

PROBLEM SET #2

1. Consider an economy in which the consumption space is the set of functions  $c : R_+ \times R_+ \rightarrow R_+$ . In  $c(x,t)$  the index  $x$  denotes the type of good and the index  $t$  denotes the date at which it is consumed. An individual consumer has preferences given by the functional

$$u(c) = \int_0^\infty e^{-\rho t} \left[ \int_0^\infty \log(c(x,t) + 1) dx \right] dt.$$

Goods are produced using a single factor of production, labor:

$$y(x,t) = \ell(x,t) / a(x,t).$$

Each consumer has an endowment of labor equal to 1, and the total number of consumers is fixed at  $\bar{\ell}$ . The unit labor requirement  $a(x,t)$  is bounded from below,  $a(x,t) > \bar{a}(x)$ , where

$$\bar{a}(x) = e^{-x}.$$

At  $t = 0$  there is a  $z(0) > 0$  such that  $a(x,0) = e^{-x}$  for all  $x < z(0)$  and that  $a(x,0) = e^{x-2z(0)}$  for all  $x \geq z(0)$ . There is learning by doing of the form

$$\frac{\dot{a}(x,t)}{a(x,t)} = \begin{cases} -\int_0^\infty b(v,t) \ell(v,t) dv & \text{if } a(x,t) > \bar{a}(x) \\ 0 & \text{if } a(x,t) = \bar{a}(x) \end{cases}.$$

Here  $\dot{a}(x,t)$  denotes the partial derivative of  $a(x,t)$  with respect to  $t$  and  $b(v,t) = b > 0$  if  $a(v,t) > \bar{a}(v)$  and  $b(v,t) = 0$  if  $a(v,t) = \bar{a}(v)$ . There is no borrowing or lending, and there is no storage.

- a) Provide a motivation for the production technology described above.
- b) Define an equilibrium for this economy. Characterize the equilibrium as much as possible.
- c) Consider now a two country world in which the two countries are identical except in their endowments of labor and their initial technology levels. In particular,  $z^1(0) > z^2(0)$ . Define an equilibrium for this economy.
- d) Describe the environment of a static Ricardian model whose equilibrium has the same values of prices and quantities as  $p(x,0)$ ,  $w^1(0)$ ,  $w^2(0)$ ,  $y^1(x,0)$ ,  $y^2(x,0)$ ,  $c^1(x,0)$ ,  $c^2(x,0)$  in the economy of part c. Illustrate and explain the (five) different possibilities

for patterns of production and consumption in this model. (To make things easy assume that  $z^1(0)$  and  $z^2(0)$  are sufficiently large so that good  $x = 0$  is not produced in equilibrium.)

e) Describe the dynamics of the model, explaining the crucial role played by the sizes of the two countries,  $\bar{\ell}^1$  and  $\bar{\ell}^2$ .

2. Consider an economy with two goods that enter both consumption and investment. The utility function of the representative consumer is

$$\sum_{t=0}^{\infty} \beta^t \log(c_{1t}^{a_1} c_{2t}^{a_2}).$$

Here  $0 < \beta < 1$ ,  $a_1 \geq 0$ ,  $a_2 \geq 0$ , and  $a_1 + a_2 = 1$ . Investment goods are produced according to

$$k_{t+1} - (1 - \delta)k_t = dx_{1t}^{a_1} x_{2t}^{a_2}.$$

Feasible consumption/investment plans satisfy the feasibility conditions

$$c_{1t} + x_{1t} = \phi_1(k_{1t}, \ell_{1t}) = k_{1t}$$

$$c_{2t} + x_{2t} = \phi_2(k_{2t}, \ell_{2t}) = \ell_{2t}$$

where

$$k_{1t} + k_{2t} = k_t$$

$$\ell_{1t} + \ell_{2t} = \ell_t.$$

The initial endowment of  $k_t$  is  $\bar{k}_0$ .  $\ell_t$  is equal to 1. (In other words, all variables are expressed in per capita terms.)

a) Carefully define a competitive equilibrium for this economy.

b) Reduce the equilibrium conditions to two difference equations in  $k_t$  and  $c_t$  and a transversality condition. Here  $c_t = dc_{1t}^{a_1} c_{2t}^{a_2}$  is aggregate consumption. [Here is one possible approach: Prove a version of the first and second welfare theorems for this economy. Show that the two-sector social planner's problem is equivalent to a one-sector social planner's problem. Derive the difference equations and transversality conditions from the one-sector social planner's problem.]

c) Suppose now that there is a world composed of  $n$  different countries, all with the same preferences and technologies, but with different initial endowments of capital per worker,  $\bar{k}_0^i$ . The countries also have different population sizes,  $L^i$ , which are constant

over time. (In other words, there is a continuum of identical consumers/workers of measure  $L^i$  in country  $i$ .) Suppose that there is no international borrowing or lending and no and no international capital flows. Define an equilibrium for this world economy.

Prove that in this equilibrium the variables  $c_{jt} = \sum_{i=1}^n L^i c_{jt}^i / \sum_{j=1}^n L^j$ ,

$k_t = \sum_{i=1}^n L^i k_t^i / \sum_{i=1}^n L^i$ ,  $p_{it}$ ,  $r_t$ , and  $w_t$  satisfy the equilibrium conditions of the economy in part a when  $\bar{k}_0 = \sum_{i=1}^n L^i \bar{k}_0^i / \sum_{i=1}^n L^i$ .

d) State and prove versions of the factor price equalization theorem, the Stolper-Samuelson theorem, the Rybczynski theorem, and the Heckscher-Ohlin theorem for this particular world economy.

e) What effects do the assumptions of no international borrowing and lending and no international capital flows have on your analysis? Try to be precise about which variables are uniquely determined in the equilibrium with borrowing and lending and/or capital flows are which would not be uniquely determined.

f) Consider the case where  $\delta = 1$ . Set  $z_0 = c_0 / (\beta r_0 k_0)$  and  $z_t = c_{t-1} / k_t$ ,  $t = 1, 2, \dots$ . Transform the two difference equations in part b into two difference equations in  $k_t$  and  $z_t$ . Prove that

$$\frac{k_t^i - k_t}{k_t} = \frac{z_t}{z_{t-1}} \left( \frac{k_{t-1}^i - k_{t-1}}{k_{t-1}} \right) = \frac{z_t}{z_0} \left( \frac{\bar{k}_0^i - \bar{k}_0}{\bar{k}_0} \right).$$

g) Consider again the case where  $\delta = 1$ . Let  $s_t = c_t / y_t$  where

$$y_t = p_{1t} k_t + p_{2t} = dk_t^{\alpha_1} = r_t k_t + w_t.$$

Transform the two difference equations in part b into two difference equations in  $k_t$  and  $s_t$ . Prove that

$$\frac{y_t^i - y_t}{y_t} = \frac{s_t}{s_{t-1}} \left( \frac{y_{t-1}^i - y_{t-1}}{y_{t-1}} \right) = \frac{s_t}{s_0} \left( \frac{y_0^i - y_0}{y_0} \right)$$

where  $y_t^i = p_{1t} y_{1t}^i + p_{2t} y_{2t}^i = r_t k_t^i + w_t$ . Calculate an expression for  $s_t$  and discuss the significance of this result.

3. Suppose again that  $\delta = 1$ , that  $c_t = dc_{1t}^{\alpha_1} c_{2t}^{\alpha_2}$  and that  $k_{t+1} = dx_{1t}^{\alpha_1} x_{2t}^{\alpha_2}$ . Now suppose that

$$c_{1t} + x_{1t} = \phi_1(k_{1t}, \ell_{1t}) = \theta_1 k_{1t}^{\alpha_1} \ell_{1t}^{1-\alpha_1}$$

$$c_{2t} + x_{2t} = \phi_2(k_{2t}, \ell_{2t}) = \theta_2 k_{2t}^{\alpha_2} \ell_{2t}^{1-\alpha_2}.$$

a) Let  $F(k, \ell)$  be the maximum value of

$$\begin{aligned} \max \quad & dy_{1t}^{\alpha_1} y_{2t}^{\alpha_2} \\ \text{s.t.} \quad & y_1 = \theta_1 k_1^{\alpha_1} \ell_1^{1-\alpha_1} \\ & y_2 = \theta_2 k_2^{\alpha_2} \ell_2^{1-\alpha_2} \\ & k_1 + k_2 = k \\ & \ell_1 + \ell_2 = \ell \\ & k_j \geq 0, \ell_j \geq 0. \end{aligned}$$

Show that  $F(k, \ell)$  has the form  $Dk^A \ell^{A_2}$ . Derive formulas for  $D$  and  $A_j$ .

b) Suppose now that there is a world made up of  $n$  different countries all with the same technologies and preferences, but different initial endowments of capital,  $\bar{k}_0^i$ , and different population sizes,  $L^i$ . Suppose that there is no international borrowing or lending. Define an equilibrium for the world economy.

c) What effects do the assumptions of no international borrowing and lending and no international capital flows have on your analysis in this case?

d) Using the answers to parts a and b, show that necessary and sufficient conditions for the integrated equilibrium approach to work for all  $t = T, T+1, \dots$ , is that

$$\kappa_1 k_t \geq k_t^i \geq \kappa_2 k_t \text{ for all } i = 1, \dots, n \text{ and all } t = T, T+1, \dots$$

For some  $\kappa_1, \kappa_2 > 0$ . [Hint: You should calculate  $\kappa_1, \kappa_2$ .]

e) Consider a model with two countries. Assign numerical values to the parameters  $\beta$ ,  $d$ ,  $a_1$ ,  $a_2$ ,  $\theta_1$ ,  $\alpha_1$ ,  $\theta_2$ ,  $\alpha_2$ . Choose values of the country sizes  $L^i$  and the initial capital-labor ratios  $\bar{k}_0^i$  so that

$$\kappa_1 \bar{k}_0 \geq \bar{k}_0^i \geq \kappa_2 \bar{k}_0 \text{ for } i = 1, 2.$$

Use the answers to parts a, b, and c and the answer to parts f and g of question 2 to calculate analytical expressions for the equilibrium values of the variables in part b for all  $t = 0, 1, \dots$ . Use an MS Excel spreadsheet or some other means to calculate the numerical values of these variables for the first 10 periods.

4. Consider now a generalization of the model in questions 2 and 3: The representative consumer has the utility function

$$\sum_{t=0}^{\infty} \beta^t \log(a_1 c_{1t}^b + a_2 c_{2t}^b)^{1/b},$$

and investment good is produced according to

$$k_{t+1} = d(a_1 x_{1t}^b + a_2 x_{2t}^b)^{1/b}.$$

Here  $b < 1$  and there is complete depreciation,  $\delta = 1$ . Again, feasible consumption/investment plans satisfy the feasibility constraints

$$c_{1t} + x_{1t} = \phi_1(k_{1t}, \ell_{1t}) = k_{1t}$$

$$c_{2t} + x_{2t} = \phi_2(k_{2t}, \ell_{2t}) = \ell_{2t}$$

where

$$k_{1t} + k_{2t} = k_t$$

$$\ell_{1t} + \ell_{2t} = \ell_t.$$

Explain how the analysis of questions 2 and 3 can be generalized. For the generalization of question 2, you should again assume that

$$\phi_1(k_{1t}, \ell_{1t}) = k_{1t}$$

$$\phi_2(k_{2t}, \ell_{2t}) = \ell_{2t}.$$

For the generalization of question 3, you should assume that

$$\phi_1(k_{1t}, \ell_{1t}) = \theta_1 \left( \alpha_1 k_{1t}^b + (1 - \alpha_1) \ell_{1t}^b \right)^{1/b}$$

$$\phi_2(k_{2t}, \ell_{2t}) = \theta_2 \left( \alpha_2 k_{2t}^b + (1 - \alpha_2) \ell_{2t}^b \right)^{1/b}.$$

Here  $b$  is the same parameter as in the utility function and the production function for investment goods. You do not need to go through the calculus and algebra of explicitly proving anything, but you should carefully define equilibrium concepts and you should carefully state all results. Discuss your generalizations of part g of question 2 and part e of question 3.

5. Consider a version of the model in question 4 with two countries. Assign numerical values to the parameters  $\beta$ ,  $d$ ,  $a_1$ ,  $a_2$ ,  $b$ ,  $\theta_1$ ,  $\alpha_1$ ,  $\theta_2$ ,  $\alpha_2$  so that the equilibrium of the model displays sustained growth. Calculate the analogues of the parameters  $\kappa_1$  and  $\kappa_2$  for this economy. Choose  $L^i$  and the initial capital-labor ratios  $\bar{k}_0^i$  so that

$$\kappa_1 \bar{k}_0 \geq \bar{k}_0^i \geq \kappa_2 \bar{k}_0 \text{ for } i = 1, 2.$$

Compare the initial distributions of capital and income with the asymptotic distributions.