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ANSWERS TO MIDTERM EXAMINATION

1. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots$; $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \ \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t.} \ \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i$$
$$c_t^i \geq 0 .$$

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$.

(b) With sequential market markets structure, there are markets for goods and bonds open every period. Consumers trade goods and bonds among themselves.

A **sequential markets equilibrium** is sequences of interest rates $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots$; $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, and bond holdings $\hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots$; $\hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$ such that

- Given $\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots$, the consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$; $\hat{b}_0^i, \hat{b}_1^i, \hat{b}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t.} \ c_t^i + b_{t+1}^i \leq w_t^i + (1+\hat{r}_t) b_t^i , \ t = 0,1,\ldots$$
$$c_t^i \geq 0 , \ b_t^i \geq -B$$
$$b_0^i = 0 .$$

Here $b_t^i \geq -B$, where $B > 0$ is chosen large enough, rules out Ponzi schemes but does not otherwise bind in equilibrium.

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,\ldots$
- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0,1,\ldots$

(c) Using the two consumers' first order conditions

$$\frac{\beta^t}{c_t^i} = \lambda^i p_t, \; i = 1,2,$$

we can write

$$\frac{c_t^1}{c_t^2} = \frac{\lambda^2}{\lambda^1}.$$

In even periods,

$$c_t^1 + c_t^2 = 3$$

$$c_t^1 + \frac{\lambda^1}{\lambda^2} c_t^1 = 3$$

$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 3.$$

Similarly, in odd periods,

$$c_t^1 = \frac{\lambda^2}{\lambda^1 + \lambda^2} 5.$$

Normalizing $p_0 = 1$, we can use the first order condition to write

$$p_t = \begin{cases} \beta^t & \text{if } t \text{ is even} \\ \frac{3}{5}\beta^t & \text{if } t \text{ is odd} \end{cases},$$

which implies that

$$p_t c_t^1 = \beta^t \frac{3\lambda^2}{\lambda^1 + \lambda^2}.$$

Consequently,

$$\sum_{t=0}^{\infty} p_t c_t^1 = \frac{3\lambda^2}{\lambda^1 + \lambda^2} \sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = \sum_{t=0}^{\infty} p_t w_t^1$$

$$\frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = 2\sum_{t=0}^{\infty} p_{2t} + \sum_{t=0}^{\infty} p_{2t+1}$$

$$\frac{1}{1-\beta} \frac{3\lambda^2}{\lambda^1 + \lambda^2} = 2\sum_{t=0}^{\infty} \beta^{2t} + \frac{3}{5}\beta \sum_{t=0}^{\infty} \beta^{2t}$$

$$\frac{1}{1-\beta}\frac{3\lambda^2}{\lambda^1+\lambda^2}=\frac{2+\frac{3}{5}\beta}{1-\beta^2}$$

$$\frac{\lambda^2}{\lambda^1+\lambda^2}=\frac{\frac{2}{3}+\frac{1}{5}\beta}{1+\beta}=\frac{10+3\beta}{15(1+\beta)},$$

which implies that

$$\frac{\lambda^1}{\lambda^1+\lambda^2}=\frac{\frac{1}{3}+\frac{4}{5}\beta}{1+\beta}=\frac{5+12\beta}{15(1+\beta)}.$$

$$c_t^1=\begin{cases}\dfrac{10+3\beta}{5(1+\beta)} & \text{if } t \text{ is even}\\ \dfrac{10+3\beta}{3(1+\beta)} & \text{if } t \text{ is odd}\end{cases}$$

$$c_t^2=\begin{cases}\dfrac{5+12\beta}{5(1+\beta)} & \text{if } t \text{ is even}\\ \dfrac{5+12\beta}{3(1+\beta)} & \text{if } t \text{ is odd}\end{cases}.$$

(We can even work out λ^1 and λ^2, although the question does not require this and it would be a waste of precious time to do so during the exam.

$$\lambda^1=\frac{1}{c_0^1}=\frac{5(1+\beta)}{10+3\beta}$$

$$\lambda^2=\frac{1}{c_0^2}=\frac{5(1+\beta)}{5+12\beta}.$$

Check:

$$\frac{\lambda^1}{\lambda^1+\lambda^2}=\frac{\frac{5(1+\beta)}{10+3\beta}}{\frac{5(1+\beta)}{10+3\beta}+\frac{5(1+\beta)}{5+12\beta}}=\frac{\frac{1}{10+3\beta}}{\frac{1}{10+3\beta}+\frac{1}{5+12\beta}}$$

$$\frac{\lambda^1}{\lambda^1+\lambda^2}=\frac{\frac{5+12\beta}{(10+3\beta)(5+12\beta)}}{\frac{5+12\beta}{(10+3\beta)(5+12\beta)}+\frac{10+3\beta}{(10+3\beta)(5+12\beta)}}=\frac{5+12\beta}{15(1+\beta)}.)$$

(d) A **sequential markets equilibrium** is sequences of rental rates on capital $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$; wages $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$; interest rates on bonds $\hat{r}_0^b, \hat{r}_1^b, \hat{r}_2^b, \ldots$, consumption levels $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots$; $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$; capital holdings $\hat{k}_0^1, \hat{k}_1^1, \hat{k}_2^1, \ldots$; $\hat{k}_0^2, \hat{k}_1^2, \hat{k}_2^2, \ldots$; bond holdings $\hat{b}_0^1, \hat{b}_1^1, \hat{b}_2^1, \ldots$; $\hat{b}_0^2, \hat{b}_1^2, \hat{b}_2^2, \ldots$; and production plans $(\hat{y}_0, \hat{k}_0, \hat{\ell}_0), (\hat{y}_1, \hat{k}_1, \hat{\ell}_1), (\hat{y}_2, \hat{k}_2, \hat{\ell}_2), \ldots$ such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$, $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \hat{r}_2^b, \ldots$, consumer i, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots, \hat{k}_0^i, \hat{k}_1^i, \hat{k}_2^i, \ldots$, $\hat{b}_0^i, \hat{b}_1^i, \hat{b}_2^i, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t^i$$
$$\text{s.t. } c_t^i + k_{t+1}^i + b_{t+1}^i \leq \hat{w}_t^i \bar{\ell}_t^i + (1 + \hat{r}_t^k - \delta) k_t^i + (1 + \hat{r}_t^b) b_t^i,\ t = 0,1,\ldots$$
$$c_t^i \geq 0,\ k_t^i \geq 0\ b_t^i \geq -B$$
$$k_0^i = \bar{k}_0^i,\ b_0^i = 0.$$

- Given $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$, $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$, firms choose $(\hat{y}_0, \hat{k}_0, \hat{\ell}_0), (\hat{y}_1, \hat{k}_1, \hat{\ell}_1), (\hat{y}_2, \hat{k}_2, \hat{\ell}_2), \ldots$ to minimize costs, and $\hat{r}_0^k, \hat{r}_1^k, \hat{r}_2^k, \ldots$, $\hat{w}_0, \hat{w}_1, \hat{w}_2, \ldots$ are such that firms earn 0 profits:

$$\hat{r}_t^k = \alpha\theta \hat{k}_t^{\alpha-1} \hat{\ell}_t^{1-\alpha}$$
$$\hat{w}_t = (1-\alpha)\theta \hat{k}_t^{\alpha} \hat{\ell}_t^{-\alpha}$$

- $\hat{c}_t^1 + \hat{c}_t^2 + \hat{k}_{t+1} - (1-\delta)\hat{k}_t = \hat{y}_t = \theta \hat{k}_t^{\alpha} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$
- $\hat{k}_t^1 + \hat{k}_t^2 = \hat{k}_t$, $t = 0,1,\ldots$
- $\bar{\ell}_t^1 + \bar{\ell}_t^2 = \hat{\ell}_t$, $t = 0,1,\ldots$
- $\hat{b}_t^1 + \hat{b}_t^2 = 0$, $t = 0,1,\ldots$

2. (a) With an Arrow-Debreu markets structure futures markets for goods are open in period 1. Consumers trade futures contracts among themselves.

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \ldots$ such that

- Given $\hat{p}_1$, consumer 0 chooses $\hat{c}_1^0$ to solve

$$\max\ \log c_1^0$$
$$\text{s.t } \hat{p}_1 c_1^0 \leq \hat{p}_1 w_2 + m$$
$$c_1^0 \geq 0.$$

- Given $\hat{p}_t, \hat{p}_{t+1}$, consumer *t*, $t = 1, 2, \ldots$, chooses $\left(\hat{c}_t^t, \hat{c}_{t+1}^t\right)$ to solve

$$\begin{gathered} \max \;\; \log c_t^t + \log c_{t+1}^t \\ \text{s.t.} \;\; \hat{p}_t c_t^t + \hat{p}_{t+1} c_{t+1}^t \leq \hat{p}_t w_1 + \hat{p}_{t+1} w_2 \\ c_t^t, c_{t+1}^t \geq 0 . \end{gathered}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1,\ 2,\ \ldots$.

(b) With sequential market markets structure, there are markets for goods and assets open every period. The consumers in generations $t-1$ and t trade goods and assets among themselves.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \ldots$, an allocation $\hat{c}_1^0, \left(\hat{c}_1^1, \hat{c}_2^1\right), \left(\hat{c}_2^2, \hat{c}_3^2\right), \ldots$, and asset holdings $\hat{s}_2^1, \hat{s}_3^2, \ldots$ such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\begin{gathered} \max \;\; \log c_1^0 \\ \text{s.t} \;\; c_1^0 \leq w_2 + m \\ c_1^0 \geq 0 . \end{gathered}$$

- Given $\hat{r}_{t+1}$, consumer *t*, $t = 1, 2, \ldots$, chooses $\left(\hat{c}_t^t, \hat{c}_{t+1}^t\right)$ and $\hat{s}_{t+1}^t$ to solve

$$\begin{gathered} \max \;\; \log c_t^t + \log c_{t+1}^t \\ \text{s.t.} \;\; c_t^t + s_{t+1}^t \leq w_1 \\ c_{t+1}^t \leq w_2 + (1 + \hat{r}_{t+1}) s_{t+1}^t \\ c_t^t, c_{t+1}^t \geq 0 . \end{gathered}$$

(Alternativly, we could use an alternative notational convention and label assets by the period in which they are purchased, rather than the period in which their return is realized. This convention also has us label interest rates consistently with the asset holdings. Given $\hat{r}_t$, consumer *t*, $t = 1, 2, \ldots$, chooses $\left(\hat{c}_t^t, \hat{c}_{t+1}^t\right)$ and $\hat{s}_t^t$ to solve

$$\begin{gathered} \max \;\; \log c_t^t + \log c_{t+1}^t \\ \text{s.t.} \;\; c_t^t + s_t^t \leq w_1 \\ c_{t+1}^t \leq w_2 + (1 + \hat{r}_t) s_t^t \\ c_t^t, c_{t+1}^t \geq 0 . \end{gathered}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t = w_2 + w_1$, $t = 1, 2, \dots$.

- $\hat{s}_2^1 = m$

  $\hat{s}_{t+1}^t = \left[\prod_{\tau=2}^t (1+\hat{r}_\tau)\right] m$, $t = 2, 3, \dots$.

(c) Since there is no fiat money, there is only one good per period, there is only one consumer type in each generation, and consumers live for only two periods, the equilibrium allocation is autarky:

$$\hat{c}_1^0 = w_2$$
$$(\hat{c}_t^t, \hat{c}_{t+1}^t) = (w_1, w_2)$$

The first order conditions from the consumers' problems in the Arrow-Debreu equilibrium imply that

$$\frac{\hat{p}_{t+1}}{\hat{p}_t} = \frac{\hat{c}_t^t}{\hat{c}_{t+1}^t} = \frac{w_1}{w_2}.$$

Normalizing $\hat{p}_1 = 1$, we obtain $\hat{p}_t = (w_1 / w_2)^{t-1}$. Similarly, the first order conditions from the consumers' problems in the sequential markets equilibrium, imply that

$$\frac{1}{1+\hat{r}_{t+1}} = \frac{\hat{c}_t^t}{\hat{c}_{t+1}^t} = \frac{w_1}{w_2}$$

or $\hat{r}_{t+1} = (w_2 / w_1) - 1$. Since the equilibrium allocation is autarky, $\hat{s}_{t+1}^t = 0$.

A **sequential markets equilibrium** is a sequence of interest rates $\hat{r}_2, \hat{r}_3, \dots$, an allocation $\hat{c}_1^0, (\hat{c}_1^1, \hat{c}_2^1), (\hat{c}_2^2, \hat{c}_3^2), \dots$, nominal asset holdings $\hat{s}_2^1, \hat{s}_3^2, \dots$, and storage holdings $\hat{x}_2^1, \hat{x}_3^2, \dots$, such that

- Consumer 0 chooses $\hat{c}_1^0$ to solve

$$\max \ \log c_1^0$$
$$\text{s.t } c_1^0 \le w_2 + m$$
$$c_1^0 \ge 0.$$

- Given $\hat{r}_t$, consumer *t*, $t = 1,2,\ldots$, chooses $\left(\hat{c}_t^t, \hat{c}_{t+1}^t\right)$, $\hat{s}_{t+1}^t$ and $\hat{x}_{t+1}^t$ to solve

$$\begin{gathered} \max \;\; \log c_t^t + \log c_{t+1}^t \\ \text{s.t.} \quad c_t^t + s_{t+1}^t + x_{t+1}^t \leq w_1 \\ c_{t+1}^t \leq w_2 + (1+\hat{r}_{t+1}) s_{t+1}^t + \theta x_{t+1}^t \\ c_t^t, c_{t+1}^t, x_{t+1}^t \geq 0 . \end{gathered}$$

- $\hat{c}_t^{t-1} + \hat{c}_t^t + \hat{x}_{t+1}^t = w_2 + w_1 + \theta \hat{x}_t^{t-1}$, $t = 1,2,\ldots$.

- $\hat{s}_2^1 = m$

  $\hat{s}_{t+1}^t = \left[\prod_{\tau=2}^{t} (1+\hat{r}_\tau)\right] m$, $t = 2,3,\ldots$.

The consumer will utilize the storage technology if it provides a higher return than the nominal asset:

$$\theta > 1 + \hat{r}_{t+1} = \frac{w_2}{w_1} .$$

If this condition does not hold, the no-storage autarky equilibrium is still the unique equilibrium.

3. (a) With Arrow-Debreu markets, there are futures markets of goods, capital, labor services, capital services open in period 0. Consumers sell labor services to firms. They buy goods from firms. Who makes the capital accumulation decision can be modeled different ways. We could have consumers buy and sell future claims to capital and sell claims to capital services to firms, or we could have consumers sell their initial capital to firms and have firms buy and sell future claims to capital and sell claims to capital services to other firms.

An **Arrow-Debreu equilibrium** is sequences of prices of goods $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, rental rates $\hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, and capital stocks $\hat{k}_0, \hat{k}_1, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, $\hat{w}_0, \hat{w}_1, \ldots$, and $\hat{r}_0$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$ to solve

$$\begin{gathered} \max \sum_{t=0}^{\infty} \beta^t \log c_t \\ \text{s.t.} \;\; \sum_{t=0}^{\infty} \hat{p}_t c_t \leq \sum_{t=0}^{\infty} \hat{w}_t + \hat{r}_0 \bar{k}_0 \\ c_t \geq 0 . \end{gathered}$$

(Here we have the consumers sell their initial capital to firms and have firms make capital accumulation decisions. If we have consumers make capital accumulation decisions, then consumers choose $\hat{k}_0, \hat{k}_1, \ldots$ and the budget constraint is
$\sum_{t=0}^{\infty} \hat{p}_t \left(c_t + k_{t+1}\right) \leq \sum_{t=0}^{\infty} \left(\hat{w}_t + \hat{r}_t k_t\right)$.)

- $\hat{r}_t = \hat{p}_t \alpha \theta \hat{k}_t^{\alpha-1}$, $t = 0,1,\ldots$
  $\hat{w}_t = \hat{p}_t (1-\alpha) \theta \hat{k}_t^{\alpha}$, $t = 0,1,\ldots$
  $\hat{r}_{t+1} - \hat{p}_t \leq 0$, $= 0$ if $\hat{k}_{t+1} > 0$, $t = 0,1,\ldots$

(A good answer would explain that these are the profit maximization conditions for constant returns. Notice that, if we have consumers make capital accumulation decisions, then the zero profit condition on accumulating capital is a first order condition for utility maximization and does not need to be included as a separate equilibrium condition.)

- $\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}$, $t = 0,1,\ldots$

(b) With sequential market markets structure, there are markets for goods, labor services, capital services, and bonds open every period. Consumers sell labor services and rent capital to the firm. They buy goods from the firm, some of which they consume and some of which they save as capital. They trade bonds among themselves.

A **sequential markets equilibrium** is sequences of rental rates $\hat{r}_0^k, \hat{r}_1^k, \ldots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, capital stocks $\hat{k}_0, \hat{k}_1, \ldots$, and bond holdings $\hat{b}_0, \hat{b}_1, \ldots$, such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \ldots$, and $\hat{w}_0, \hat{w}_1, \ldots$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$, $\hat{k}_0, \hat{k}_1, \ldots$, and $\hat{b}_0, \hat{b}_1, \ldots$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \log c_t$$
$$\text{s.t. } c_t + k_{t+1} + b_{t+1} \leq \hat{w}_t + \hat{r}_t^k k_t + \left(1 + \hat{r}_t^b\right) b_t, \; t = 0,1,\ldots$$
$$k_0 = \bar{k}_0, \; b_0 = 0$$
$$b_t \geq -B, \; c_t, k_t \geq 0.$$

- $\hat{r}_t^k = \alpha \theta \hat{k}_t^{\alpha-1}$, $t = 0,1,\ldots$
  $\hat{w}_t = (1-\alpha) \theta \hat{k}_t^{\alpha}$, $t = 0,1,\ldots$.

- $\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha}$, $t = 0,1,\ldots$

- $\hat{b}_t = 0$, $t = 0,1,\ldots$

(c) **Proposition 1:** Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{w}_0, \hat{w}_1, \ldots; \hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots; \hat{c}_0, \hat{c}_1, \ldots; \hat{k}_0, \hat{k}_1, \ldots$ is an Arrow-Debreu equilibrium. Then $\hat{r}_0^k, \hat{r}_1^k, \ldots; \hat{r}_0^b, \hat{r}_1^b, \ldots; \tilde{w}_0, \tilde{w}_1, \ldots; \hat{c}_0, \hat{c}_1, \ldots; \hat{k}_0, \hat{k}_1, \ldots;$ $\hat{b}_0, \hat{b}_1, \ldots$ is a sequential markets equilibrium where

$$\hat{r}_t^k = \frac{\hat{r}_t}{\hat{p}_t}$$
$$\hat{r}_t^b = \hat{r}_t^k - 1$$
$$\tilde{w}_t = \frac{\hat{w}_t}{\hat{p}_t}$$
$$\hat{b}_t = 0, \; t = 0, 1, \ldots.$$

Proposition 2: Suppose that $\hat{r}_0^k, \hat{r}_1^k, \ldots; \hat{r}_0^b, \hat{r}_1^b, \ldots; \hat{w}_0, \hat{w}_1, \ldots; \hat{c}_0, \hat{c}_1, \ldots; \hat{k}_0, \hat{k}_1, \ldots;$ $\hat{b}_0, \hat{b}_1, \ldots$ is a sequential markets equilibrium. Then $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \tilde{w}_0, \tilde{w}_1, \ldots; \hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots;$ $\hat{c}_0, \hat{c}_1, \ldots;$ $\hat{k}_0, \hat{k}_1, \ldots$ is an Arrow-Debreu equilibrium where

$$\hat{p}_0 = 1$$
$$\hat{p}_t = \prod_{s=1}^{t} \frac{1}{\hat{r}_s^k}, \; t = 1, 2, \ldots$$
$$\hat{r}_t = \hat{p}_t \hat{r}_t^k$$
$$\tilde{w}_t = \hat{p}_t \hat{w}_t.$$

(Notice that, according to the way in which we have done things, we need to use separate symbols for the Arrow-Debreu wage, the price of labor services in period t in terms of period 0 goods, and the sequential markets wage, of labor services in period t in terms of period t goods.)

(d) An **Arrow-Debreu equilibrium** is sequences of prices of goods $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, rental rates $\hat{r}_0, \hat{r}_1, \hat{r}_2, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, leisure levels $\hat{x}_0, \hat{x}_1, \ldots$, labor supplies $\hat{\ell}_0, \hat{\ell}_1, \ldots$, and capital stocks $\hat{k}_0, \hat{k}_1, \ldots$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, $\hat{w}_0, \hat{w}_1, \ldots$, and $\hat{r}_0$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$, $\hat{x}_0, \hat{x}_1, \ldots$, and $\hat{\ell}_0, \hat{\ell}_1, \ldots$ to solve

$$\begin{gathered}\sum_{t=0}^{\infty} \beta^t \left(\gamma \log c_t + (1-\gamma) \log x_t\right) \\ \text{s.t.} \quad \sum_{t=0}^{\infty} \hat{p}_t c_t \leq \sum_{t=0}^{\infty} \hat{w}_t \ell_t + \hat{r}_0 \bar{k}_0 \\ x_t + \ell_t = 1 \\ c_t, x_t, \ell_t \geq 0 .\end{gathered}$$

- $\hat{r}_t = \hat{p}_t \alpha \theta \hat{k}_t^{\alpha-1} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$
  $\hat{w}_t = \hat{p}_t (1-\alpha) \theta \hat{k}_t^{\alpha} \hat{\ell}_t^{-\alpha}$, $t = 0,1,\ldots$
  $\hat{r}_{t+1} - \hat{p}_t \leq 0$, $= 0$ if $\hat{k}_{t+1} > 0$, $t = 0,1,\ldots$

- $\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$

(e) A **sequential markets equilibrium** is sequences of rental rates $\hat{r}_0^k, \hat{r}_1^k, \ldots$, interest rates $\hat{r}_0^b, \hat{r}_1^b, \ldots$, wages $\hat{w}_0, \hat{w}_1, \ldots$, consumption levels $\hat{c}_0, \hat{c}_1, \ldots$, leisure levels $\hat{x}_0, \hat{x}_1, \ldots$, labor supplies $\hat{\ell}_0, \hat{\ell}_1, \ldots$, capital stocks $\hat{k}_0, \hat{k}_1, \ldots$, and bond holdings $\hat{b}_0, \hat{b}_1, \ldots$, such that

- Given $\hat{r}_0^k, \hat{r}_1^k, \ldots$, $\hat{r}_0^b, \hat{r}_1^b, \ldots$, and $\hat{w}_0, \hat{w}_1, \ldots$, the consumer chooses $\hat{c}_0, \hat{c}_1, \ldots$, $\hat{x}_0, \hat{x}_1, \ldots$, $\hat{\ell}_0, \hat{\ell}_1, \ldots$, $\hat{k}_0, \hat{k}_1, \ldots$, and $\hat{b}_0, \hat{b}_1, \ldots$ to solve

$$\begin{gathered}\sum_{t=0}^{\infty} \beta^t \left(\gamma \log c_t + (1-\gamma) \log x_t\right) \\ \text{s.t.} \quad c_t + k_{t+1} + b_{t+1} \leq \hat{w}_t \ell_t + \hat{r}_t^k k_t + \left(1 + \hat{r}_t^b\right) b_t, \; t = 0,1,\ldots \\ x_t + \ell_t = 1 \\ k_0 = \bar{k}_0, \; b_0 = 0 \\ b_t \geq -B, \;\; c_t, x_t, \ell_t, k_t \geq 0 .\end{gathered}$$

- $\hat{r}_t^k = \alpha \theta \hat{k}_t^{\alpha-1} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$
  $\hat{w}_t = (1-\alpha) \theta \hat{k}_t^{\alpha} \hat{\ell}_t^{-\alpha}$, $t = 0,1,\ldots$.

- $\hat{c}_t + \hat{k}_{t+1} = \theta \hat{k}_t^{\alpha} \hat{\ell}_t^{1-\alpha}$, $t = 0,1,\ldots$

- $\hat{b}_t = 0$, $t = 0,1,\ldots$