Blackwell's sufficient conditions

Let $K \subset R^n$ and $B(K)$ be the set of bounded functions $V : K \to R$. Suppose that the operator $T : B(K) \to B(K)$ satisfies

(monotonicity) If $W(k) \geq V(k)$ for all $k \in K$ where $V, W \in B(K)$, then $T(W)(k) \geq T(V)(k)$.

(discounting) There exists γ, $1 > \gamma > 0$, such that

$$T(V + a)(k) \leq T(V)(k) + \gamma a \text{ for all } V \in B(K),\ a \geq 0, \text{ and } k \in K.$$

[Here $(V + a)(k) = V(k) + a$.]

Then T is a contraction with modulus γ.

Proof:

For all $V, W \in B(K)$,

$$V(k) \leq W(k) + \sup_{k \in K} \left| V(k) - W(k) \right| = W(k) + \left\| V - W \right\| \text{ for all } k \in K.$$

Monotonicity implies that

$$T(W)(k) \leq T(V + \left\| V - W \right\|)(k) \text{ for all } k \in K,$$

and discounting implies that

$$T(W)(k) \leq T(V)(k) + \gamma \left\| V - W \right\| \text{ for all } k \in K.$$

Similarly,

$$T(V)(k) \leq T(W)(k) + \gamma \left\| V - W \right\| \text{ for all } k \in K.$$

Consequently,

$$\left| T(V)(k) - T(W)(k) \right| \leq \gamma \left\| V - W \right\| \text{ for all } k \in K.$$

$$\sup_{k \in K} \left| T(V)(k) - T(W)(k) \right| \leq \gamma \left\| V - W \right\|$$

$$\left\| T(V) - T(W) \right\| \leq \gamma \left\| V - W \right\|$$

Proof that value function iteration for one-sector growth model is a contraction:

$$T(V)(k) = \max \ u(c) + \beta V(k')$$
$$\text{s.t.} \quad c + k' - (1-\delta)k \le f(k)$$
$$c, k' \ge 0.$$

Monotonicity:

Suppose that $W(k) \ge V(k)$ for all $k \in K$. Then

$$u(f(k) + (1-\delta)k - k') + \beta W(k') \ge u(f(k) + (1-\delta)k - k') + \beta V(k')$$

for all $k \in K$ such that $f(k) + (1-\delta)k \ge k' \ge 0$, which implies that

$$T(W)(k) \geq T(V)(k).$$

Discounting:

$$T(V+a)(k) = \max u(c) + \beta(V(k') + a)$$
$$\text{s.t. } c + k' - (1-\delta)k \leq f(k)$$

$$T(V+a)(k) = T(V)(k) + \beta a.$$

$\gamma = \beta$ is the modulus of the contraction mapping.