Further Notes on Micro Heterogeneity and Macro Slow Convergence

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Abstract

This recaps the key idea in my *Federal Reserve Bank of Minneaplis* working paper 696 (March 2012) and relates it to more recent work.

1. The Basic Example

This example is taken from Luttmer [2012]. Consider an economy in which firms enter with size $k_0 > 0$ and then grow exponentially at the rate μ, conditional on survival. Firms exit randomly at the rate strictly positive rate δ. Suppose the flow of new firms at time t is E_t and define $I_t = k_0 E_t$ to be aggregate "investment" by entrants. Then the aggregate measure of "capital" evolves according to

$$\mathrm{d}K_t = -(\delta - \mu)K_t \mathrm{d}t + I_t \mathrm{d}t. \tag{1}$$

Assume $\delta > \mu > 0$. If $E_t = E$, so that $I_t = k_0 E$ is constant, then K_t converges to $k_0 E/(\delta - \mu)$. The speed of adjustment is measured by $\delta - \mu$.

The size of a surviving firm of age a is $k_a = k_0 e^{\mu a}$. With $E_t = E$, the long-run age distribution will be exponential with right tail $e^{-\delta a}$. This means that the size distribution is determined by

$$1 - F(k) = e^{-\delta \ln(k/k_0)/\mu} = \left(\frac{k}{k_0}\right)^{-\delta/\mu}, \text{ for all } k \geq k_0.$$

That is, the distribution is Pareto on $[k_0, \infty)$ with right tail index $\zeta = \delta/\mu > 1$. Benhabib and Bisin [2007] trace this interpretation of the Pareto distribution back to Francesco

Cantelli and Enrico Fermi. The speed of adjustment of the aggregate can now be written as

$$\delta - \mu = \left(1 - \frac{1}{\zeta}\right)\delta. \tag{2}$$

The key observation made in Luttmer [2012] is that this speed converges to zero precisely when $\zeta \downarrow 1$. That is, if the size distribution is close to Zipf's law (as is the case for firm employment in US data), then the aggregate K_t will converge very slowly.

In Luttmer [2012], the flow of new entrants E_t is, in fact, an endogenous variable that responds to K_t. The resulting speed of convergence is proportional to $(1 - 1/\zeta)\delta$ but also depends on the elasticity of E_t with respect to K_t. Importantly, because the equilibrium response of E_t to K_t is non-linear, the speed of convergence far away from the steady state can be much faster than it is near the steady state. The formula (2) is best thought of as an indication of what happens relatively close to the steady state.

2. Adding Brownian Shocks

An obvious extension is to suppose that firm size evolves with age according to

$$\mathrm{d}\ln(k_a) = \mu \mathrm{d}a + \sigma \mathrm{d}W_a,$$

where W_a is a firm-specific standard Brownian motion and $\sigma^2 > 0$. Ito's lemma says that

$$\mathrm{d}k_a = \left(\mu + \frac{1}{2}\sigma^2\right) k_a \mathrm{d}a + \sigma k_a \mathrm{d}W_a.$$

This is linear in firm size. As before, suppose there is a flow of entrants E_t, and that every entrant starts with the same $k_0 > 0$. Firms exit randomly at the rate $\delta > 0$. Taking into account entry and exit results in an aggregate K_t that evolves according to

$$\mathrm{d}K_t = -\left(\delta - \left(\mu + \frac{1}{2}\sigma^2\right)\right) K_t \mathrm{d}t + I_t \mathrm{d}t,$$

where $I_t = k_0 E_t$. The condition for K_t to settle down when $I_t = k_0 E_t$ is constant is now $\delta > \mu + \sigma^2/2$.

2.1 The Stationary Distribution

The condition $\delta > \mu + \sigma^2/2$ also implies that there is a stationary density with a finite mean. To see this, write f for the density of $z = \ln(k)$. The forward equation says that

$$0 = -\mu \frac{\partial f(z)}{\partial z} + \frac{1}{2}\sigma^2 \frac{\partial^2 f(z)}{\partial z^2} - \delta f(z),$$

for all $z \neq \ln(k_0)$. One possible solution is $e^{-\alpha z}$ with $0 = \mu\alpha + \frac{1}{2}\sigma^2\alpha^2 - \delta$. This yields roots $\alpha \in \{\alpha_-, \alpha_+\}$, where

$$\alpha_{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\delta}{\sigma^2/2}}. \tag{3}$$

Note that $\delta > 0$ implies that $\alpha_- < 0 < \alpha_+$. The resulting density $f(z)$ is proportional to $e^{-\alpha_- z}$ for $z < \ln(k_0)$, to $e^{-\alpha_+ z}$ for $z > \ln(k_0)$, and continuous at $\ln(k_0)$. As is well known, the above process of entry, growth, and exit implies a distribution of firm size that is double Pareto.

In particular, the right tail is Pareto with tail index $\zeta = \alpha_+$,

$$\zeta = -\frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2}\right)^2 + \frac{\delta}{\sigma^2/2}}.$$

To see that the mean of $k = e^z$ will be finite, observe that

$$\mu + \frac{1}{2}\sigma^2 = \frac{1}{2}\sigma^2\left(\left(1 + \frac{\mu}{\sigma^2}\right)^2 - \left(\frac{\mu}{\sigma^2}\right)^2\right),$$

and so $\delta > \mu + \sigma^2/2$ implies $\zeta > 1$. Given $\delta > 0$, the converse is also true. The limit $\zeta \downarrow 1$ corresponds to $\mu + \sigma^2/2 \uparrow \delta$, and so we have the same result as before: the mean reversion rate of K_t goes to zero precisely when the stationary distribution of firm size approaches Zipf's law.

3. Other Moments

The aggregate K_t is the first moment of the distribution of firm size. It is very easy to calculate the speed of adjustment of alternative moments. After all, a geometric Brownian motion raised to a power is again a geometric Brownian motion, to which the above reasoning applies.

To make this explicit, define

$$x_a = k_a^{-\xi},$$

and suppose $\alpha_+ + \xi > 0$ and $-(\alpha_- + \xi) > 0$, so that the mean of the stationary distribution of $k^{-\xi}$ is finite. Ito's lemma says that

$$\mathrm{d}x_a = \left(-\mu\xi + \frac{1}{2}\sigma^2\xi^2\right)x_a\mathrm{d}a - \xi\sigma x_a\mathrm{d}W_a.$$

Just like the evolution of k_a itself, this is linear in x_a. Write X_t for the aggregate at time t. Accounting for entry and exit gives

$$\mathrm{d}X_t = \left(-\mu\xi + \frac{1}{2}\sigma^2\xi^2 - \delta\right) X_t \mathrm{d}t + E_t k_0^{-\xi} \mathrm{d}t.$$

So the mean reversion rate of X_t is

$$\lambda(\xi) = \delta + \mu\xi - \frac{1}{2}\sigma^2\xi^2. \tag{4}$$

This generalizes the above computation for $\xi = -1$.

3.1 Mean Reversion and Existence of Moments

Recall that $\delta > 0$ ensures that $\alpha_- < 0 < \alpha_+$. The definition of the characteristic roots $\alpha_\pm$ implies that $\lambda(-\alpha_\pm) = 0$. The parabola $\lambda(\xi)$ defined in (4) is therefore positive if and only if $\xi \in (-\alpha_+, -\alpha_-)$, a non-empty interval. This restriction on ξ can be written as the combination of $\alpha_+ + \xi > 0$ and $-(\alpha_- + \xi) > 0$. So the mean reversion rate $\lambda(\xi)$ is strictly positive if and only the stationary distribution of k is such that the mean of $k^{-\xi}$ is finite.

In any equilibrium, the aggregate capital stock (or aggregate firm employment, or aggregate income, wealth) will have to be finite. So we have to have $-1 \in (-\alpha_+, -\alpha_-)$ in any equilibrium. The parabola $\lambda(\xi)$ attains its maximum at μ/σ^2, and so $0 < \lambda(\xi) < \lambda(-1)$ for all $\xi \in (-\alpha_+, -1)$ if $-1 < \mu/\sigma^2$. In other words, if $\mu + \sigma^2$ is positive, then the speed of convergence $\lambda(\xi)$ will be less than $\lambda(-1)$ for any $\xi < -1$ (that is, higher moments of k) for which the mean is still finite. More generally, one can take $\xi \in (-\alpha_+, -1)$ close enough to $-\alpha_+$ to obtain an arbitrarily slow rate of convergence.

It is worth noting that X_t is likely to be a poor approximation for a finite-population aggregate when the parameters are such that $\xi \in (-\alpha_+, -1)$ is close to $-\alpha_+$.

4. Two Applications

To learn about the convergence properties of the cross-sectional distribution of income, Gabaix, Lasry, Lions, and Moll [2016] propose studying the rate at which its Laplace transforms converge. This is the same as studying the convergence properties of X_t for various ξ, and so they obtain the convergence rates $\lambda(\xi)$. For values of $\xi < -1$, the aggregate X_t puts more weight on what happens in the right tail than the aggregate K_t. A rough estimate of the tail index of the US (non-human) wealth distribution is

$\alpha_+ = 1.5$ in recent years and about $\alpha_+ = 2$ in the 1960s. The authors point out that a one-time increase in σ^2 would have lead to a much slower change in X_t than is observed in the data.

Luttmer [2012] considers economies in which K_t turns out to be a sufficient state variable for the aggregate economy and studies the mean reversion of K_t (that is, $\xi = -1$). The firm size distribution in US data is very close to Zipf's law. This implies very slow recoveries following a destruction of some part of K_t. This is proposed as an ingredient for models that attempt to explain slow recoveries. The interval $(-\alpha_+, -1)$ is very small in this application, and so there is barely any scope for considering higher moments.

5. Breaking the Link

Gabaix, Lasry, Lions, and Moll [2016] break the link between Zipf's law and slow aggregate mean reversion by introducing heterogeneity and dynamics in the drift parameter μ. This device was used earlier in Luttmer [2011] to avoid the counterfactual firm age implications of the random growth model: given Gibrat's law, a calibration based on the firm size distribution and observed entry and exit rates implies that large firms would have to be centuries old.

The following discussion describes an alternative model that can also generate relatively fast convergence of aggregates, even when the implied stationary distribution is close to Zipf's law.[1]

5.1 Stationary Markov Diffusions with a Linear Drift

Abstract from entry and exit and suppose there is fixed population of firms (or dynastic households) that live forever. Suppose that the size k_t of a typical firm is determined by the stochastic differential equation

$$\mathrm{d}k_t = \lambda(\mu - k_t)\mathrm{d}t + \sigma(k_t)\mathrm{d}W_t, \tag{5}$$

where W_t is, as before, a firm-specific Brownian motion. The parameters λ, μ and $\sigma(\cdot)$ are common to all firms. The key simplifying assumption is that the drift $\lambda(\mu - k_t)$ is linear in k_t. The shape of $\sigma(\cdot)$ will be critical for determining the properties of k_t.

[1] I thank Alberto Bisin for reminding me of the Kesten interpretation of Pareto-like distributions and pointing me to the quantitative results in Section 6 of Benhabib, Bisin and Luo [2015]. This prompted the following investigation.

The random growth process considered previously had $\lambda < 0$, $\mu = 0$ and $\sigma(k) = \sigma$. The result was a geometric Brownian motion. Geometric Brownian motions are not stationary, and stationarity was restored by entry and random exit. If entrants are identified with specific firms that have just exited, then one can think of k_t as the size of an infinitely lived (dynastic) firm that has its size return to some baseline value k_0 at random times. In fact, such processes are known as return processes (Karlin and Taylor [1981, p. 260]).

Take (5) as a continuous-time approximation of a process with stochastic increments $\sigma(k_t)(W_{t+\Delta} - W_t)$ at discrete time intervals Δ. In a large population of firms, with Gaussian increments $W_{t+\Delta} - W_t$ that are independent across firms, the aggregate size K_t will satisfy

$$\mathrm{d}K_t = \lambda(\mu - K_t)\mathrm{d}t.$$

The assumed linearity of the drift, combined with the fact that λ is the same for everyone,[2] implies that the aggregate size K_t is "Markov" again, and that we can interpret λ as the aggregate mean reversion coefficient.

5.2 The Conditional Mean

Suppose that (5) produces a well-defined Markov diffusion. To begin studying its properties, write (5) in integral form

$$k_t = k_0 + \int_0^t \lambda(\mu - k_s)\mathrm{d}s + \int_0^t \sigma(k_s)\mathrm{d}W_s.$$

Define $m_t = \mathrm{E}[k_t|k_0]$. Taking an expectation conditional on k_0 gives

$$m_t = k_0 + \int_0^t \lambda(\mu - m_s)\mathrm{d}s.$$

This implies $\partial m_t/\partial t = -\lambda(m_t - \mu)$, starting from the initial value $m_0 = k_0$. Solving this differential equation gives

$$\mathrm{E}[k_t|k_0] = \mu + (k_0 - \mu)e^{-\lambda t}.$$

That is, as is well known, the linearity of $\mu(k) = \lambda(\mu - k)$ implies that the conditional mean of k_t given k_0 is linear as well. And the mean of k_t given k_0 it converges to μ if and only if λ is positive, at an exponential rate. If there is a stationary distribution, then μ must be its mean.

[2] Recall from Granger (1980) that heterogeneity in λ can be a source of long memory in the aggregate.

5.3 Stationary Distributions

The shape of $\sigma(\cdot)$ is key, for ensuring that there actually is a stationary distribution on $(0, \infty)$, and for determining the tail properties of the stationary distribution.

5.3.1 Two Thin-Tailed Examples

Suppose that $\lambda > 0$ and that $\sigma(\cdot) = \sigma$, a constant. Then (5) becomes an Ornstein-Uhlenbeck process. One can write

$$k_{t+\Delta} - \mu = e^{-\lambda\Delta}(k_t - \mu) + \sigma \int_0^\Delta e^{\lambda a} \mathrm{d}W_{t+a}.$$

So the discretely sampled process is just an AR(1) process with Gaussian innovations. In fact, the implied stationary distribution is Gaussian, and it will not have the thick-tailed tail that we see in firm-size, income, and wealth data.

Another classic example is obtained by taking $\lambda > 0$ and $\sigma(k) = \sigma\sqrt{k}$. This is the square-root process used by Cox, Ingersoll and Ross [1985] as a model for interest rates. The resulting stationary distribution is known to be a Gamma distribution with a density $f(k) \propto k^{\gamma-1}e^{-\beta k}$, where $\beta = \lambda/(\sigma^2/2)$ and $\gamma = \beta\mu$. Even though this begins to look somewhat like a power law, the right tail is still thin, because of the exponential factor $e^{-\beta k}$.

5.3.2 A Thick-Tailed Example

If it exists, the stationary density $f(k)$ associated with a Markov diffusion on $(0, \infty)$ with drift $\mu(k)$ and diffusion coefficient $\sigma(k)$ has to satisfy the Kolmogorov Forward Equation

$$0 = \frac{\partial[-\mu(k)f(k)]}{\partial k} + \frac{1}{2}\frac{\partial^2[\sigma^2(k)f(k)]}{\partial k^2}, \tag{6}$$

on $(0, \infty)$. And $f(k)$ has to integrate to 1. The forward equation (6) can be integrated explicitly to construct candidate stationary densities (for example, see Karlin and Taylor [1981].)

For a thick-tailed example, consider the same diffusion coefficient $\sigma(k) = \sigma k$ as in the case of the geometric Brownian motion. But now, as in the two thin-tailed examples, the drift is $\lambda(\mu - k)$ instead of $(\mu + \sigma^2/2)k$. The resulting process can be interpreted as a continuous-time version of the Kesten [1973] process. A slightly laborious but straightforward calculation shows that

$$f(k) \propto \frac{1}{k^{2+\beta}e^{\gamma/k}}$$

solves (6) when $\beta = \lambda/(\sigma^2/2)$ and $\gamma = \beta\mu$. One can show that this is the only solution to (6) that is integrable. The result is a density with a strictly positive mode, equal to 0 at $k = 0$, and with right tail probabilities that behave like $1/k^{1+\beta}$. Therefore, the right tail index of the stationary distribution is

$$\zeta = 1 + \frac{\lambda}{\sigma^2/2} > 0,$$

and the mean-reversion coefficient is $\lambda > 0$.

Holding fixed $\sigma > 0$, one can take λ close to zero and obtain what happens in the random growth model: letting aggregate mean reversion go to zero results in a tail index that approaches 1 from above, an approximate version of Zipf's law. But the converse is no longer true. A stationary distribution that is close to Zipf's law in the right tail does not imply slow aggregate mean reversion. All an approximate version of Zipf's law says is that $\lambda/(\sigma^2/2)$ is small. This can be achieved not only with a sufficiently small $\lambda > 0$, but also with a large λ combined with a large σ^2. Or in other words, given a certain aggregate rate of mean reversion, a sufficiently large diffusion coefficient will produce a stationary distribution that is close to Zipf's law.

5.4 The Idiosyncratic Returns Interpretation

A natural interpretation of (5) with $\sigma(k) = \sigma k$ is as follows. Suppose infinitely lived dynastic households earn constant income flows $y > 0$ and face investment opportunities that produce idiosyncratic cumulative return processes R_t. Suppose that these returns are not predictable. Specifically, take

$$\mathrm{d}R_t = R_t \left(\mu_{\mathrm{R}}\mathrm{d}t + \sigma_{\mathrm{R}}\mathrm{d}W_t\right), \tag{7}$$

for some parameters μ_{R} and σ_{R}, common across households, and idiosyncratic Brownian motions W_t. If k_t represents household assets, and households consume $c_t = \phi + \psi k_t$ for some $\phi \in [0, y)$ and $\psi > 0$, then their assets will evolve according to

$$\mathrm{d}k_t = (y - [\phi + \psi k_t])\mathrm{d}t + k_t \mathrm{d}R_t/R_t. \tag{8}$$

Combining (7) and (8) gives

$$\mathrm{d}k_t = (y - \phi - (\psi - \mu_{\mathrm{R}})k_t)\mathrm{d}t + \sigma_{\mathrm{R}} k_t \mathrm{d}W_t.$$

This is an example of (5), with mean reversion parameter $\lambda = \psi - \mu_{\mathrm{R}}$, mean $\mu = (y - \phi)/(\psi - \mu_{\mathrm{R}})$ and diffusion parameter $\sigma = \sigma_{\mathrm{R}}$. Aggregate mean reversion will be relatively quick if $\psi - \mu_{\mathrm{R}}$ is well above zero, and the stationary distribution of household assets will have a thick right tail if σ_{R} is also large.

Given a tail index ζ, the implied mean reversion coefficient is $\lambda = \frac{1}{2}\sigma^2(\zeta - 1)$. Suppose the distribution of household assets has a tail index of 1.5 and $\sigma_{\mathrm{R}} = 0.15$ on an annual basis, like the NYSE. Then $\lambda = (.15/2)^2 = 0.005625$. The resulting half life for aggregate assets is $\ln(2)/\lambda \approx 123$ years. As in the basic random growth model, this is very slow. Idiosyncratic returns will have to be substantially more volatile to account for much more rapid aggregate mean reversion. Figure 1 shows how volatile, for the two scenarios $\zeta \in \{1.5, 2.0\}$.

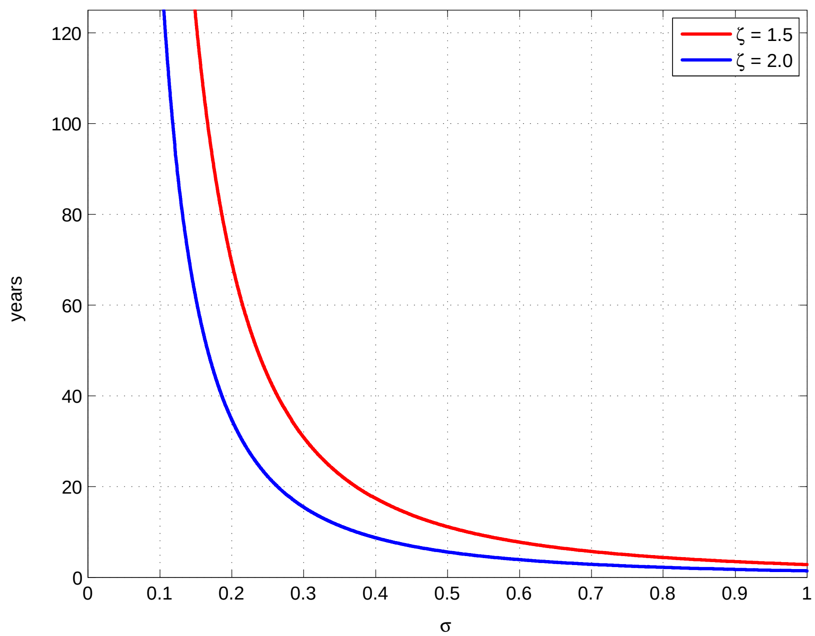

FIGURE 1 The Half-Life $\ln(2)/\lambda$, $\zeta = 1 + \lambda/(\sigma^2/2)$.

Entrepreneurial returns are likely to be more volatile than the NYSE, and so Figure 1 opens up the possibility of fairly rapid mean reversion in combination with a wealth distribution with a rather fat right tail.

Idiosyncratic returns are the key mechanism for wealth inequality emphasized by Benhabib, Bisin and Zhu [2011]. They present an overlapping generations economy with bequests and idiosyncratic returns drawn from a distribution with finite support at the beginning of each new life. From one generation to the next, wealth follows a (discrete-time) Kesten process, and the tail index of the stationary distribution is given explicitly. In a similar economy, Benhabib, Bisin and Luo [2015] find relatively rapid convergence to the steady state.

The example given here also uses idiosyncratic returns. But wealth is dynastic asset wealth of an infinitely lived household. The implicit bequests and intergenerational mobility implications are no doubt counterfactual. The payoff is an explicit analysis of the relation between aggregate mean reversion and tail indices that may shed some further light on the calibration and quantitative findings of Benhabib, Bisin and Luo [2015].

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