INTERNATIONAL TRADE AND PAYMENTS THEORY **T. J. KEHOE**
ECON 8401 **FALL 2005**

PROBLEM SET #3

1. Consider an economy with two goods that enter both consumption and investment. The utility function of the representative consumer is

$$\sum_{t=0}^{\infty} \beta^t \log(c_{1t}^{a_1} c_{2t}^{a_2}) .$$

Here $0 < \beta < 1$, $a_1 \geq 0$, $a_2 \geq 0$, and $a_1 + a_2 = 1$. Investment goods are produced according to

$$k_{t+1} - (1-\delta)k_t = dx_{1t}^{a_1} x_{2t}^{a_2} .$$

Feasible consumption/investment plans satisfy the feasibility conditions

$$c_{1t} + x_{1t} = \phi_1(k_{1t}, \ell_{1t}) = k_{1t}$$
$$c_{2t} + x_{2t} = \phi_2(k_{2t}, \ell_{2t}) = \ell_{2t}$$

where

$$k_{1t} + k_{2t} = k_t$$
$$\ell_{1t} + \ell_{2t} = \ell_t .$$

The initial endowment of k_t is $\bar{k}_0$. ℓ_t is equal to 1. (In other words, all variables are expressed in per capita terms.)

a) Carefully define a competitive equilibrium for this economy.

b) Reduce the equilibrium conditions to two difference equations in k_t and c_t and a transversality condition. Here $c_t = dc_{1t}^{a_1} c_{2t}^{a_2}$ is aggregate consumption.

c) Suppose now that there is a world composed of m different countries, all with the same preferences and technologies, but with different initial endowments of capital per worker, $\bar{k}_0^j$. The countries also have different population sizes, $\bar{L}^j$, which are constant over time. (In other words, there is a continuum of identical consumers/workers of measure $\bar{L}^j$ in country j.) Suppose that there is no international borrowing or lending and no and no international capital flows. Define an equilibrium for this world economy. Prove that in this equilibrium the variables $c_{it} = \sum_{j=1}^{m} \bar{L}^j c_{it}^j \Big/ \sum_{j=1}^{m} \bar{L}^j$,

$k_t = \sum_{j=1}^{m} \overline{L}^j k_t^j \Big/ \sum_{j=1}^{m} \overline{L}^j$, p_{it} , r_t , and w_t satisfy the equilibrium conditions of the economy in part a when $\overline{k}_0 = \sum_{j=1}^{m} \overline{L}^j \overline{k}_0^j \Big/ \sum_{j=1}^{m} \overline{L}^j$.

d) State and prove versions of the factor price equalization theorem, the Stolper-Samuelson theorem, the Rybczynski theorem, and the Heckscher-Ohlin theorem for this particular world economy.

e) What effects do the assumptions of no international borrowing and lending and no international capital flows have on your analysis? Try to be precise about which variables are uniquely determined in the equilibrium with borrowing and lending and/or capital flows are which would not be uniquely determined.

f) Consider the case where $\delta = 1$. Set $z_0 = c_0 / (\beta r_0 k_0)$ and $z_t = c_{t-1} / k_t$, $t = 1, 2, \ldots$. Transform the two difference equations in part b into two difference equations in k_t and z_t . Prove that

$$\frac{k_t^i - k_t}{k_t} = \frac{z_t}{z_{t-1}} \left(\frac{k_{t-1}^i - k_{t-1}}{k_{t-1}} \right) = \frac{z_t}{z_0} \left(\frac{\overline{k}_0^i - \overline{k}_0}{\overline{k}_0} \right).$$

g) Consider again the case where $\delta = 1$. Let $s_t = c_t / y_t$ where

$$y_t = p_{1t} k_t + p_{2t} = d k_t^{a_1} = r_t k_t + w_t .$$

Transform the two difference equations in part b into two difference equations in k_t and s_t . Prove that

$$\frac{y_t^i - y_t}{y_t} = \frac{s_t}{s_{t-1}} \left(\frac{y_{t-1}^i - y_{t-1}}{y_{t-1}} \right) = \frac{s_t}{s_0} \left(\frac{y_0^i - y_0}{y_0} \right)$$

where $y_t^i = p_{1t} y_{1t}^i + p_{2t} y_{2t}^i = r_t k_t^i + w_t$. Calculate an expression for s_t and discuss the significance of this result.

2. Suppose again that $\delta = 1$, that $c_t = d c_{1t}^{a_1} c_{2t}^{a_2}$ and that $k_{t+1} = d x_{1t}^{a_1} x_{2t}^{a_2}$. Now suppose that

$$c_{1t} + x_{1t} = \phi_1(k_{1t}, \ell_{1t}) = \theta_1 k_{1t}^{\alpha_1} \ell_{1t}^{1-\alpha_1}$$
$$c_{2t} + x_{2t} = \phi_2(k_{2t}, \ell_{2t}) = \theta_2 k_{2t}^{\alpha_2} \ell_{2t}^{1-\alpha_2} .$$

a) Let $F(k, \ell)$ be the maximum value of

$$
\begin{aligned}
&\max \ d y_{1t}^{a_1} y_{2t}^{a_2} \\
&\text{s.t. } \ y_1 = \theta_1 k_1^{\alpha_1} \ell_1^{1-\alpha_1} \\
&\qquad y_2 = \theta_2 k_2^{\alpha_2} \ell_2^{1-\alpha_2} \\
&\qquad k_1 + k_2 = k \\
&\qquad \ell_1 + \ell_2 = \ell \\
&\qquad k_j, \ell_j \geq 0.
\end{aligned}
$$

Show that $F(k,\ell)$ has the form $Dk^A \ell^{1-A}$.

b) Suppose now that there is a world made up of m different countries all with the same technologies and preferences, but different initial endowments of capital, $\bar{k}_0^j$, and different population sizes, $\bar{L}^j$. Suppose that there is no international borrowing or lending. Define an equilibrium for the world economy.

c) What effects do the assumptions of no international borrowing and lending and no international capital flows have on your analysis in this case?

d) Using the answers to parts a and b, show that necessary and sufficient conditions for the integrated equilibrium approach to work for all $t = T, T+1, \ldots$, is that

$$\kappa_1 k_t \geq k_t^i \geq \kappa_2 k_t \ \text{ for all } i = 1, \ldots, m \text{ and all } t = T, T+1, \ldots.$$

For some $\kappa_1, \kappa_2 > 0$. [Hint: You should calculate κ_1, κ_2.]

e) Suppose that, in some period T,

$$\kappa_1 k_T \geq k_T^i \geq \kappa_2 k_T \ \text{ for all } i = 1, \ldots, m.$$

Use the answers to parts a, b, and c and the answer to parts f and g of question 1 to calculate analytical expressions for the equilibrium values of the variables in part b for all $t = T, T+1, \ldots$. [Hint: You can show that $\kappa_1 k_t \geq k_t^i \geq \kappa_2 k_t$ for all $i = 1, \ldots, m$ and all $t = T, T+1, \ldots$.]

3. Consider now a generalization of the model in questions 1 and 2: The representative consumer has the utility function

$$\sum_{t=0}^{\infty} \beta^t \log(a_1 c_{1t}^b + a_2 c_{2t}^b)^{1/b},$$

and investment good is produced according to

$$k_{t+1} - (1-\delta)k_t = d(a_1 x_{1t}^b + a_2 x_{2t}^b)^{1/b}.$$

Here $b < 1$. Again, feasible consumption/investment plans satisfy the feasibility constraints

$$c_{1t} + x_{1t} = \phi_1(k_{1t}, \ell_{1t})$$
$$c_{2t} + x_{2t} = \phi_2(k_{2t}, \ell_{2t})$$

where

$$k_{1t} + k_{2t} = k_t$$
$$\ell_{1t} + \ell_{2t} = \ell_t \ .$$

Explain how the analysis of questions 1 and 2 can be generalized. For the generalization of question 1, you should again assume that

$$\phi_1(k_{1t}, \ell_{1t}) = k_{1t}$$
$$\phi_2(k_{2t}, \ell_{2t}) = \ell_{2t} \ .$$

For the generalization of question 2, you should again assume that

$$\phi_1(k_{1t}, \ell_{1t}) = \theta_1 \left(\alpha_1 k_{1t}^b + (1-\alpha_1)\ell_{1t}^b \right)^{1/b}$$
$$\phi_2(k_{2t}, \ell_{2t}) = \theta_2 \left(\alpha_2 k_{2t}^b + (1-\alpha_2)\ell_{2t}^b \right)^{1/b}.$$

Here b is the same parameter as in the utility function and the production function for investment goods. You do not need to go through the calculus and algebra of explicitly proving anything, but you should carefully define equilibrium concepts and you should carefully state all results. Discuss your generalizations of part g of question 1 and part e of question 2.