we wanted. $\square$

Problem #3: Chapter 5, Problem #12 of Rudin.
If $f(x) = |x|^3$ compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.

Solution: Observe that $f(x) = g(u(x))$ where $g(y) = y^3$ and $u(x) = |x|$. Note that g' exists everywhere and that $u'(x)$ exists for all $x \neq 0$, so we may apply (5.5 Theorem) for computing $f'(x)$ and $f''(x)$ where $x \neq 0$ and compute directly when $x = 0$. Note that:

$$g'(x) = 3x^2 \text{ and } u'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Now $f'(0) = \lim_{h \to 0} \frac{|h|^3}{h} = 0$, since $h^2, -h^2 \to 0$ as $h \to 0$. Therefore applying (5.5 Theorem) with this, we get:

$$f'(x) = \begin{cases} 0 & \text{if } x = 0 \\ 3|x|^2 & \text{if } x > 0 \\ -3|x|^2 & \text{if } x < 0. \end{cases}$$

We may now compute $f''(x)$ by observing that $f''(0) = \lim_{h \to 0} \frac{\pm 3|h|^2}{h} = 0$, since $h, -h \to 0$ as $h \to 0$ and applying (5.5 Theorem) to $f'(x)$ for $x \neq 0$ to get:

$$f''(x) = \begin{cases} 0 & \text{if } x = 0 \\ 6|x| & \text{if } x \neq 0. \end{cases}$$

We may conclude from this that $f^{(3)}(0)$ does not exist since $u'(0)$ does not exist (i.e. $|x|$ has no derivative at zero). □

Problem #4: Chapter 6, Problem #2 of Rudin.
Suppose $f \geq 0$, f continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof: By Contradiction suppose that $f(x) > 0$ for some $x \in [a, b]$. Since f is continuous, it is then strictly positive on an interval. Since f is never negative, any Riemann sum which places two partition points in this interval will have a strictly positive value since there is no negative term to cancel out. Thus once the mesh size is less than the length of the interval where $f > 0$, all Riemann sums will be strictly positive and thus $\int_a^b f(x)dx > 0$, contradiction. Thus $f(x) = 0$ for all $x \in [a, b]$. □