

A Version of Kei-Mu Yi's Model

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Environment:

Goods are produced using labor and intermediate goods:

$$\begin{aligned} y_j(z) &= \left[\theta v_j(z)^\eta + (1-\theta) \ell_j(z)^\eta \right]^{\frac{1}{\eta}} / a_j(z) \\ &= v_j(z)^\theta \ell_j(z)^{1-\theta} / a_j(z) \quad \text{if } \eta = 0. \end{aligned}$$

where

$$\begin{aligned} a_1(z) &= e^{\alpha z} \\ a_2(z) &= e^{\alpha(1-z)}. \end{aligned}$$

Here $y_j(z)$ is the production of good z in country j ; $v_j(z)$ is the input of the aggregate intermediate good; and $\ell_j(z)$ is the input of labor.

The utility function is

$$\left(\int_0^1 c_j(z)^\rho dz - 1 \right) / \rho = \int_0^1 \log c_j(z) dz \quad \text{if } \rho = 0.$$

The budget constraint is

$$\int_0^1 p_j(z) c_j(z) dz = w_j \bar{\ell}_j + T_j,$$

where T_j is the lump-sum rebate of tariff revenue.

The technology for producing intermediate goods is

$$\begin{aligned} v_j &= \left(\int_0^1 x_j(z)^\rho dz \right)^{\frac{1}{\rho}} \\ &= \exp \int_0^1 \log x_j(z) dz \quad \text{if } \rho = 0, \end{aligned}$$

where v_j is the total production of intermediate goods in country j .

Cost minimization:

We calculate some price indices by solving cost minimization problems:

For the aggregate consumption/intermediate good in country j :

$$\begin{aligned} \min & \int_0^1 p_j(z) x_j(z) dz \\ \text{s. t.} & \left(\int_0^1 x_j(z)^\rho dz \right)^{\frac{1}{\rho}} = \bar{x}_j \end{aligned}$$

$$x_j(z) = \frac{\bar{x}_j \left(\int_0^1 p_j(\zeta)^{-\frac{\rho}{1-\rho}} d\zeta \right)^{\frac{1}{\rho}}}{p_j(z)^{\frac{1}{1-\rho}}} = \frac{\bar{x}_j \left(\exp \int_0^1 \log p_j(z) dz \right)}{p_j(z)} \quad \text{if } \rho = 0$$

$$\int_0^1 p_j x_j(z) dz = \left(\int_0^1 p_j(z)^{-\frac{\rho}{1-\rho}} dz \right)^{-\frac{1-\rho}{\rho}} \bar{x}_j = \left(\exp \int_0^1 \log p_j(z) dz \right) \bar{x}_j \quad \text{if } \rho = 0$$

$$\int_0^1 p_j x_j(z) dz = q_j \bar{x}_j$$

$$x_j(z) = \frac{q_j^{\frac{1}{1-\rho}} \bar{x}_j}{p_j(z)^{\frac{1}{1-\rho}}} = \frac{q_j \bar{x}_j}{p_j(z)} \quad \text{if } \rho = 0.$$

Here q_j is the price of the aggregate consumption/intermediate good in country j .

For the good z produced in country j :

$$\min q_j v_j(z) + w_j \ell_j(z)$$

$$\text{s. t. } \left[\theta v_j(z)^\eta + (1-\theta) \ell_j(z)^\eta \right]^{\frac{1}{\eta}} = a_j(z) y_j(z)$$

$$\begin{aligned} q_j(z) y_j(z) = q_j v_j(z) + w_j \ell_j(z) &= a_j(z) y_j(z) \left[\theta^{\frac{1}{1-\eta}} q_j^{-\frac{\eta}{1-\eta}} + (1-\theta)^{\frac{1}{1-\eta}} w_j^{-\frac{\eta}{1-\eta}} \right]^{\frac{1-\eta}{\eta}} \\ &= \frac{a_j(z) y_j(z) q_j^\theta w_j^{1-\theta}}{\theta^\theta (1-\theta)^{1-\theta}} \quad \text{if } \eta = 0 \end{aligned}$$

$$q_j(z) y_j(z) = q_j v_j(z) + w_j \ell_j(z) = a_j(z) \bar{p}_j y_j(z)$$

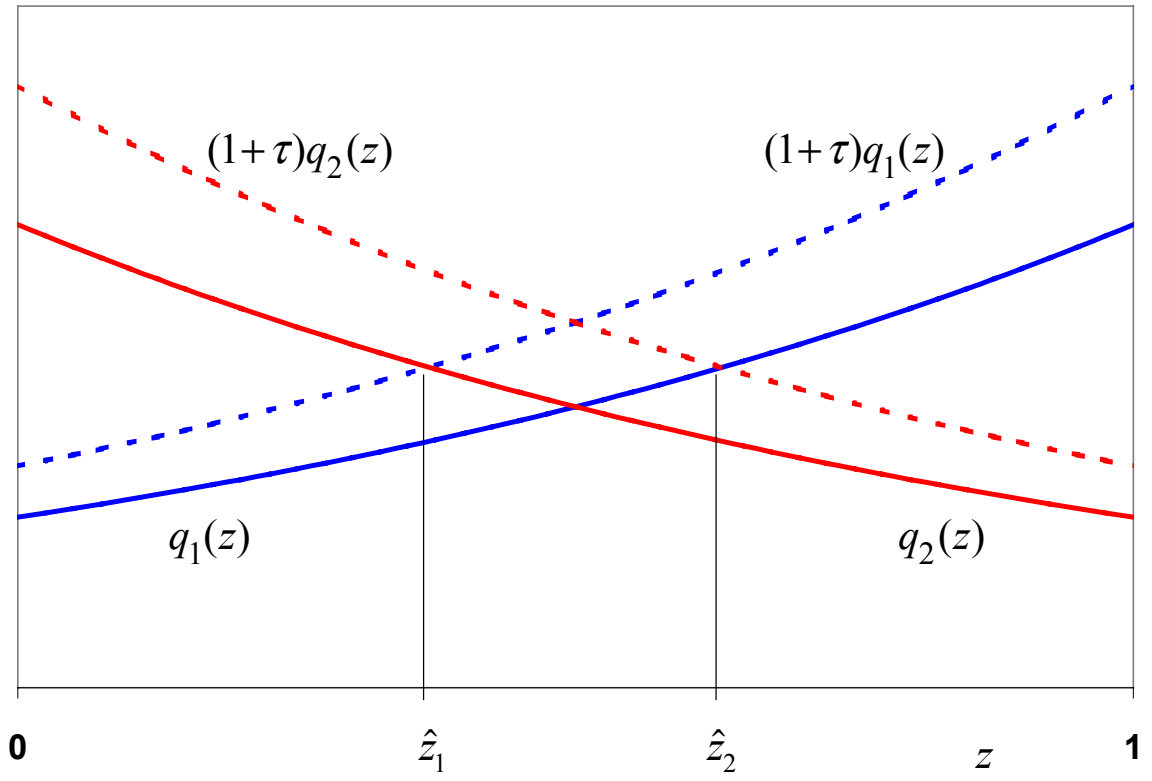
$$q_j(z) = a_j(z) \bar{p}_j$$

where

$$\begin{aligned}
 q_j(z) &= a_j(z) \left[\theta^{\frac{1}{1-\eta}} q_j^{\frac{-\eta}{1-\eta}} + (1-\theta)^{\frac{1}{1-\eta}} w_j^{\frac{-\eta}{1-\eta}} \right]^{\frac{1-\eta}{\eta}} \\
 &= \frac{a_j(z) q_j^\theta w_j^{1-\theta}}{\theta^\theta (1-\theta)^{1-\theta}} \quad \text{if } \eta = 0. \\
 v_j(z) &= \frac{\theta^{\frac{1}{1-\eta}} \bar{p}_j^{\frac{1}{1-\eta}} a_j(z) y_j(z)}{q_j^{\frac{1}{1-\eta}}} = \frac{\theta \bar{p}_j a_j(z) y_j(z)}{q_j} \quad \text{if } \eta = 0. \\
 \ell_j(z) &= \frac{(1-\theta)^{\frac{1}{1-\eta}} \bar{p}_j^{\frac{1}{1-\eta}} a_j(z) y_j(z)}{w_j^{\frac{1}{1-\eta}}} = \frac{(1-\theta) \bar{p}_j a_j(z) y_j(z)}{w_j} \quad \text{if } \eta = 0.
 \end{aligned}$$

Here $q_j(z)$ is the price of good z produced in country j and \bar{p}_j is the price of the aggregate of the intermediate good and labor. Notice that, if $\theta = 0$, $\bar{p}_j = w_j$.

Patterns of production and trade



Symmetric Cobb-Douglas case:

Suppose that $\eta = \rho = 0$. Further suppose that labor endowments and tariffs are equal across countries. Then

$$p_1(z) = \min[q_1(z), (1 + \tau)q_2(z)] = \min\left[\frac{a_1(z)q_1^\theta w_1^{1-\theta}}{\theta^\theta(1-\theta)^{1-\theta}}, (1 + \tau)\frac{a_2(z)q_2^\theta w_2^{1-\theta}}{\theta^\theta(1-\theta)^{1-\theta}}\right].$$

Here $p_1(z)$ is the price paid for good z in country j .

Symmetry implies that

$$\begin{aligned} w_1 &= w_2 = 1 \\ q_1 &= q_2 = q \quad \hat{z}_1 = 1 - \hat{z}_2 \\ (1 + \tau)e^{\alpha\hat{z}_1} &= e^{\alpha(1-\hat{z}_1)} \\ \hat{z}_1 &= \frac{\alpha - \log(1 + \tau)}{2\alpha} \\ \hat{z}_2 &= \frac{\alpha + \log(1 + \tau)}{2\alpha}. \end{aligned}$$

We can calculate equilibrium prices:

$$\begin{aligned} q &= q_1 = \exp\left[\int_0^1 \log p_1(z) dz\right] = \exp\left[\int_0^{\hat{z}_2} \log \bar{p}e^{\alpha z} dz + \int_{\hat{z}_2}^1 \log(1 + \tau)\bar{p}e^{\alpha(1-z)} dz\right] \\ &= \bar{p}(1 + \tau)^{1-\hat{z}_2} e^{\alpha(2\hat{z}_2-1)} = \bar{p}(1 + \tau)^{2-\hat{z}_2} \end{aligned}$$

$$q = \bar{p}(1 + \tau)^{2-\hat{z}_2} = \frac{q^\theta}{\theta^\theta(1-\theta)^{1-\theta}}(1 + \tau)^{2-\hat{z}_2} = \frac{(1 + \tau)^{\frac{2-\hat{z}_2}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}(1-\theta)}.$$

$$\bar{p} = \frac{(1 + \tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}(1-\theta)}$$

$$q_j(z) = a_j(z) \frac{(1 + \tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}{\theta^{\frac{\theta}{1-\theta}}(1-\theta)}$$

$$p_1(z) = \begin{cases} \frac{e^{az} (1+\tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}{\theta^{1-\theta} (1-\theta)} & \text{if } z \leq \hat{z}_2 \\ \frac{(1+\tau)e^{\alpha(1-z)} (1+\tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}{\theta^{1-\theta} (1-\theta)} & \text{if } z \geq \hat{z}_2 \end{cases}$$

$$p_2(z) = \begin{cases} \frac{(1+\tau)e^{az} (1+\tau)^{\frac{(1+\hat{z}_1)\theta}{1-\theta}}}{\theta^{1-\theta} (1-\theta)} & \text{if } z \leq \hat{z}_1 \\ \frac{e^{\alpha(1-z)} (1+\tau)^{\frac{(1+\hat{z}_1)\theta}{1-\theta}}}{\theta^{1-\theta} (1-\theta)} & \text{if } z \geq \hat{z}_1 \end{cases}$$

Let $u_1(z) = a_1(z)y_1(z) = v_1(z)^\theta \ell_1(z)^{1-\theta}$ be the aggregate input of intermediate goods and labor in the production of good z in country 1. We know that

$$\begin{aligned} u_1 &= \int_0^{\hat{z}_1} u_1(z) dz + \int_0^{\hat{z}_2} u_1(z) dz = v_1^\theta \ell_1^{1-\theta} \\ v_1 &= \frac{\theta \bar{p} u_1}{q} = \theta (1+\tau)^{-(2-\hat{z}_2)} u_1 \\ \ell_1 &= (1-\theta) \bar{p} u_1 = \frac{(1+\tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}}{\theta^{1-\theta}} u_1. \end{aligned}$$

Therefore

$$\begin{aligned} u_1 &= \frac{\theta^{\frac{\theta}{1-\theta}}}{(1+\tau)^{\frac{(2-\hat{z}_2)\theta}{1-\theta}}} \bar{\ell} \\ v_1 &= \frac{\theta^{\frac{1}{1-\theta}}}{(1+\tau)^{\frac{2-\hat{z}_2}{1-\theta}}} \bar{\ell}. \end{aligned}$$

The value of exports is

$$\begin{aligned}
 X &= \int_0^{\hat{z}_1} q_1(z) [c_1(z) + x_1(z)] dz \\
 &= \int_0^{\hat{z}_1} q_1(z) \left[\frac{\bar{\ell} + T}{(1 + \tau)q_1(z)} + \frac{qv_1}{(1 + \tau)q_1(z)} \right] dz \\
 &= \int_0^{\hat{z}_1} q_1(z) \left[\frac{\bar{\ell} + T}{(1 + \tau)q_1(z)} + \frac{\theta\bar{\ell}}{(1 - \theta)(1 + \tau)q_1(z)} \right] dz \\
 &= \frac{\hat{z}_1}{(1 + \tau)} \left[\frac{\bar{\ell}}{1 - \theta} + T \right].
 \end{aligned}$$

Tariff revenues are

$$\begin{aligned}
 T &= \tau X = \frac{\tau \hat{z}_1}{(1 + \tau)} \left[\frac{\bar{\ell}}{1 - \theta} + T \right] \\
 T &= \frac{\tau \hat{z}_1 \bar{\ell}}{(1 - \theta)(1 + \tau - \tau \hat{z}_1)} = \frac{\tau [\alpha - \log(1 + \tau)] \bar{\ell}}{(1 - \theta)[2\alpha + \alpha\tau + \tau \log(1 + \tau)]}.
 \end{aligned}$$

The value of exports is

$$X = \frac{[\alpha - \log(1 + \tau)] \bar{\ell}}{(1 - \theta)[2\alpha + \alpha\tau + \tau \log(1 + \tau)]}.$$

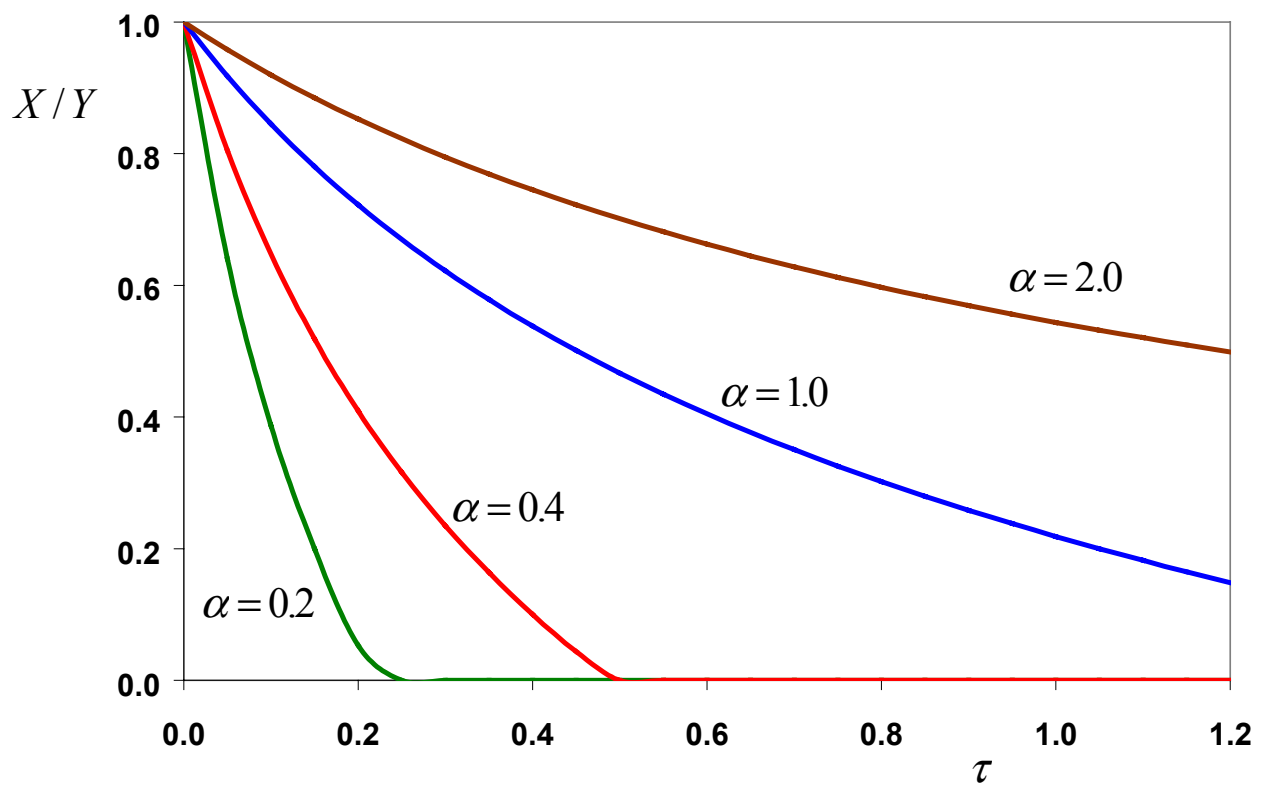
GDP is

$$\begin{aligned}
 Y = \ell + T &= \frac{(1 - \theta)[2\alpha + \alpha\tau + \tau \log(1 + \tau)] \bar{\ell} + \tau [\alpha - \log(1 + \tau)] \bar{\ell}}{(1 - \theta)[2\alpha + \alpha\tau + \tau \log(1 + \tau)]} \\
 &= \frac{[2(1 - \theta)\alpha + (2 - \theta)\alpha\tau - \theta \tau \log(1 + \tau)] \bar{\ell}}{(1 - \theta)[2\alpha + \alpha\tau + \tau \log(1 + \tau)]}.
 \end{aligned}$$

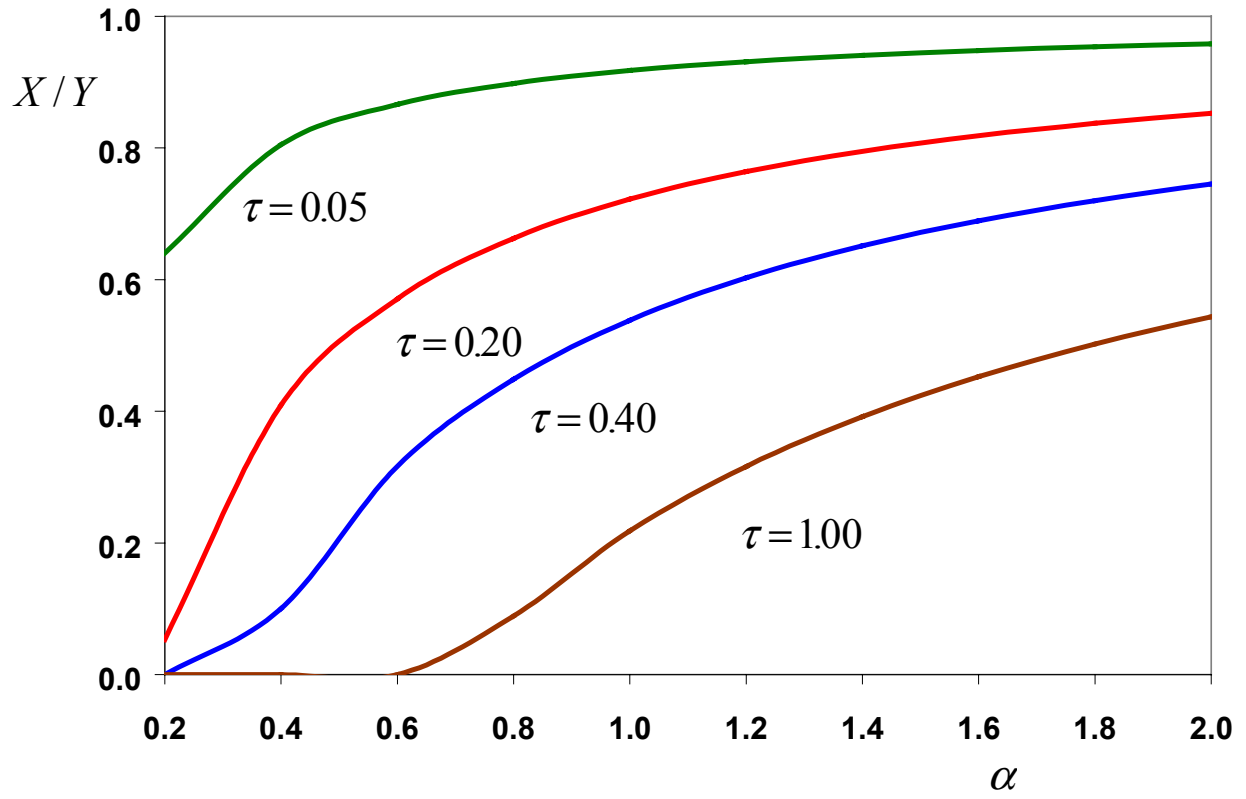
The ratio Exports/GDP is

$$X = \frac{[\alpha - \log(1 + \tau)]}{[2(1 - \theta)\alpha + (2 - \theta)\alpha\tau - \theta \tau \log(1 + \tau)]}.$$

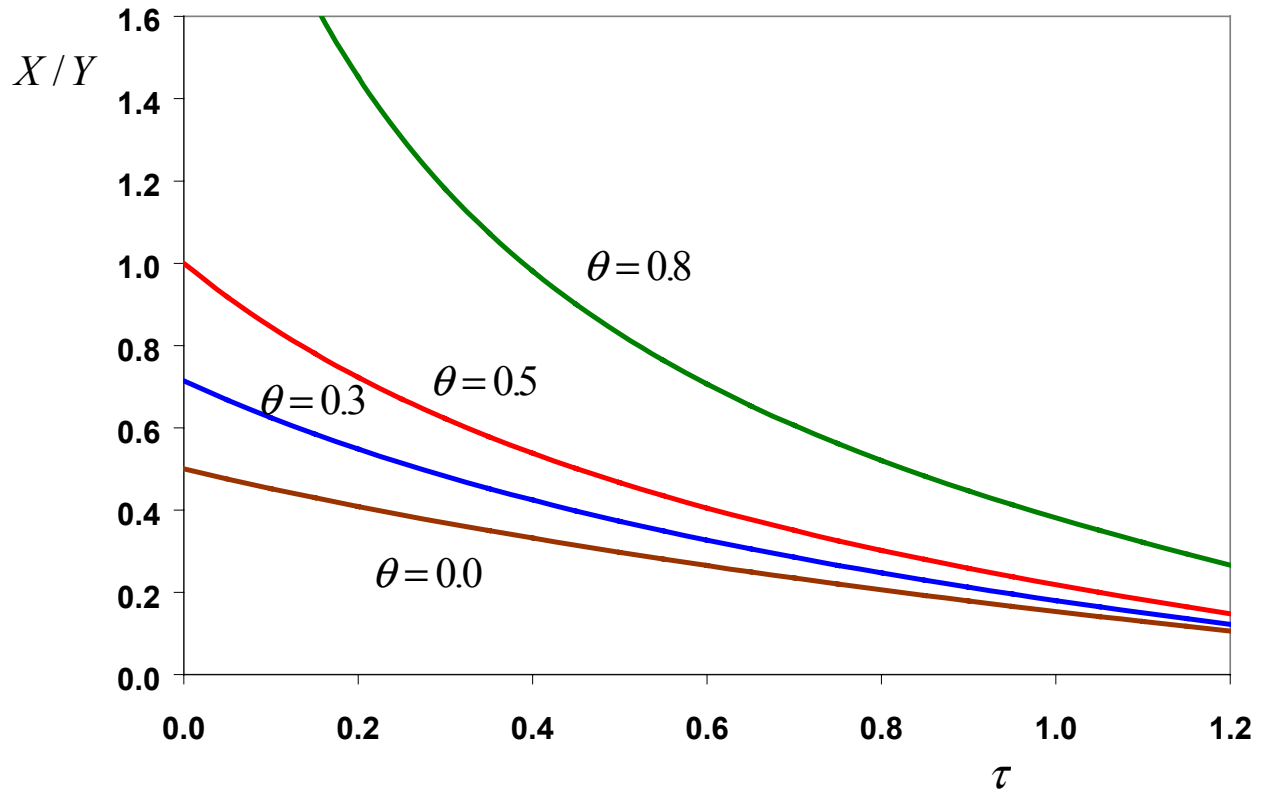
Fraction of world trade ($\theta=0.5$)



Fraction of world trade ($\theta=0.5$)



Fraction of world trade ($\alpha = 1.0$)



Fraction of world trade ($\alpha = 1.0$)

