Bellman equation:

$$
\begin{aligned}
V(k) &= \max \; u(c) + \beta V(k') \\
\text{s.t.} \quad c + k' - (1-\delta)k &\le f(k) \\
c, k' &\ge 0.
\end{aligned}
$$

Value function iteration:

$$T(V)(k) = \max \; u(c) + \beta V(k')$$
$$\text{s.t.} \quad c + k' - (1-\delta)k \le f(k)$$
$$c, k' \ge 0.$$

Start with a simple guess for V, for example, V_0 such that

$$V_0(k) = 0 \text{ for all } k \in K.$$

Let $V_1 = T(V_0)$ and continue to iterate $V_{n+1} = T(V_n)$. We will prove that the sequence $V_0, V_1, V_2, \dots$ converges to a function $\hat{V}$ such that $\hat{V} = T(\hat{V})$. Consequently, $\hat{V}$ satisfies the Bellman equation.

Let K be the space of possible capital stocks, and let $C(K)$ be the space of continuous, bonded functions defined on K.

For $V,W \in C(K)$, let

$$d(V,W) = \|V - W\| = \sup_{k \in K} |V(k) - W(k)|.$$

With this definition of a metric, $C(K)$ is a Banach space, a complete normed vector space.

Let

$$T : C(K) \to C(K).$$

Suppose that for any $V, W \in C(K)$,

$$\|T(V) - T(W)\| \leq \gamma \|V - W\|$$

for some fixed γ, $1 > \gamma > 0$.

Then we call T a contraction mapping with modulus γ.

We want to show that mapping T defined by

$$
\begin{aligned}
T(V)(k) = \max\ & u(c) + \beta V(k') \\
\text{s.t.}\quad & c + k' - (1-\delta)k \le f(k) \\
& c, k' \ge 0
\end{aligned}
$$

maps continuous bounded functions into continuous bounded functions, that is,

$$T : C(K) \to C(K)$$

and that T is a contraction mapping with modulus β.

Then

$$\left\|V_{n+2} - V_{n+1}\right\| = \left\|T(V_{n+1}) - T(V_n)\right\| \leq \beta \left\|V_{n+1} - V_n\right\|$$

$$\left\|V_{n+2} - V_{n+1}\right\| \leq \beta^{n+1} \left\|V_1 - V_0\right\|.$$