

Problem #5: Chapter 6, Problem #1(a) of Wheeden & Zygmund

Let E be a measurable subset of \mathbb{R}^2 such that for *a.e.* $x \in \mathbb{R}^1$, $\{y : (x, y) \in E\}$ has \mathbb{R}^1 -measure zero. Show that (i) E has measure zero, and that (ii) for *a.e.* $y \in \mathbb{R}^1$, $\{x : (x, y) \in E\}$ has \mathbb{R}^1 -measure zero.

Proof: (i) First define $E_x := \{y : (x, y) \in E\}$ and $E_y := \{x : (x, y) \in E\}$. Now take any rectangle $I = I_1 \times I_2 \subseteq \mathbb{R}^2$ and regard $\chi_E(x, y)$ as a function on I (i.e., restrict it to I). Clearly $\chi_E(x, y) \in L(I)$ since the function is bounded. We may now apply all the consequences of Fubini's Theorem (6.1): For *a.e.* $x \in I_1$, $\chi_E(x, y) = \chi_{E_x}(y)$ is measurable and integrable on I_2 as a function of y . Also, for *a.e.* $x \in I_1$, $\chi_{E_x}(y) = 0$ *a.e.* since for *a.e.* $x \in \mathbb{R}^1$, $|E_x| = 0$. So we may say that $h(x) := \int_{I_2} \chi_{E_x}(y) dy = 0$ for *a.e.* $x \in I_1$ and as we continue to apply Fubini we get:

$$\int \int_I \chi_E(x, y) dx dy = \int_{I_1} \left[\int_{I_2} \chi_{E_x}(y) dy \right] dx = 0.$$

Now since $\chi_E(x, y)$ is non-negative, $\chi_E(x, y) = 0$ *a.e.* on I (by 5.11). Now $I \subseteq \mathbb{R}^2$ was an arbitrary interval; therefore, $\chi_E(x, y) = 0$ *a.e.* on \mathbb{R}^2 , so $|E| = 0$.

(ii) Now we may apply Fubini again in a similar way and use the prior result to get:

$$0 = \int \int_I \chi_E(x, y) dx dy = \int_{I_2} \left[\int_{I_1} \chi_{E_y}(x) dx \right] dy.$$

This implies that $\int_{I_1} \chi_{E_y}(x) dx = 0$ for *a.e.* $y \in I_2$. This implies that for *a.e.* $y \in I_2$, $\chi_{E_y}(x) = 0$ *a.e.* on I_1 and thus for *a.e.* $y \in I_2$, $|E_y| = 0$. Now $I_2 \subseteq \mathbb{R}^1$ was arbitrary thus for *a.e.* $y \in \mathbb{R}^1$, $|E_y| = 0$.