Further Problem #2
Math 8602

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Further Problem #2: Suppose that $T : X \to Y$, where X and Y are normed spaces and T is linear. Let $S_T := \{C \in \mathbb{R} : ||Tx||_Y \leq C||x||_X \text{ for all } x \in X\}$ (S_T may be empty!).
i) Prove that $||T|| :=\inf S_T = \sup_{||x||_X \leq 1} ||Tx||_Y$.
ii) Let $X = Y$ be the collection of all sequences of real numbers that are eventually zero, both having the ℓ^2 norm. Find an example of a linear $T : X \to Y$ such that $S_T = \emptyset$

i)*Proof:* Note that because norms are non-negative we may assume that the definition of S_T is for $C \geq 0$. Letting $S_{T_1} := \{C \in [0, \infty) : ||Tx||_Y \leq C \text{ for all } x \in X \text{ with } ||x||_X = 1\}$ we shall show that $S_{T_1} = S_T$:

$$\begin{aligned}
C \in S_{T_1} &\iff ||Tx||_Y \leq C \text{ for all } x \in X \text{ with } ||x||_X = 1 && (1)\\
&\iff ||T(x/||x||_X)||_Y \leq C \text{ for all } x \in X \text{ with } ||x||_X > 0 && (2)\\
&\iff ||Tx||_Y \leq C||x||_X \text{ for all } x \in X \text{ with } ||x||_X > 0 && (3)\\
&\iff ||Tx||_Y \leq C||x||_X \text{ for all } x \in X && (4)\\
&\iff C \in S_T && (5)
\end{aligned}$$

Explanation: (1)-(3) hold by linearity of T and properties of norm. (3) implies (4) because $||x||_X = 0$ yields $x = 0$ and $Tx = 0$, thus $||Tx||_Y = 0 \leq C||x||_X = 0$.

Now we have that $||T|| :=\inf S_T =\inf S_{T_1}$. Suppose by contradiction that $||T|| \neq \sup_{||x||_X \leq 1} ||Tx||_Y$.

- Case 1: $||T|| > \sup_{||x||_X \leq 1} ||Tx||_Y$. Using $\inf S_{T_1}$ we have that $||Tx||_Y > \sup_{||x||_X \leq 1} ||Tx||_Y$ for all $x \in X$ with $||x||_X = 1$, this is a clear contradiction.
- Case 2: $||T|| < \sup_{||x||_X \leq 1} ||Tx||_Y$. Using $\inf S_T$ we have that there exists $C \in S_T$ such that $\inf S_T \leq C < \sup_{||x||_X \leq 1} ||Tx||_Y$, so by definition of sup we have some $x \in X$ with $||x||_X \leq 1$ so that $C < ||Tx||_Y$, but $C \in S_T$ implies $||Tx||_Y \leq C||x||_X \leq C$, since $||x||_X \leq 1$, which establishes the contradiction. □

ii)*Example:* For $x \in X$ define $Tx := (\sum_{k=1}^{\infty} x_k, 0, 0, \dots)$. T is linear: take $x, y \in X$ and $a, b \in \mathbb{R}$

then we have:

$$
\begin{aligned}
T(ax+by) &= (\sum_{k=1}^{\infty} ax_k + by_k, 0, 0, \dots) && (6)\\
&= (a\sum_{k=1}^{\infty} x_k + b\sum_{k=1}^{\infty} y_k, 0, 0, \dots) && (7)\\
&= a(\sum_{k=1}^{\infty} x_k, 0, 0, \dots) + b(\sum_{k=1}^{\infty} y_k, 0, 0, \dots) && (8)\\
&= aTx + bTy && (9)
\end{aligned}
$$

Explanation: (7) follows because there are finitely many non-zero terms and thus all series converge. To show $S_T = \emptyset$ take any $C \in \mathbb{R}$ and pick an $N \in \mathbb{N}$ so that $\sqrt{N} > C$, now choose a $z \in X$ so that the first N terms are equal to one and the rest are zero. We have $||z||_X = \sqrt{\sum_{k=1}^{N} 1^2} = \sqrt{N}$ and $Tz = (N, 0, 0, \dots)$, thus $||Tz||_Y = N = \sqrt{N}\sqrt{N} > \sqrt{N}C = C||z||_X$. So for this T we have $S_T = \emptyset$. $\square$