MACROECONOMIC THEORY **T. J. KEHOE**
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ANSWERS TO MIDTERM EXAMINATION

1. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \ \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \le \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$
$$c_t^i \ge 0.$$

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$.

(b) With sequential market markets structure, there are markets for goods and bonds open every period. Consumers trade goods and bonds among themselves.

A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, and asset holdings $\hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ such that

- Given $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, the consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots; \hat{b}_1^i, \hat{b}_2^i, \hat{b}_3^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_0^i + b_1^i \le w_0^i$$
$$c_t^i + b_{t+1}^i \le w_t^i + \left(1 + \hat{r}_t\right) b_t^i,\ t = 1,2,\ldots$$
$$b_t^i \ge -B$$
$$c_t^i \ge 0.$$

Here $b_t^i \ge -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$

- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0,1,\ldots$

(c) **Proposition 1:** Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1$$
$$\hat{b}_1^i = w_0^i - \hat{c}_0^i$$
$$\hat{b}_{t+1}^i = w_t^i + (1+\hat{r}_t)\hat{b}_t^i - \hat{c}_t^i,\ t = 1, 2, \ldots.$$

Proposition 2: Suppose that $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium where

$$\hat{p}_0 = 1$$
$$\hat{p}_t = \prod_{s=1}^{t} \frac{1}{(1+\hat{r}_s)},\ t = 1, 2, \ldots.$$

(d) Using the two consumers' first order conditions

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t\,,$$

we can write

$$\frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1}.$$

In even periods,

$$c_t^1 + c_t^2 = 4$$
$$c_t^1 + \frac{\lambda^1}{\lambda^2} c_t^1 = 4$$
$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 4\,.$$

Similarly, in odd periods,

$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 5\,.$$

Normalizing $p_0 = 1$, we can use the first order condition to write

$$p_t = \begin{cases} \beta^t & \text{if } t \text{ is even} \\ \frac{4}{5}\beta^t & \text{if } t \text{ is odd} \end{cases},$$

which implies that

$$p_t c_t^1 = \beta^t \frac{4\lambda^2}{\lambda^1 + \lambda^2}.$$

Consequently,

$$\sum_{t=0}^{\infty} p_t c_t^1 = \frac{4\lambda^2}{\lambda^1 + \lambda^2} \sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta} \frac{4\lambda^2}{\lambda^1 + \lambda^2} = \sum_{t=0}^{\infty} p_t w_t^1$$

$$\frac{1}{1-\beta} \frac{4\lambda^2}{\lambda^1 + \lambda^2} = 3\sum_{t=0}^{\infty} p_{2t} + \sum_{t=0}^{\infty} p_{2t+1}$$

$$\frac{1}{1-\beta} \frac{4\lambda^2}{\lambda^1 + \lambda^2} = 3\sum_{t=0}^{\infty} \beta^{2t} + \frac{4}{5}\beta \sum_{t=0}^{\infty} \beta^{2t}$$

$$\frac{1}{1-\beta} \frac{4\lambda^2}{\lambda^1 + \lambda^2} = \frac{3 + \frac{4}{5}\beta}{1-\beta^2}$$

$$\frac{\lambda^2}{\lambda^1 + \lambda^2} = \frac{\frac{3}{4} + \frac{1}{5}\beta}{1+\beta} = \frac{15 + 4\beta}{20(1+\beta)},$$

which implies that

$$\frac{\lambda^1}{\lambda^1 + \lambda^2} = \frac{5 + 16\beta}{20(1+\beta)}.$$

$$c_t^1 = \begin{cases} \frac{15 + 4\beta}{5(1+\beta)} & \text{if } t \text{ is even} \\ \frac{15 + 4\beta}{4(1+\beta)} & \text{if } t \text{ is odd} \end{cases}$$

$$c_t^2 = \begin{cases} \frac{5 + 16\beta}{5(1+\beta)} & \text{if } t \text{ is even} \\ \frac{5 + 16\beta}{4(1+\beta)} & \text{if } t \text{ is odd} \end{cases}.$$

(We can even work out λ^1 and λ^2, but the question does not ask for this, and it would be a waste of precious time to do so during the exam.

$$\lambda^1 = \frac{1}{c_0^1} = \frac{5(1+\beta)}{15+4\beta}$$
$$\lambda^2 = \frac{1}{c_0^2} = \frac{5(1+\beta)}{5+16\beta}.$$

Check:

$$\frac{\lambda^1}{\lambda^1+\lambda^2} = \frac{\frac{5(1+\beta)}{15+4\beta}}{\frac{5(1+\beta)}{15+4\beta}+\frac{5(1+\beta)}{5+16\beta}} = \frac{\frac{1}{15+4\beta}}{\frac{1}{15+4\beta}+\frac{1}{5+16\beta}} = \frac{5+16\beta}{5+16\beta+15+4\beta} = \frac{5+16\beta}{20(1+\beta)}.)$$

To calculate the sequential markets equilibrium, we just use the formulas from proposition 1 in part c. For example,

$$r_t = \frac{\hat{p}_{t-1}}{\hat{p}_t} - 1 = \begin{cases} \frac{5}{4\beta} - 1 & \text{if } t \text{ is odd} \\ \frac{4}{5\beta} - 1 & \text{if } t \text{ is even} \end{cases}.$$

Notice that, in $t = 0$,

$$\hat{b}_1^1 = 3 - \frac{15+4\beta}{5(1+\beta)} = \frac{11\beta}{5(1+\beta)}.$$

That is, in even periods, consumer 1 lends $\frac{11\beta}{5(1+\beta)}$ to consumer 2, who pays back $\frac{11}{4(1+\beta)}$ in odd periods.

(e) A **Pareto efficient allocation** is an allocation $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots; \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ that is **feasible**,

$$\hat{c}_t^1 + \hat{c}_t^2 \leq w_t^1 + w_t^2,\ t = 0,1,\ldots,$$

and is such that there is no other feasible allocation $\bar{c}_0^1, \bar{c}_1^1, \bar{c}_2^1, \ldots; \bar{c}_0^2, \bar{c}_1^2, \bar{c}_2^2, \ldots$ that is also feasible,

$$\bar{c}_t^1 + \bar{c}_t^2 \leq w_t^1 + w_t^2,\ t = 0,1,\ldots,$$

and satisfies

$$\sum_{t=0}^{\infty} \beta^t \log \bar{c}_t^i \geq \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i,\ i = 1,2,$$

with at least one of the two inequalities being strict.

Proposition 3. Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is a Pareto efficient allocation.

Proof. Suppose not, that there is an allocation $\bar{c}_0^1, \bar{c}_1^1, \bar{c}_2^1, \ldots;\ \bar{c}_0^2, \bar{c}_1^2, \bar{c}_2^2, \ldots$ that is feasible and Pareto superior. $\sum_{t=0}^{\infty} \beta^t \log \bar{c}_t^i > \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i$ implies that

$$\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i > \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$

because, otherwise, $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ would not be utility maximizing.
$\sum_{t=0}^{\infty} \beta^t \log \bar{c}_t^i \geq \sum_{t=0}^{\infty} \beta^t \log \hat{c}_t^i$ implies that

$$\sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i \geq \sum_{t=0}^{\infty} \hat{p}_t w_t^i .$$

Otherwise, we could set $\tilde{c}_0^i = \bar{c}_0^i + \left(\sum_{t=0}^{\infty} \hat{p}_t w_t^i - \sum_{t=0}^{\infty} \hat{p}_t \bar{c}_t^i \right) / \hat{p}_0$ and $\tilde{c}_t^i = \bar{c}_t^i$, $t = 1, 2, \ldots$ and obtain a consumption plan $\tilde{c}_0^i, \tilde{c}_1^i, \tilde{c}_2^i, \ldots$ that satisfies the budget constraint and yields strictly higher utility than $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$. Adding the inequalities for the two consumers together yields

$$\sum_{t=0}^{\infty} \hat{p}_t (\bar{c}_t^1 + \bar{c}_t^2) > \sum_{t=0}^{\infty} \hat{p}_t (w_t^1 + w_t^2) .$$

Notice that $\sum_{t=0}^{\infty} \hat{p}_t w_t^i < \infty$, $i = 1, 2$, for utility maximization to make sense, so that this last inequality makes sense. (This is, we are not saying $\infty > \infty$, which is nonsense.) Since utility is strictly increasing, prices $\hat{p}_t$ are strictly positive. Multiply the condition that $\bar{c}_0^1, \bar{c}_1^1, \bar{c}_2^1, \ldots;\ \bar{c}_0^2, \bar{c}_1^2, \bar{c}_2^2, \ldots$ be feasible in period t by $\hat{p}_t$ and adding up $t = 0, 1, \ldots$, we obtain

$$\sum_{t=0}^{\infty} \hat{p}_t (\bar{c}_t^1 + \bar{c}_t^2) \leq \sum_{t=0}^{\infty} \hat{p}_t (w_t^1 + w_t^2) ,$$

which is a contradiction.■

Proposition 4. Suppose that $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{b}_1^1, \hat{b}_2^1, \hat{b}_3^1, \ldots;$ $\hat{b}_1^2, \hat{b}_2^2, \hat{b}_3^2, \ldots$ is a sequential markets equilibrium. Then $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is a Pareto efficient allocation.

Proof: Proposition 2 implies that $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is the equilibrium allocation of an Arrow-Debreu equilibrium. Proposition 3 implies that it is Pareto efficient.■

We could also answer this question using first order conditions from the consumers' problems and first order conditions from the Pareto problem.

2. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 1. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max \ \log c_1^0 \\ &\text{s.t } \ \hat{p}_1 c_1^0 \leq \hat{p}_1 w_2 + m \\ &c_1^0 \geq 0 . \end{aligned}$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer *t*, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ to solve

$$\begin{aligned} &\max \ \log c_t^t + \log c_{t+1}^t \\ &\text{s.t. } \ \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \leq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 \\ &c_t^t, c_{t+1}^t \geq 0 . \end{aligned}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, \ 2, \ \ldots$.

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t-1$ and t trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, and asset holdings $\hat{b}_2^1, \hat{b}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{aligned} &\max \ \log c_1^0 \\ &\text{s.t } \ c_1^0 \leq w_2 + m \\ &c_1^0 \geq 0 . \end{aligned}$$

- Given $\hat{r}_{t+1}$, consumer *t*, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{b}_{t+1}^t$ to solve

$$\begin{aligned} &\max \ \log c_t^t + \log c_{t+1}^t \\ &\text{s.t. } \ c_t^t + b_{t+1}^t \leq w_1 \end{aligned}$$

$$c_{t+1}^t \le w_2 + (1+\hat{r}_{t+1})b_{t+1}^t$$
$$c_t^t, c_{t+1}^t \ge 0\,.$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \ldots$.
- $\hat{b}_2^1 = m$, $\hat{b}_{t+1}^t = \left[\prod_{\tau=2}^{t}(1+\hat{r}_\tau)\right]m$, $t = 2, 3, \ldots$.

(c) Since there is no fiat money, there is only one good per period, there is only one consumer type in each generation, and consumers live for only two periods, the equilibrium allocation is autarky:

$$\hat{c}_1^0 = w_2$$
$$(\hat{c}_t^t, \hat{c}_{t+1}^t) = (w_1, w_2)$$

The first order conditions from the consumers' problems in the Arrow-Debreu equilibrium imply that

$$\frac{\hat{c}_{t+1}^t}{\hat{c}_t^t} = \frac{\hat{p}_t}{\hat{p}_{t+1}}\,.$$

Normalizing $\hat{p}_1 = 1$, we obtain $\hat{p}_t = (w_1 / w_2)^{t-1}$. Similarly, the first order conditions from the consumers' problems in the sequential markets equilibrium, imply that

$$1 + \hat{r}_{t+1} = \frac{\hat{c}_{t+1}^t}{\hat{c}_t^t} = \frac{w_2}{w_1}$$

or $\hat{r}_t = w_2 / w_1 - 1$. Since the equilibrium allocation is autarky, $\hat{b}_{t+1}^t = 0$.

(d) An allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ is **feasible** if

$$\hat{c}_t^{t-1} + \hat{c}_t^t \le w_2 + w_1,\ t = 1,\ 2,\ \ldots.$$

An allocation is **Pareto efficient** if it is feasible and there exists no other allocation $\overline{c}_1^0, (\overline{c}_1^1, \overline{c}_2^1), (\overline{c}_2^2, \overline{c}_3^2), \ldots$ that is also feasible and satisfies

$$\log \overline{c}_1^0 \ge \log \hat{c}_1^0$$
$$\log \overline{c}_t^t + \log \overline{c}_{t+1}^t \ge \log \hat{c}_t^t + \log \hat{c}_{t+1}^t,\ t = 1, 2, \ldots,$$

with at least one inequality strict.

If $w_2 > w_1$, the equilibrium allocation is Pareto efficient. Suppose not. Then there exists a feasible allocation that is Pareto superior. If

$$\log \overline{c}_t^t + \log \overline{c}_{t+1}^t > \log \hat{c}_t^t + \log \hat{c}_{t+1}^t,$$

then

$$\hat{p}_t \overline{c}_t^t + \hat{p}_{t+1} \overline{c}_{t+1}^t > \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Otherwise, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ would not solve the maximization problem of generation t. Similarly, $\log \overline{c}_1^0 > \log \hat{c}_1^0$ implies $\hat{p}_1 \overline{c}_1^0 > \hat{p}_1 w_2$.

Suppose that

$$\log \overline{c}_t^t + \log \overline{c}_{t+1}^t \geq \log \hat{c}_t^t + \log \hat{c}_{t+1}^t$$

but that

$$\hat{p}_t \overline{c}_t^t + \hat{p}_{t+1} \overline{c}_{t+1}^t < \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Then let

$$\tilde{c}_t^t = \overline{c}_t^t + \frac{\hat{p}_t w_1 + \hat{p}_{t+1} w_2 - \hat{p}_t \overline{c}_t^t + \hat{p}_{t+1} \overline{c}_{t+1}^t}{\hat{p}_t} > \overline{c}_t^t$$

and $\tilde{c}_{t+1}^t = \overline{c}_{t+1}^t$. Then

$$\log \tilde{c}_t^t + \log \tilde{c}_{t+1}^t > \log \hat{c}_t^t + \log \hat{c}_{t+1}^t$$

but

$$\hat{p}_t \tilde{c}_t^t + \hat{p}_{t+1} \tilde{c}_{t+1}^t = \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Once again, this would imply that $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ would not solve the maximization problem of generation t, which is impossible. Consequently,

$$\hat{p}_t \overline{c}_t^t + \hat{p}_{t+1} \overline{c}_{t+1}^t \geq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 .$$

Similarly, $\log \overline{c}_1^0 \geq \log \hat{c}_1^0$ implies $\hat{p}_1 \overline{c}_1^0 \geq \hat{p}_1 w_2$. Therefore

$$\hat{p}_1 \overline{c}_1^0 \geq \hat{p}_1 w_2$$
$$\hat{p}_t \overline{c}_t^t + \hat{p}_{t+1} \overline{c}_{t+1}^t \geq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 ,\ t = 1, 2, \ldots ,$$

with at least one inequality strict. Adding these inequalities up, we obtain

$$\sum_{t=1}^{\infty} \hat{p}_t (\overline{c}_t^{t-1} + \overline{c}_t^t) > \sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) .$$

It is here that $\hat{p}_t = (w_1 / w_2)^{t-1}$ plays its role in ensuring that these series converge.

$$\sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) = \sum_{t=1}^{\infty} \left(\frac{w_1}{w_2} \right)^{t-1} (w_1 + w_2) = \frac{w_1 + w_2}{1 - w_1 / w_2} < \infty$$

Multiplying the feasibility condition in period t by $\hat{p}_t > 0$ and adding up yields

$$\sum_{t=1}^{\infty} \hat{p}_t (\overline{c}_t^{t-1} + \overline{c}_t^t) \leq \sum_{t=1}^{\infty} \hat{p}_t (w_1 + w_2) < \infty ,$$

which is a contradiction.

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(e) A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$, asset holdings $\hat{b}_2^1, \hat{b}_3^2, \ldots$, and storage holdings $\hat{x}_2^1, \hat{x}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{gathered} \max \ \log c_1^0 \\ \text{s.t } c_1^0 \le w_2 + m + \theta \overline{x}_1^0 \\ c_1^0 \ge 0 . \end{gathered}$$

Here $\overline{x}_1^0$ are the initial storage holdings brought into period 1 of generation 0.

- Given $\hat{r}_t$, consumer *t*, $t = 1, 2, \ldots$, chooses $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ and $\hat{b}_{t+1}^t$ to solve

$$\begin{gathered} \max \ \log c_t^t + \log c_{t+1}^t \\ \text{s.t. } c_t^t + b_{t+1}^t + x_{t+1}^t \le w_1 \\ c_{t+1}^t \le w_2 + (1 + \hat{r}_{t+1}) b_{t+1}^t + \theta x_{t+1}^t \\ c_t^t, c_{t+1}^t, x_{t+1}^t \ge 0 . \end{gathered}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t + \hat{x}_{t+1}^t = w_2 + w_1 + \theta \hat{x}_t^{t-1}$, $t = 1, 2, \ldots$.
- $\hat{b}_2^1 = m$, $\hat{b}_{t+1}^t = \left[\prod_{\tau=2}^t (1 + \hat{r}_\tau) \right] m$, $t = 2, 3, \ldots$.

3. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is sequence of prices $\hat{p}_1, \hat{p}_2, \hat{p}_3, \ldots$ and consumption levels $\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1, \ldots$; $\hat{c}_1^{20}, (\hat{c}_1^{21}, \hat{c}_2^{21}), (\hat{c}_2^{22}, \hat{c}_3^{22}), (\hat{c}_3^{23}, \hat{c}_4^{23}), \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer 1 chooses $\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1, \ldots$ to solve

$$\begin{gathered} \max \ \sum_{t=1}^\infty \beta^{t-1} \log c_t^1 \\ \text{s.t. } \sum_{t=1}^\infty \hat{p}_t c_t^1 \le \sum_{t=1}^\infty \hat{p}_t w_t^1 \\ c_t^1 \ge 0 . \end{gathered}$$

- Given $\hat{p}_1$, consumer 20 chooses $\hat{c}_1^{20}$ to solve

$$\max \ \log c_1^{20}$$
$$\text{s.t } \hat{p}_1 c_1^{20} \le \hat{p}_1 w_1^{20} = \hat{p}_1$$
$$c_1^{20} \ge 0 .$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer $2t$, $t = 1,2,\ldots$, chooses $(\hat{c}_t^{2t}, \hat{c}_{t+1}^{2t})$ to solve

$$\max \ \log c_t^{2t} + \log c_{t+1}^{2t}$$
$$\text{s.t. } \hat{p}_t c_t^{2t} + \hat{p}_{t+1} c_{t+1}^{2t} \le \hat{p}_t w_t^{2t} + \hat{p}_{t+1} w_{t+1}^{2t} = 3\hat{p}_t + \hat{p}_{t+1}$$
$$c_t^{2t}, c_{t+1}^{2t} \ge 0 .$$

- $\hat{c}_t^1 + \hat{c}_t^{2t-1} + \hat{c}_t^{2t} = w_t^1 + w_t^{2t-1} + w_t^{2t}$, $t = 1,2,\ldots$.

(b) With sequential market markets structure, there are markets for goods and bonds open every period. Consumers trade goods and bonds among themselves.

A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_2, \hat{r}_3, \hat{r}_4, \ldots$, consumption levels $\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1, \ldots$; $\hat{c}_1^{20}, (\hat{c}_1^{21}, \hat{c}_2^{21}), (\hat{c}_2^{22}, \hat{c}_3^{22}), (\hat{c}_3^{23}, \hat{c}_4^{23}), \ldots$, and asset holdings $\hat{b}_2^1, \hat{b}_3^1, \hat{b}_4^1, \ldots$; $\hat{b}_2^{21}, \hat{b}_3^{22}, \hat{b}_4^{23}, \ldots$ such that

- Given $\hat{r}_2, \hat{r}_3, \hat{r}_4, \ldots$, the consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots; \hat{b}_1^i, \hat{b}_2^i, \hat{b}_3^i, \ldots$ to solve

$$\max \sum_{t=1}^{\infty} \beta^{t-1} \log c_t^1$$
$$\text{s.t. } c_1^1 + b_2^1 \le w_1^1$$
$$c_t^1 + b_{t+1}^1 \le w_t^1 + \left(1 + \hat{r}_t\right) b_t^1, \ t = 2,3,\ldots$$
$$b_t^1 \ge -B$$
$$c_t^1 \ge 0 .$$

Here $b_t^1 \ge -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- Consumer 20 chooses $\hat{c}_1^{20}$ to solve

$$\max \ \log c_1^{20}$$
$$\text{s.t } c_1^{20} \le w_1^{20} = 1$$
$$c_1^{20} \ge 0 .$$

- Given $\hat{r}_{t+1}$, consumer $2t$, $t = 1,2,\ldots$, chooses $(\hat{c}_t^{2t}, \hat{c}_{t+1}^{2t})$ and $\hat{b}_{t+1}^{2t}$ to solve

$$\max \ \log c_t^{2t} + \log c_{t+1}^{2t}$$
$$\text{s.t. } c_t^{2t} + b_{t+1}^{2t} \le w_t^{2t} = 3$$
$$c_{t+1}^{2t} \le w_{t+1}^{2t} + (1+\hat{r}_{t+1})b_{t+1}^{2t} = 1 + (1+\hat{r}_{t+1})b_{t+1}^{2t}$$
$$c_t^{2t}, c_{t+1}^{2t} \ge 0 .$$

- $\hat{c}_t^1 + \hat{c}_t^{2t-1} + \hat{c}_t^{2t} = w_t^1 + w_t^{2t-1} + w_t^{2t}$, $t = 1, 2, \ldots$.

- $\hat{b}_t^1 + \hat{b}_t^{2t-1} = 0$, $t = 2, 3, \ldots$

(c) An allocation $\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1, \ldots;\ \hat{c}_1^{20}, (\hat{c}_1^{21}, \hat{c}_2^{21}), (\hat{c}_2^{22}, \hat{c}_3^{22}), (\hat{c}_3^{23}, \hat{c}_4^{23}), \ldots$, is **feasible** if

$$\hat{c}_t^1 + \hat{c}_t^{2t-1} + \hat{c}_t^{2t} = w_t^1 + w_t^{2t-1} + w_t^{2t},\ t = 1, 2, \ldots .$$

An allocation is **Pareto efficient** if it is feasible and there exists no other allocation $\overline{c}_1^1, \overline{c}_2^1, \overline{c}_3^1, \ldots;\ \overline{c}_1^{20}, (\overline{c}_1^{21}, \overline{c}_2^{21}), (\overline{c}_2^{22}, \overline{c}_3^{22}), (\overline{c}_3^{23}, \overline{c}_4^{23}), \ldots$ that is also feasible and satisfies

$$\sum_{t=1}^{\infty} \beta^{t-1} \log \overline{c}_t^1 \ge \sum_{t=1}^{\infty} \beta^{t-1} \log \hat{c}_t^1$$
$$\log \overline{c}_1^{20} \ge \log \hat{c}_1^{20}$$
$$\log \overline{c}_t^{2t} + \log \overline{c}_{t+1}^{2t} \ge \log \hat{c}_t^{2t} + \log \hat{c}_{t+1}^{2t},\ t = 1, 2, \ldots,$$

with at least one inequality strict.

The equilibrium allocation is Pareto.

The key step in the proof is to note that, since $\sum_{t=1}^{\infty} \hat{p}_t w_t^1$ must be finite for the maximization problem of consumer 2 to have a solution, the present discounted value of the aggregate endowment must be finite:

$$\sum_{t=1}^{\infty} \hat{p}_t (w_t^1 + w_t^{2t-1} + w_t^{2t}) < \infty .$$

To see this notice that, since

$$\sum_{t=1}^{\infty} \hat{p}_t w_t^1 = 3\sum_{t=1}^{\infty} \hat{p}_{2t-1} + \sum_{t=1}^{\infty} \hat{p}_{2t}$$

is finite and $\hat{p}_t > 0$ for all t, both $\sum_{t=1}^{\infty} \hat{p}_{2t-1}$ and $\sum_{t=1}^{\infty} \hat{p}_{2t}$ are finite. This implies that

$$a\sum_{t=1}^{\infty} \hat{p}_{2t-1} + b\sum_{t=1}^{\infty} \hat{p}_{2t}$$

is finite for any constants $a, b \ge 0$. Consequently,

$$\sum_{t=1}^{\infty} \hat{p}_t (w_t^1 + w_t^{2t-1} + w_t^{2t}) = 7\sum_{t=1}^{\infty} \hat{p}_{2t-1} + 5\sum_{t=1}^{\infty} \hat{p}_{2t}$$

is finite.

Now let us prove that the equilibrium allocation is part a is Pareto efficient.

Suppose not. Then there exists a feasible allocation that is Pareto superior. If $\sum_{t=1}^{\infty} \beta^{t-1} \log \bar{c}_t^1 > \sum_{t=1}^{\infty} \beta^{t-1} \log \hat{c}_t^1$, then

$$\sum_{t=1}^{\infty} \hat{p}_t \bar{c}_t^1 > \sum_{t=1}^{\infty} \hat{p}_t w_t^1$$

because, otherwise, $\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1, \ldots$ would not be utility maximizing. If $\sum_{t=1}^{\infty} \beta^{t-1} \log \bar{c}_t^i \geq \sum_{t=1}^{\infty} \beta^{t-1} \log \hat{c}_t^i$, then

$$\sum_{t=1}^{\infty} \hat{p}_t \bar{c}_t^1 \geq \sum_{t=1}^{\infty} \hat{p}_t w_t^1 .$$

Otherwise, we could set $\tilde{c}_1^1 = \bar{c}_1^1 + \left(\sum_{t=1}^{\infty} \hat{p}_t w_t^1 - \sum_{t=1}^{\infty} \hat{p}_t \bar{c}_t^1 \right) / \hat{p}_1$ and $\tilde{c}_t^1 = \bar{c}_t^1$, $t = 2,3,...$ and obtain a consumption plan $\tilde{c}_1^1, \tilde{c}_2^i, \tilde{c}_3^i, \ldots$ that satisfies the budget constraint and yields strictly higher utility than $\hat{c}_1^1, \hat{c}_2^1, \hat{c}_3^1, \ldots$.

If $\log \bar{c}_t^{2t} + \log \bar{c}_{t+1}^{2t} > \log \hat{c}_t^{2t} + \log \hat{c}_{t+1}^{2t}$, then

$$\hat{p}_t \bar{c}_t^{2t} + \hat{p}_{t+1} \bar{c}_{t+1}^{2t} > \hat{p}_t w_t^{2t} + \hat{p}_{t+1} w_{t+1}^{2t} .$$

Otherwise, $(\hat{c}_t^{2t}, \hat{c}_{t+1}^{2t})$ would not solve the maximization problem of generation t. If $\log \bar{c}_t^{2t} + \log \bar{c}_{t+1}^{2t} \geq \log \hat{c}_t^{2t} + \log \hat{c}_{t+1}^{2t}$, then

$$\hat{p}_t \bar{c}_t^{2t} + \hat{p}_{t+1} \bar{c}_{t+1}^{2t} \geq \hat{p}_t w_t^{2t} + \hat{p}_{t+1} w_{t+1}^{2t} .$$

Otherwise, we could set $\tilde{c}_t^t = \bar{c}_t^t + (\hat{p}_t w_1 + \hat{p}_{t+1} w_2 - \hat{p}_t \bar{c}_t^t + \hat{p}_{t+1} \bar{c}_{t+1}^t) / \hat{p}_t$ and $\tilde{c}_{t+1}^t = \bar{c}_{t+1}^t$ and obtain a consumption plan $(\tilde{c}_t^t, \tilde{c}_{t+1}^t)$ that satisfies the budget constraint and yields strictly higher utility than $(\hat{c}_t^t, \hat{c}_{t+1}^t)$.

Similarly, if $\log \bar{c}_1^{20} > \log \hat{c}_1^{20}$,

$$\hat{p}_1 \bar{c}_1^{20} > \hat{p}_1 w_1^{20} ,$$

and, if $\log \bar{c}_1^{20} \geq \log \hat{c}_1^{20}$,

$$\hat{p}_1 \bar{c}_1^{20} \geq \hat{p}_1 w_1^{20} .$$

Consequently,

$$\sum_{t=1}^{\infty} \hat{p}_t \overline{c}_t^1 \geq \sum_{t=1}^{\infty} \hat{p}_t w_t^1$$
$$\hat{p}_1 \overline{c}_1^{20} \geq \hat{p}_1 w_1^{20}$$
$$\hat{p}_t \overline{c}_t^{2t} + \hat{p}_{t+1} \overline{c}_{t+1}^{2t} \geq \hat{p}_t w_t^{2t} + \hat{p}_{t+1} w_{t+1}^{2t},\ t = 1, 2, \ldots,$$

with at least one inequality strict. Adding these inequalities up, we obtain

$$\sum_{t=1}^{\infty} \hat{p}_t (\overline{c}_t^1 + \overline{c}_t^{t-1} + \overline{c}_t^t) > \sum_{t=1}^{\infty} \hat{p}_t (w_t^1 + w_t^{t-1} + w_t^t).$$

Multiplying the feasibility condition in period t by $\hat{p}_t > 0$ and adding up yields

$$\sum_{t=1}^{\infty} \hat{p}_t (\overline{c}_t^1 + \overline{c}_t^{t-1} + \overline{c}_t^t) \leq \sum_{t=1}^{\infty} \hat{p}_t (w_t^1 + w_t^{t-1} + w_t^t) < \infty,$$

which is a contradiction.