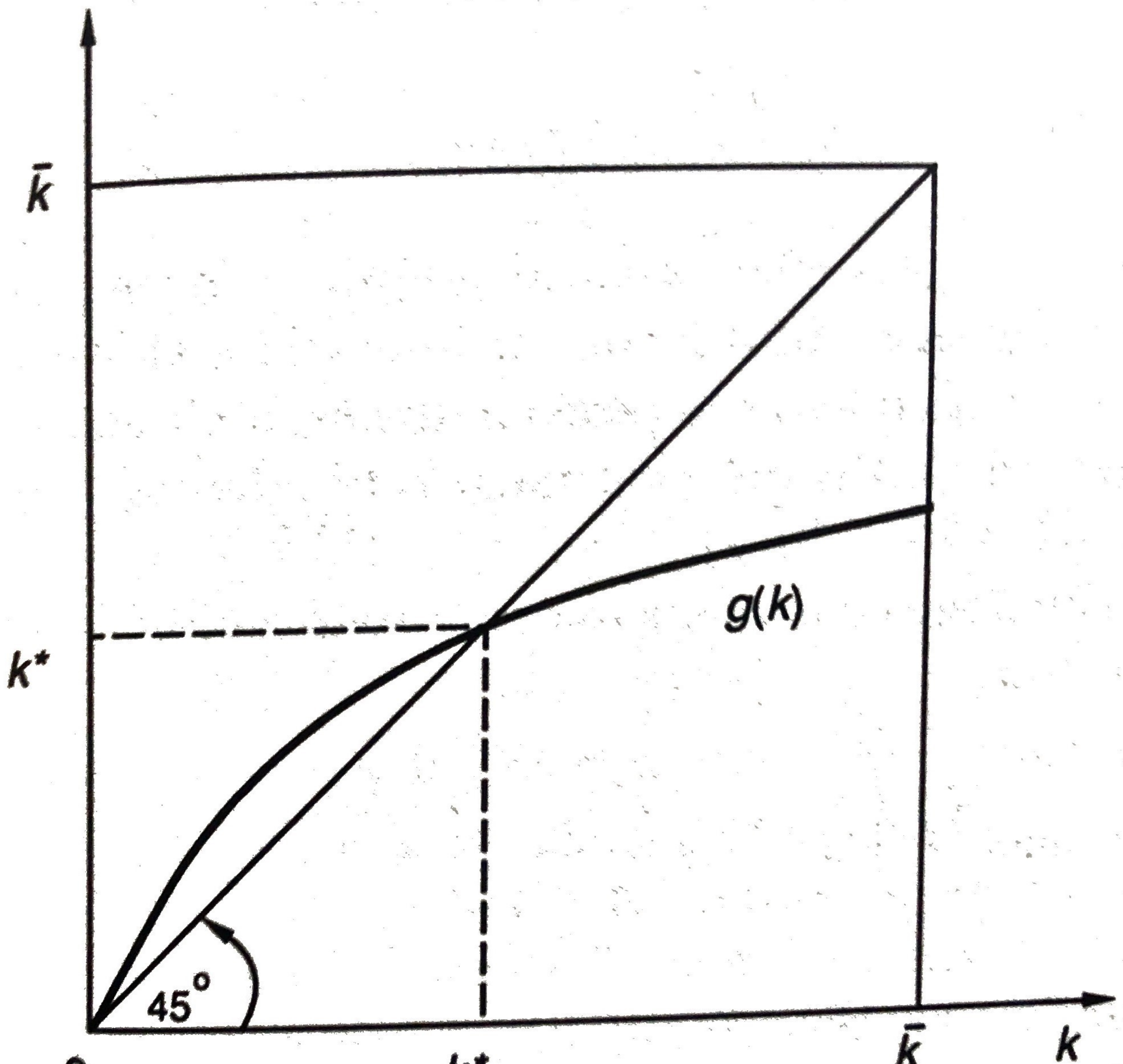

$\bar{k}$
k^*
$g(k)$
45°
0
k^*
$\bar{k}$
k

$$\lim_{t\to\infty} \beta^t F_x(x_t^*, x_{t+1}^*) \cdot x_t^* = 0. \tag{3}$$

This condition has the following interpretation. Since the vector of derivatives F_x is the vector of marginal returns from increases in the current state variables, the inner product $F_x \cdot x$ is a kind of total value in period t of the vector of state variables. For example, in the many-sector growth model, F_x is the vector of capital goods prices. In this case (3) requires that the present discounted value of the capital stock in period t, evaluated using period t market prices, tends to zero as t tends to infinity. Whether or not one finds these market interpretations helpful, we have the following result.

THEOREM 4.15 (*Sufficiency of the Euler and transversality conditions*) *Let $X \subset \mathbf{R}^l_+$, and let F satisfy Assumptions 4.3–4.5, 4.7, and 4.9. Then the sequence $\{x_{t+1}^*\}_{t=0}^{\infty}$, with $x_{t+1}^* \in \text{int}\, \Gamma(x_t^*)$, $t = 0, 1, \ldots$, is optimal for the problem* (SP), *given x_0, if it satisfies* (2) *and* (3).

Proof. Let x_0 be given; let $\{x_t^*\} \in \Pi(x_0)$ satisfy (2) and (3); and let $\{x_t\} \in \Pi(x_0)$ be any feasible sequence. It is sufficient to show that the difference, call it D, between the objective function in (SP) evaluated at $\{x_t^*\}$ and at $\{x_t\}$ is nonnegative.

Since F is continuous, concave, and differentiable (Assumptions 4.4, 4.7, and 4.9),

$$D = \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F(x_t^*, x_{t+1}^*) - F(x_t, x_{t+1})]$$

$$\geq \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [F_x(x_t^*, x_{t+1}^*) \cdot (x_t^* - x_t) + F_y(x_t^*, x_{t+1}^*) \cdot (x_{t+1}^* - x_{t+1})].$$

Since $x_0^* - x_0 = 0$, rearranging terms gives

$$D \geq \lim_{T\to\infty} \left\{ \sum_{t=0}^{T-1} \beta^t [F_y(x_t^*, x_{t+1}^*) + \beta F_x(x_{t+1}^*, x_{t+2}^*)] \cdot (x_{t+1}^* - x_{t+1}) \right.$$
$$\left. + \beta^T F_y(x_T^*, x_{T+1}^*) \cdot (x_{T+1}^* - x_{T+1}) \right\}.$$

Since $\{x_t^*\}$ satisfies (2), the terms in the summation are all zero. Therefore, substituting from (2) into the last term as well and then using (3) gives

$$D \geq - \lim_{T\to\infty} \beta^T F_x(x_T^*, x_{T+1}^*) \cdot (x_T^* - x_T)$$

$$\geq - \lim_{T\to\infty} \beta^T F_x(x_T^*, x_{T+1}^*) \cdot x_T^*,$$

where the last line uses the fact that $F_x \geq 0$ (Assumption 4.5) and $x_t \geq 0$, all t. It then follows from (3) that $D \geq 0$, establishing the desired result. ■