

$$(3) \quad \lim_{t \rightarrow \infty} \beta^t F_x(x_t^*, x_{t+1}^*) \cdot x_t^* = 0.$$

This condition has the following interpretation. Since the vector of derivatives F_x is the vector of marginal returns from increases in the current state variables, the inner product $F_x \cdot x$ is a kind of total value in period t of the vector of state variables. For example, in the many-sector growth model, F_x is the vector of capital goods prices. In this case (3) requires that the present discounted value of the capital stock in period t , evaluated using period t market prices, tends to zero as t tends to infinity. Whether or not one finds these market interpretations helpful, we have the following result.

THEOREM 4.15 (*Sufficiency of the Euler and transversality conditions*) Let $X \subset \mathbf{R}_+^l$, and let F satisfy Assumptions 4.3–4.5, 4.7, and 4.9. Then the sequence $\{x_{t+1}^*\}_{t=0}^\infty$, with $x_{t+1}^* \in \text{int } \Gamma(x_t^*)$, $t = 0, 1, \dots$, is optimal for the problem (SP), given x_0 , if it satisfies (2) and (3).

Proof. Let x_0 be given; let $\{x_t^*\} \in \Pi(x_0)$ satisfy (2) and (3); and let $\{x_t\} \in \Pi(x_0)$ be any feasible sequence. It is sufficient to show that the difference, call it D , between the objective function in (SP) evaluated at $\{x_t^*\}$ and at $\{x_t\}$ is nonnegative.

Since F is continuous, concave, and differentiable (Assumptions 4.4, 4.7, and 4.9),

$$\begin{aligned} D &= \lim_{T \rightarrow \infty} \sum_{i=0}^T \beta^i [F(x_i^*, x_{i+1}^*) - F(x_i, x_{i+1})] \\ &\geq \lim_{T \rightarrow \infty} \sum_{i=0}^T \beta^i [F_x(x_i^*, x_{i+1}^*) \cdot (x_i^* - x_i) + F_y(x_i^*, x_{i+1}^*) \cdot (x_{i+1}^* - x_{i+1})]. \end{aligned}$$

Since $x_0^* - x_0 = 0$, rearranging terms gives

$$\begin{aligned} D &\geq \lim_{T \rightarrow \infty} \left\{ \sum_{i=0}^{T-1} \beta^i [F_y(x_i^*, x_{i+1}^*) + \beta F_x(x_{i+1}^*, x_{i+2}^*)] \cdot (x_{i+1}^* - x_{i+1}) \right. \\ &\quad \left. + \beta^T F_y(x_T^*, x_{T+1}^*) \cdot (x_{T+1}^* - x_{T+1}) \right\}. \end{aligned}$$

Since $\{x_i^*\}$ satisfies (2), the terms in the summation are all zero. Therefore, substituting from (2) into the last term as well and then using (3) gives

$$\begin{aligned} D &\geq - \lim_{T \rightarrow \infty} \beta^T F_x(x_T^*, x_{T+1}^*) \cdot (x_T^* - x_T) \\ &\geq - \lim_{T \rightarrow \infty} \beta^T F_x(x_T^*, x_{T+1}^*) \cdot x_T^*, \end{aligned}$$

where the last line uses the fact that $F_x \geq 0$ (Assumption 4.5) and $x_t \geq 0$, all t . It then follows from (3) that $D \geq 0$, establishing the desired result. ■