ECON 8105 Problem Set IV

Kemal Badur

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1) *Consider a two period economy, $t = 0, 1$. At time $t = 0$, there is no uncertainty, but at time $t = 1$, there are two possible states R and NR. There are also two assets, S and B, whose price at time $t = 0$ has been normalized to 1. (i.e. $q_S(0) = 1, q_B(0) = 1$). Their payoffs at time 1 are the following: $D^S(NR) = 5$ and $D^S(R) = 2$ for asset S and $D^B(NR) = D^B(R) = 3$ for asset B. All these quantities are units of the only good that there is at each period and each state of nature.*

a. *Compute the set of state prices for which the fundamental theorem of finance will hold.*

The fundamental theorem of finance requires

$$q_t = \frac{q_{t+1} + p_{t+1} D_{t+1}}{1 + r_t}$$

or, the current price of an asset is equal to the discounted value of future incomes (p_{t+1} is the price of the numeraire good at time $t + 1$ and D_{t+1} is the dividend payment at $t + 1$). As the time horizon ends at $t = 1$, and the asset will have no returns afterwards, $q_{t+1} = 0$. Then, for $t = 0$,

$$q_0^A = \frac{\left(p^R, p^{NR}\right) \cdot \left(D^A(R), D^A(NR)\right)}{1 + r_t}$$

for $A = S, B$. Plugging in the values given,

$$\begin{aligned} 1 &= \frac{\left(p^R, p^{NR}\right) \cdot (2, 5)}{1 + r_t} \\ 1 &= \frac{\left(p^R, p^{NR}\right) \cdot (3, 3)}{1 + r_t} \end{aligned}$$

Setting these two equations equal after multiplying with $1+r_t$, we get

$$\begin{aligned} 2p^R + 5p^{NR} &= 3p^R + 3p^{NR} \\ p^R &= 2p^{NR} \end{aligned}$$

as the set of state prices that satisfy the fundamental theroem of finance. If we write the dividends to the assets in terms of p^{NR}

	R	NR
S	$4p^{NR}$	$5p^{NR}$
B	$6p^{NR}$	$3p^{NR}$

shows that no asset dominates the other in both states.

b. *Compute the price of a contingent claim that pays 1 in state R and 0 in state NR and the price of a contingent claim that pays 0 in state R and 1 in state NR.*

Denote by q^R and q^{NR} the prices of the claims that pay $(1,0)$ and $(0,1)$, respectively at time $t=0$. Then we have

$$\begin{aligned} q_S &= 2q^R + 5q^{NR} = 1 \\ q_B &= 3q^R + 3q^{NR} = 1 \end{aligned}$$

Then, again setting those two equal, we get $q^R = 2q^{NR}$, and consequently, $q^R = \frac{1}{9}$ and $q^{NR} = \frac{2}{9}$.

c. *More generally, compute the price of an asset that pays M units in state R and N units in state NR.*

From the fundamental theorem of finance and the results obtained above, we can write

$$q = \frac{2}{9}M + \frac{1}{9}N$$

d. *What would happen in this economy if we introduce an asset that costs 1.5 units and pays 6 units in state R and 7 units in state NR?*

Given the state prices (we can now explicitly calculate them, $p^R = \frac{6}{38}, p^{NR} = \frac{3}{38}$) this asset is "too cheap" when compared to the initial two assets.

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2) Suppose that in the above economy, the consumer is endowed with $w(0)$, and $w(R)$ and $w(NR)$ units of the good (at each respective period and state). At time $t = 0$, the consumer is also endowed with a portfolio of assets $(B(0), S(0))$.

a. *Write down the sequential budget constraint, assuming that the consumer must pay all debts by the end of the horizon.*

Noting that the prices of B and S are normalized to 1 at $t = 0$.

$$\begin{aligned} t &= 0 \rightarrow p_0 c(0) + B(1) + S(1) \leq p_0 w(0) + B(0) + S(0) \\ t &= 1 \rightarrow \begin{cases} c^R(1) \leq 3B(1) + 2S(1) + w(R) & if\ R \\ c^{NR}(1) \leq 3B(1) + 5S(1) + w(NR) & if\ NR \end{cases} \end{aligned}$$

b. *Is there any overall budget constraint which is equivalent to that in part (a)?*

Multiply the budget constraints for $t = 1$ with q^R and q^{NR} respectively and add the three of them

$$\begin{aligned} & p_0 c(0) + B(1) + S(1) + q^R c^R(1) + q^{NR} c^{NR}(1) \\ \leq\ & B(0) + S(0) + q^R 3B(1) + q^R 2S(1) + q^{NR} 3B(1) + q^{NR} 5S(1) + \end{aligned}$$

Given $q^R = \frac{2}{9}$ and $q^{NR} = \frac{1}{9}$,we know $B(1) + S(1) = 3q^R B(1) + 2q^R S(1) + 3q^{NR} B(1) + 5q^{NR} S(1)$ and hence can cancel those from the inequality to get

$$p_0 c(0) + q^R c^R(1) + q^{NR} c^{NR}(1) \leq B(0) + S(0) + p_0 w(0) + q^R w(R) + q^{NR} w(NR)$$