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ON THE GENERAL STRUCTURE OF RICARDIAN MODELS WITH A CONTINUUM OF GOODS: APPLICATIONS TO GROWTH, TARIFF THEORY, AND TECHNICAL CHANGE[1]

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A continuum of goods is introduced into the general Ricardian model of international trade. By looking at the derived demand for labor, it is demonstrated that the analysis of the model can be reduced to the analysis of an equivalent model of pure exchange in which each country essentially trades its own labor for the labor of other countries. Furthermore, unlike the case where the number of goods is finite, the derived demand for labor becomes a differentiable function of the relative wages of the different countries. How this facilitates the analysis of comparative statics exercises is illustrated by establishing a number of propositions in the theory of growth, technical change, and tariffs.

1. INTRODUCTION

THE RICARDIAN MODEL IS perhaps the simplest formulation in which the technology can be explicitly incorporated into an analysis of international trade. In a general form, it consists of an arbitrary number of countries each of whom use only one factor of production, called labor, to produce an arbitrary number of goods. Each country has a constant returns to scale technology but they differ in the relative amounts of labor required to produce different goods. This generates an incentive for each country to specialize in the production of only certain goods which in turn generates the gains from trade.

Although the model is frequently employed to illustrate many of the basic principles of international trade, it is not commonly used to examine those issues which require a detailed analysis of comparative statics. Questions such as how a shift in demand affects the pattern of trade and the relative prices of goods, or the corresponding impact of a tariff, technical change, or growth in the labor force are generally analyzed either with simpler models which do not explicitly incorporate the technology at all or else more sophisticated models which include a technology with several factors of production. The problem with the Ricardian model is that the qualitative properties of the results typically depend upon the pattern of specialization. In order to determine the general equilibrium effect of a small change in the tariff rates, for instance, we must know precisely which countries are completely specialized in the production of which goods and which goods are jointly produced by more than one country. A general analysis of any of these issues, therefore, will require a separate analysis for each possible pattern of specialization. Even with two countries and two goods, there are generally several cases to examine. An even more serious defect is the fact that the first order effect in any one of these cases tells only part of the story of what happens in a world with many goods and discrete parameter changes. In general, a change in some

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parameter will affect both the relative price of labor (and hence the relative price of goods produced in different countries) *and* the pattern of specialization. When there are only a finite number of goods in the model, however, we will generally capture only one or the other of these effects by looking at infinitesimal changes.

Recently, Dornbusch, Fisher, and Samuelson (D-F-S) [**4**] have suggested a way of formulating the Ricardian model so that these limitations can be avoided. Rather than working with a finite number of goods, they assume a continuum of goods represented by the unit interval. Restricting their attention to a model with only two countries, each with identical log linear demand functions, and imposing some relatively weak restrictions on the technologies of the two countries, they proceed to analyze many of the standard topics in both the pure and monetary theory of international trade. Not only does the analysis prove to be unexpectedly simple, but they also obtain some very strong comparative statics results.

The D-F-S paper represents a significant contribution in demonstrating how one might modify the standard Ricardian model in order to make it more tractable for comparative statics analysis. Their assumptions are so restrictive, however, that the extent to which their approach can be generalized is not readily apparent. Besides the possibility of relaxing their assumptions on demand, it is not at all clear from their examples how the analysis would proceed if we wished to allow for more than two countries. The purpose of this paper is to present a general analysis of the Ricardian model with a continuum of goods which generalizes the D-F-S model in a number of respects. Not only is the theory developed for a much larger class of demand functions, but the model is also formulated to include any finite number of countries. Preferences need not be identical across countries and I allow for variations in the transport costs of shipping goods to different countries. Even with these generalizations, I demonstrate that a number of sharp comparative statics results are still possible.

The central theme of my approach is that, in the absence of initial tax distortions, the analysis of the Ricardian model can always be reduced to that of a very special model of pure exchange. Since the prices of the products in each country are strictly tied to the price of labor in that country, the wage rates become a proxy for prices of final output. Similarly, the demand for final products implies a derived demand for the labor of each country. Therefore, by concentrating on the derived demand for labor as a function of the wage rates, the original n-country Ricardian model is reduced to an n-agent, n-good exchange model in which each agent's endowment consists of the total supply of one of the goods, its own labor.

Looked at in this way, many of the problems that arise when analyzing the standard finite commodity Ricardian model can be traced to the fact that the derived demand for labor is not single-valued or differentiable at those wage rates at which two or more countries can produce the same good at least cost. By introducing a continuum of goods, this problem can be avoided because, at any given wage vector, only a negligible set of goods will be produced by more than one country. If sufficient regularity conditions are then imposed on the technology and demand, the derived demand for labor also becomes a differentiable function of the wage rates. By defining equilibrium in terms of the supply and derived demand for labor, such questions as the impact of a tariff or a change in the technology can then be analyzed by first determining the change in the equilibrium wage vector. Using information about the technology in each country, this implies the impact on the pattern of specialization and the prices of final output which in turn implies the impact on the demand for final products and the volume of trade.

In the analysis which follows, this basic idea is developed in more detail. In the next two sections, I will demonstrate how the properties of the derived demand for labor are related to the assumptions we make on the technology and the demand for final goods. Once the structure of the model is fully developed, I turn my attention to some of the topics investigated by D-F-S. In Section 6, I examine the impact of a change in either the supply of labor or the technology on the relative wage rates among different countries and their level of welfare. In Section 7, I look at the effect of a tariff. In each of these cases, I am able to find assumptions which generate reasonably sharp results and, in some cases, duplicate the D-F-S results exactly. Moreover, because of the general framework in which each of these problems are considered, the role of various assumptions in obtaining our results can be readily identified. In some instances, the two-country assumption is essential. Frequently, however, sufficient restrictions on demand will permit a straightforward generalization to the n-country case.

2. THE MODEL

Suppose there are I countries indexed by $\{1, \ldots, I\}$ and suppose the set of commodities can be represented by the unit interval, which I will denote by X. Generally i, j, k will denote countries and x, y, z will denote commodity types. Each country is endowed with a fixed supply of labor L_i and a constant returns to scale technology for producing each good. Let $a_j^i(x)$ be labor required in country j to produce one unit of good x for delivery in country i. Differences between $a_j^i(x)$ and $a_k^i(x)$ reflect the difference in the labor required in countries j and k to produce one unit of good x for delivery in country i. In addition, we may allow for differences in the cost of transporting a good produced in country j to two different countries i and k. This is reflected in the difference between $a_j^i(x)$ and $a_j^k(x)$.

The analysis of the model proceeds almost exactly as it would if the number of goods were finite. Let $p^i(x)$ be the price of good x in country i and let $w = (w_1, \ldots, w_I)$[2] be the vector of wage rates for each country. Finally, let $t_j^i(x)$ be one plus the tariff rate in country i on good x when it is produced in country j. Then, assuming perfect competition, prices will adjust so that the per unit cost of producing any good in country j for delivery in country i is greater than or equal to the price of the good in country i:

$$(2.1) \qquad w_j t_j^i(x) a_j^i(x) \geqslant p^i(x) \qquad \text{with } (=) \text{ if } q_j^i(x) > 0,$$

where $q_j^i(x)$ is quantity of good x produced in country j for delivery in country i.

[2] For the present, we will not normalize prices.

Assuming a positive amount of each good is demanded in each country when the economy is in equilibrium, relation (2.1) then implies

$$p^i(x) = \min_j \,[w_j t_j^i(x) a_j^i(x)]. \tag{2.2}$$

Let $X_j^i = \{x: w_j t_j^i(x) a_j^i(x) \leqslant w_k t_k^i(x) a_k^i(x),\ k = 1, \ldots, I\}$ be the set of commodities in country i for which country j is the least cost producer. Note that it depends upon the wage vector w, tariff functions t_k^i, and technology a_k^i.[3] If we assume that the set of goods for which more than one country is a least cost producer is negligible, then X_j^i may be regarded as the set of commodities consumed in country i which are produced in country j. Note that because we are allowing for differences in transport costs, the price of any good need not be identical across countries. It is also possible that different countries will purchase the same good from different sources. In particular, the model allows quite naturally for the existence of nontraded goods.

In order to get an expression for the derived demand for labor, all that remains is to specify how the demand for labor depends upon the demand for final goods. Let $c^i(x)$ denote the level of consumption of good x in country i. Then integrating the labor required to produce each good over the set X_j^i gives the quantity of labor in country j which is demanded to produce the goods consumed in country i, f_j^i:

$$f_j^i = \int_{X_j^i} a_j^i(x) c^i(x)\, dx.^4 \tag{2.3}$$

3. THE DERIVED DEMAND FOR LABOR

We turn now to a detailed examination of the derived demand for labor. The impact of any parameter change on a country's derived demand for labor can always be decomposed into two effects. The first, which I will call the *output effect*, works through a change in either the technology or the level of demand for final goods, assuming the pattern of specialization remains unchanged. In equation (2.3), this corresponds to a change in the value of f_j^i in response to a change in $a_j^i c^i$, holding X_j^i constant. The second effect, which I call the *specialization effect*, works through a change in the pattern of specialization, assuming the labor demanded for the production of each good remains unchanged. It corresponds to the effect of a change in X_j^i holding $a_j^i c^i$ constant.

For more than one reason, the analysis will be considerably simplified if we assume from the outset that all tariff rates are initially set equal to zero. Because I want to analyze the first order effect of changes in the tariff rate, however, it will be convenient to explicitly include the tariff factor in our calculations. I will discuss

[3] When the x argument is deleted, t_k^i refers to the entire function of tariff rates $t_k^i(x)$. Similarly for a_j^i, c^i, and p^i.

[4] Clearly some restrictions are required on a_j^i and c^i in order for equation (2.3) to be well defined. It should be understood throughout that any of the functions defined over the commodity space are assumed to be integrable. Similarly for any changes in these functions.

briefly at the conclusion of Section 7 how the analysis would have to be modified if we were to allow for more general tariff structures.

3.1 *The Output Effect*

Let $\hat{f}^i_j|_{X^i_j}$ refer to the per cent change in f^i_j resulting from a change in $a^i_j c^i$, holding X^i_j constant. If we define $d^i_j(x) = a^i_j(x)c^i(x)/f^i_j$ to be the proportion of country i's demand for the labor of country j used to produce good x, then upon differentiating (2.3), we have

$$(3.1) \qquad \hat{f}^i_j|_{X^i_j} = \int_{X_j} d^i_j(x)[\hat{a}^i_j(x) + \hat{c}^i_j(x)]\, dx.$$

Here $\hat{a}^i_j(x)$ refers to a per cent change in the labor required to produce one unit of good x. For the purposes of this paper it will be assumed to be exogenous. The per cent change in the density of consumption $\hat{c}^i$, however, will generally depend upon the change in Y^i, the income of the consumers in country i, and the change in p^i, the prices they face.

Assume that the first order effect of a change in Y^i and p^i on the demand for a typical good x can be expressed as

$$(3.2) \qquad \hat{c}^i(x) = \eta_Y(x)\hat{Y}^i + \eta_p(x)\hat{p}^i(x) + \int \eta^i(x, y)\hat{p}^i(y)\, dy.^5$$

Equation (3.2) is an infinite dimensional analogue of the standard equation expressing the first order effects of changes in income and prices when the number of goods is finite. In this equation, $\eta^i_Y(x)$ refers to the value of income elasticity of demand at good x, $\eta^i_p(x)$ refers to the value of the own price elasticity of demand at good x, and $\eta^i(x, y)$ refers to the value of cross price elasticity of demand at good x with respect to the price of good y. In general, η^i_Y, η^i_p, and η^i are all functions of Y^i and p^i.

Note that the change in the price of any single good has a significant effect only on its own demand. Although at first glance, this might appear excessively restrictive, I should point out that equation (3.2) will be satisfied by any demand function which is derived from a utility function of the form $U(c) = \int u(c(x), x)\, dx$ where $u(\cdot, x)$ is strictly concave and continuously differentiable.[6] In particular, it is satisfied by the special demand function used by D-F-S. Moreover, this equation is quite consistent with my earlier remarks on how the model should be interpreted. As we consider smaller and smaller intervals of goods, the proportion of the budget which is allocated to those goods goes to zero. Consequently, as their price changes, very little adjustment in the level of consumption of other goods is required to satisfy the budget constraint. It is only when the prices of a significant

[5] If the domain of integration is not specified, $\int f(x)\, dx$ should be understood to mean $\int_X f(x)\, dx$.

[6] In fact, it is not difficult to show in this case that (1) $\eta_Y(x) > 0$ for all x and (2) there is a function $\lambda(y) > 0$ such that $\eta(x, y) = \lambda(y)\eta_Y(x)$ for all $x, y \in [0, 1]$. Neither of these assumptions are assumed in the text. D-F-S implicitly assume a utility function of the form $\int b(x) \log x$ which implies $\eta_Y(x) = -\eta_p(x) = 1$ and $\eta(x, y) = 0$ for all x, y.

fraction of goods change that we should expect a significant adjustment in the demand for all other goods. This point is reinforced in the statement of the budget constraint:

$$(3.3) \qquad Y^i = \int p^i(x) c^i(x)\, dx.$$

Having specified how changes in the level of final demand depend on changes in prices and income, the next step is to use equation (2.2) and the definition of income to express the changes in prices and income in terms of changes in the level of wages and other parameters of the model. Although it is not required for the general analysis, it will considerably simplify the notation if we restrict our consideration to only those changes in tariffs or technology which are uniform in each country. That is, the percentage change in the tariff rate in country i on goods from country j must rise by the same percentage for all goods. Similarly, the percentage change in the labor required to produce goods in country j for delivery in country i must also be the same for all goods. Then using the definition of X^i_j, we have upon differentiating equation (2.2):

$$(3.4) \qquad \hat{p}^i(x) = \hat{w}_j + \hat{t}^i_j + \hat{a}^i_j \quad \text{for all} \quad x \in X^i_j.$$

The only source of income for the consumers in any country is the wage they receive for their labor plus any tariff revenue or other transfer payments T^i. Therefore, we may write

$$(3.5) \qquad Y^i = w_i L_i + T^i.$$

Assuming all tariffs and transfer payments are initially zero, we obtain, upon differentiating (3.5),

$$(3.6) \qquad \hat{Y}^i = \hat{w}_i + \hat{L}_i + \frac{dT^i}{Y^i}.$$

Substituting (3.4) and (3.6) into (3.2) then yields

$$(3.7) \qquad \hat{f}^i_j|_{X^i_j} = \int_{X^i_j} d^i_j(x) \Big\{ \hat{a}^i_j + \eta^i_Y(x)\Big(\hat{w}_i + \hat{L}_i + \frac{dT^i}{Y^i}\Big) + \eta_p(x)(\hat{w}_j + \hat{t}^i_j + \hat{a}^i_j) + \sum_k \int_{X^i_k} \eta(x,y)(\hat{w}_k + \hat{t}^i_k + \hat{a}^i_k)\, dy \Big\}\, dx.^7$$

If we now define $\beta^i_{jY} = \int_{X^i_j} d^i_j(x)\eta^i_Y(x)\, dx$ and

$$\alpha^i_{jk} = \begin{cases} \int_{X^i_j} d^i_j(x)\Big[\eta_p(x) + \int_{X^i_j} \eta(x,y)\, dy\Big]\, dx & \text{for} \quad k = j, \\ \int_{X^i_j} d^i_j(x) \int_{X^i_k} \eta^i(x,y)\, dy\, dx & \text{for} \quad k \neq j, \end{cases}$$

[7] $\sum_k$ is to be understood to mean $\sum_{k=1}^{I}$.

then (3.7) can be written as

$$(3.8) \qquad \hat{f}^i_j|_{X^i_j} = \hat{a}^i_j + \beta^i_{jY}\left(\hat{w}_i + \hat{L}_i + \frac{dT^i}{Y^i}\right) + \sum_k \alpha^i_{jk}(\hat{w}_k + \hat{\tau}^i_k + \hat{a}^i_k).$$

It is clear from equation (3.8) that β^i_{jY} can be interpreted as country i's income elasticity of demand for the labor of country j and α^i_{jk} as the corresponding price elasticity of demand with respect to the wage in country k, holding the pattern of specialization fixed.

3.2 *The Specialization Effect*

We turn now to the effect on the derived demand for labor resulting from changes in the pattern of specialization. Here it is convenient to break the analysis into two distinct steps. The first is to examine how the change in some parameter or wage rate affects the set of commodities which each country produces. Once this is determined, the next step is to examine how a change in the set of goods produced in some country affects the derived demand for the labor of that country. Since we wish to do comparative statics by looking at the first order effects of a change in the parameters, it will be necessary to introduce conditions which guarantee that both of these effects can be represented by differentiable functions.

In a two country model the types of assumptions required are relatively straightforward. The commodities may always be ranked so that $a^i_2(x)/a^i_1(x)$ is a nonincreasing function of x. If this function is strictly decreasing, then there is a critical $\bar{x}$ such that any commodity $x < \bar{x}$ is produced in country 1, and commodity $x > \bar{x}$ is produced in country 2. Assuming that $a^i_2(x)/a^i_1(x)$ is also a continuously differentiable function of x, it is then easy to show that any change in wage rates will result in a differentiable change in $\bar{x}$. If the function $a^i_j(x)c^i(x)$ is a continuous function of x at $\bar{x}$, this will generate a differentiable change in the demand for labor. This is the approach used by D-F-S.

When there are more than two countries, however, this approach must be modified. The problem is that, in general, it is not possible to rank the commodities so that the ratio of input requirements for an arbitrary pair of countries is a monotone function of x. Consequently, there is no guarantee that the set of commodities produced by each country can be represented by a single interval. Fortunately, this is not essential for the basic method used by D-F-S to work. It is sufficient to ensure that the set of commodities which are produced in each country can always be represented by a finite number of intervals. This consideration motivates the following assumption.

ASSUMPTION (A1): For each i, j, $a^i_j(x)$ is continuously differentiable and strictly positive for all $x \in X$. Furthermore, for $j \neq k$, $d(a^i_j(x)/a^i_k(x))/dx = 0$ at only a finite number of $x \in X$.

Assumption (A1) states that the rate of change in the comparative advantage between two countries with respect to a change in x is zero at only a finite number of commodities. Geometrically, it restricts each function, a_j^i/a_k^i, to reach a local maximum or minimum at no more than a finite number of points. Given my earlier comments on what constitutes a real good in this model, this assumption does not appear to impose any substantial restrictions on the relative technologies.[8]

With Assumption (A1), I show in Appendix A that each X_j^i must be composed of a finite number of intervals. Consequently, the determination of how changes in the wage rates or other parameters affect the pattern of specialization can be reduced to determining how changes in these parameters affect the boundary points of X_j^i. Suppose x is a boundary point of X_j^i. Then there must be some other country k for which

$$w_j t_j^i(x) a_j^i(x) - w_k t_k^i(x) a_k^i(x) = 0. \tag{3.9}$$

Let

$$\mu_{jk}^i(x) = \left[\frac{da_k^i(x)/dx}{a_k^i(x)} - \frac{da_j^i(x)/dx}{a_j^i(x)}\right]^{-1}.$$

Then assuming (3.9) is satisfied for only one $k \neq j$, we obtain upon differentiation:

$$dx = \mu_{jk}^i(x)[(\hat{w}_j + \hat{t}_j^i + \hat{a}_j^i) - (\hat{w}_k + \hat{t}_k^i + \hat{a}_k^i)].^9 \tag{3.10}$$

If x is a left-hand boundary point, then we can show that

$$\frac{da_k^i(x)/dx}{a_k^i(x)} - \frac{da_j^i(x)/dx}{a_j^i(x)} \geqslant 0.$$

For the moment, suppose the inequality is strict. Then $\mu_{jk}^i(x) > 0$. If x is a right-hand boundary point, then the inequality is reversed and we have $\mu_{jk}^i(x) < 0$. From this it follows that an increase in w_j will decrease the set X_j^i while an increase in w_k will increase X_j^i.

The effect of increasing w_j is illustrated in Figure 1. For simplicity, it is assumed there are only two countries, j and k. Given wage rates w_j and w_k, the unit costs of producing goods in each country is represented by the curves $w_j a_j^i(x)$ and $w_k a_k^i(x)$. Since country i always purchases from the least cost producer, X_j^i is composed of the intervals $[0, x_1]$ and $[y_2, x_2]$. When w_j increases to w_j', the $w_j a_j^i$ curve shifts up proportionately. As a consequence, the left-hand boundary point y_2 shifts rightward to y_2' and the right-hand boundary points shift leftward to x_1' and x_2'. The

[8] The differentiability condition could be relaxed to allow for a finite number of discontinuities in the a_j^i function if, for instance, one wanted to allow for a sharp distinction in the labor requirements of different types of goods. Such a generalization would not change any of the conclusions of this section in any essential way.

[9] In general, $\mu_j^i(x)$ must include the term

$$\left[\frac{dt_k^i(x)/dx}{t_k^i(x)} - \frac{dt_j^i(x)/dx}{t_j^i(x)}\right].$$

Since we are assuming that $t_j^i(x) = 1$ for all i, j, this term has been deleted.

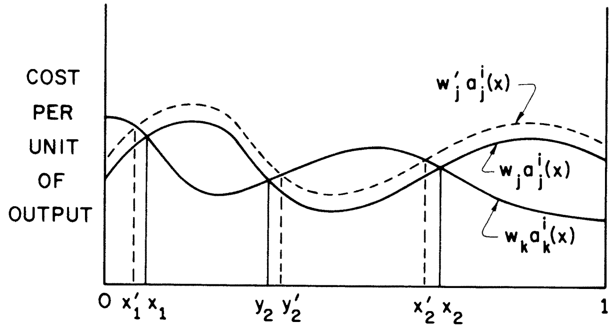

FIGURE 1

extent of the shift depends on the relative slopes of the per unit cost curves which are reflected in the magnitude of μ^i_{jk}.

In the course of the preceding analysis it was necessary to make two assumptions which, it is evident, cannot be satisfied at all wage vectors. The first was that at any boundary point of X^i_j, there is at most one other country with a cost of production equal to that of country j. The second was that if both countries j and k were least cost producers of good x, then

$$\frac{da^i_j(x)/dx}{a^i_j(x)} - \frac{da^i_k(x)/dx}{a^i_k(x)} \neq 0.$$

The problems which are created when these conditions are violated are discussed in Appendix A. I also argue there that the likelihood that either of these conditions will be violated in equilibrium is negligible. In order to proceed with the analysis, therefore, I will assume that at any equilibrium wage vector we consider, both of these conditions are satisfied.

Having determined the conditions under which we can guarantee that the boundaries of X^i_j change differentiably with changes in the wage rates and other parameters, it remains only to find conditions under which this implies that the change in the derived demand for labor resulting from changes in the boundary points are also differentiable. Given our assumptions on the tariff rates and the technology, the following assumption is sufficient:

ASSUMPTION (A2): If p^i is a continuous function of x, then $c^i(\cdot\,; p^i, Y^i)$ is also a continuous function of x.

Since $a^i_j(x)$ is continuous for each i, j by Assumption (A1), it follows from equation (2.2) (and our assumption that $t^i_j(x) = 1$ for all x) that $p^i(x)$ is also continuous. Therefore, from Assumption (A2), $p^i(x)c^i(x)$ is continuous.

Now consider the effect of changes in w, a^i_k, and t^i_k on the level of f^i_j holding $a^i_j c^i$ constant. Since X^i_j can be represented as the union of a finite number of intervals

$[y_n, x_n]$, we may write

$$(3.11) \qquad f_j^i = \sum_{n=1}^{N_j^i} \int_{y_n}^{x_n} a^i(x)c^i(x)\,dx.$$

Differentiating (3.11) with respect to (x_n, y_n), holding $a_j^i c^i$ constant then yields

$$(3.12) \qquad \hat{f}_j^i|_{a_j^i c^i} = \sum_{n=1}^{N_j^i} [d_j^i(x_n)\,dx_n - d_j^i(y_n)\,dy_n].$$

Let B_{jk}^i represent the set of commodities which are left-hand boundary points for X_j^i and right-hand boundary points for X_k^i, and define

$$g_{jk}^i = \begin{cases} \sum\limits_{\substack{x \in \bigcup B_{lj}^i \\ l \neq j}} d_j^i(x)\mu_{jl}^i(x) - \sum\limits_{\substack{x \in \bigcup B_{jl}^i \\ l \neq j}} d_j^i(x)\mu_{jl}^i(x) & \text{for} \quad k = j, \\ \sum\limits_{x \in B_{jk}^i} d_j^i(x)\mu_{jk}^i(x) - \sum\limits_{x \in B_{kj}^i} d_j^i(x)\mu_{jk}^i(x) & \text{for} \quad k \neq j. \end{cases}$$

Then since for each n, x_n is either equal to 1 or contained in B_{jk}^i for some k and y_n is either equal to 0 or contained in B_{kj}^i for some k, we may substitute (3.10) into (3.12) to obtain

$$(3.13) \qquad \hat{f}_j^i|_{a_j^i c^i} = \sum_k g_{jk}^i(\hat{w}_k + \hat{t}_k^i + \hat{a}_k^i).$$

From equation (3.13), it is clear we may interpret g_{jk}^i as country i's elasticity of demand for the labor of country j with respect to a change in the wage in country k, holding the labor requirements for final demand constant. I will sometimes refer to g_{jk}^i as the *specialization elasticity*.

3.3 *The Combined Effects*

Combining equations (3.8) and (3.13) we obtain the total first order effect on the derived demand for labor with respect to changes in the wage rates, tariffs, technology, and the supply of labor:

$$(3.14) \qquad \hat{f}_j^i = \hat{f}_j^i|_{X_j^i} + \hat{f}_j^i|_{a_j^i c^i}$$
$$= \hat{a}_j^i + \beta_{jY}^i \hat{w}_i + \hat{L}_i + \frac{dT^i}{Y^i} + \sum_k (\alpha_{jk}^i + g_{jk}^i)(\hat{w}_k + \hat{t}_k^i + \hat{a}_k^i).$$

Equation (3.14) decomposes the effect on the derived demand for labor in any country into a direct effect resulting from changes in the technology, an income effect resulting from changes in the country's income, and a substitution effect which combines both the effect on final demand and the effect on the pattern of specialization resulting from a change in the cost of producing goods. This equation will form the basis for all of the analysis which follows.

In addition to this equation, we will also need a number of cross-equation restrictions implied by our assumptions on the properties of the technology and some additional assumptions on the demand for final goods. Let $\theta_j^i = w_j f_j^i / Y^i$ be the proportion of income in country i which is spent on goods produced in country j. Then it follows upon substituting equations (2.2) and (2.3) into the budget constraint (3.3) and setting $t_j^i(x) = 1$ for all x and $T^i = 0$ that

$$(3.15) \quad \sum_j \theta_j^i = \sum_j \frac{w_j f_j^i}{w_i L_i} = 1.$$

Equation (3.15) essentially translates the budget constraint defined over final output into a budget constraint defined over the supply and demand for labor.

Define $\bar{\alpha}_{jk}^i = \alpha_{jk}^i + \theta_k^i \beta_{jY}^i$. Then assuming that the demand for final goods is homogeneous of degree zero in prices and income and the budget constraint (3.3) is always satisfied, the results of Appendices A and B can be combined to imply the following relations:

$$(3.16) \quad \sum_j \theta_j^i \beta_{jY}^i = 1;$$

$$(3.17) \quad \sum_j \theta_j^i (\bar{\alpha}_{jk}^i + g_{jk}^i) = 0 \quad \text{for all } k; \quad \text{and}$$

$$(3.18) \quad \sum_k (\bar{\alpha}_{jk}^i + g_{jk}^i) = 0 \quad \text{for all } j.$$

If, in addition, we assume that the demand for final goods can be derived from utility maximization, then the following two relationships can also be obtained from the results of Appendices A and B:

$$(3.19) \quad \theta_j^i (\bar{\alpha}_{jk}^i + g_{jk}^i) = \theta_k^i (\bar{\alpha}_{kj}^i + g_{kj}^i) \quad \text{for all } j, k; \quad \text{and}$$

$$(3.20) \quad \sum_j \sum_k v_j \theta_j^i (\bar{\alpha}_{jk}^i + g_{jk}^i) v_k \leqslant 0 \quad \text{for all } v \in R^I,$$

where the inequality is strict if $v_j = 0$ for some j but $v \neq 0$.

The point to note about these relationships is that the combined terms $(\bar{\alpha}_{jk}^i + g_{jk}^i)$ satisfy all the properties satisfied by the pure substitution elasticities of a demand function which is derived directly from a preference ordering defined over a finite number of goods. Changing all wage rates proportionately must leave the level of demand unchanged (equation (3.18)); changing any wage rate, leaving utility constant, must leave the value of demand in terms of the initial wage rates unchanged (equation (3.17)); finally, the pure substitution matrix must be symmetric (equation (3.19)) and negative semi-definite (relation (3.20)).

I should point out that each of these relationships are satisfied by the specialization elasticities, g_{jk}^i, and the output elasticities, $\bar{\alpha}_{jk}^i$, individually. For the output elasticities, they are a consequence of utility maximization; for the specialization elasticities, they are a consequence of the fact that each country purchases its goods from the least cost producers. In fact, from the assumption that each country specializes its production only in those goods for which it is the least cost producer, we can impose still another restriction on the matrix of specialization elasticies.

When the wage rate of any country is increased, its costs of production are also increased. Consequently, it ceases to produce those goods which were previously at its margin of specialization. For any other country, however, the set of goods it produces can only increase. It continues to be the least cost producer for all the goods it produced before the wage increase and may add some of the goods previously produced by the country experiencing the wage increase. Therefore, we may conclude

$$(3.21) \qquad g^i_{jj} < 0 \qquad (\text{unless } X^i_j = \phi \text{ or } X) \quad \text{and} \quad g^i_{jk} \geqslant 0 \quad \text{for all } j, k.$$

The direction of the inequalities in relation (3.21) generate an important bias on the properties of the pure substitution matrix of the derived demand for labor. I note in Section 5 that if this bias is sufficiently strong, we may be able to extend many two-country comparative statics results to the case where there are three or more countries.

4. WELFARE ANALYSIS

In order to determine the welfare implications of changes in the various parameters, it is also necessary to derive an explicit expression for how the level of welfare depends upon the level of prices and income. Since we are assuming that the demand for goods in each country can be derived from the maximization of a single utility function, the change in the welfare of any country can be expressed by taking the first derivative of the income compensation function. For initial income Y^i and price function p^i, let $-\hat{V}^i$ denote the per cent change in income necessary to maintain a constant level of utility given changes in income and prices, $\hat{Y}^i$ and $\hat{p}^i$. If we let $b^i(x) = p^i(x)c^i(x)/Y^i$ represent the proportion of income in country i spent on good x, the infinite dimensional analogue to the first derivative of the income compensation function may be written:

$$(4.1) \qquad \hat{V}^i = \hat{Y}^i - \int b^i(x)\hat{p}^i(x)\,dx.$$

The welfare of country i rises with an increase in its income and falls with an increase in the price of the goods it purchases in proportion to their share of the expenditure of country i.

Assuming $T^i = 0$ and $t^i_j(x) = 1$ for all x, it follows by definition that $\theta^i_j = \int_{X^i_j} b^i(x)\,dx$. Then upon substituting equations (3.4) and (3.6) into (4.1) we obtain:

$$(4.2) \qquad \hat{V}^i = \hat{w}_i + \hat{L}_i + \frac{dT^i}{Y^i} - \sum_j \theta^i_j(\hat{w}_j + \hat{t}^i_j + a^i_j)$$

$$= \sum_j \theta^i_j(\hat{w}_i - \hat{w}_j) + \hat{L}_i + \frac{dT^i}{Y^i} - \sum_j \theta^i_j(\hat{t}^i_j + \hat{a}^i_j).$$

Equation (4.2) will be the basis for all of the welfare analysis which follows.

5. EQUILIBRIUM AND CONDITIONS FOR COMPARATIVE STATICS

Let $f_j = \Sigma_i f_j^i$ denote the total demand for the labor of country j. Then the economy is in *equilibrium* when the wage rates have adjusted so that the demand for labor in each country is equal to the supply:

$$f_j = L_j. \tag{5.1}$$

The remainder of this paper will be devoted to illustrating how the model may be used to address some of the standard issues in the theory of international trade. As with any pure exchange model, the general procedure will be to determine how some change in the parameters affects the supply and demand for labor and then use the relation between the wage rates and the derived demand for labor to determine how the wage rates must adjust to bring the economy back into equilibrium. For the remainder of this section, therefore, we will focus on the properties of the aggregate derived demand for labor as a function of the wage rates, holding all other parameters constant.

The first step is to relate the properties of the aggregate demand for labor to the properties of the demand coming from each individual country. Define $\lambda_j^i = f_j^i/f_j$ to be the proportion of the demand for country j's labor which comes from country i. Then the percentage change in the total demand for the labor of country j can be written as a weighted sum of the percentage change in the demand of each individual country

$$\hat{f}_j = \sum_i \lambda_j^i \hat{f}_j^i. \tag{5.2}$$

From equations (3.14) and (3.15) we may define

$$\varepsilon_{jk}^i = \begin{cases} \beta_{jY}^i + \alpha_{ji}^i + g_{ji}^i \equiv (1-\theta_i)\beta_{jY}^i + \bar{\alpha}_{ji}^i + g_{jk}^i & \text{for } k = i, \\ \alpha_{jk}^i + g_{jk}^i = -\theta_k^i \beta_{jY}^i + \bar{\alpha}_{jk}^i + g_{jk}^i & \text{for } k \neq i, \end{cases}$$

to be the total elasticity of demand in country i for the labor of country j with respect to a change in w_k. It includes not only the effect of a change in the price of goods but also the effect of any change in the income of country i. If we then define $\varepsilon_{jk} = \Sigma_i \lambda_j^i \varepsilon_{jk}^i$ to be the aggregate elasticity of demand for the labor of country j with respect to a change in the wage of country k, equation (5.2) implies (holding all other parameters constant)

$$\hat{f}_j = \sum_k \varepsilon_{jk} \hat{w}_k. \tag{5.3}$$

It will frequently be convenient to express this relation in matrix form. By Walras Law (equations (3.16) and (3.17)) and the homogeneity of demand with respect to wage rates (equation (3.18)), we may eliminate the equation for f_I and normalize w so that $w_I = 1$. Then let E denote the $(I-1)\times(I-1)$ matrix whose element in the jth row of the kth column is ε_{jk}, and let $\hat{f} = (\hat{f}_1, \ldots, \hat{f}_{I-1})$. Then (5.3) can be written as

$$\hat{f} = E\hat{w}. \tag{5.4}$$

It is clear from equation (5.4) that the general equilibrium implications of any parameter change will depend critically upon the properties of the matrix of demand elasticities, E. In the remainder of this section, I will briefly discuss three different conditions which are sometimes imposed on this matrix to obtain qualitative comparative statics results. The results presented in this section will then provide the basis for most of the analysis which follows.

The first assumption permits us to treat the excess demand for labor as if it were derived from a single utility maximizing consumer. Suppose that the preferences for final goods are identical and homothetic across countries and that, in each country, the transport costs for any good is independent of its destination, i.e., $a_j^i = a_j^k$ for all i, j, k. Under these conditions, we can show that each country's derived demand for labor will behave as if it were generated by an identical homothetic preference ordering defined directly over the labor of each country. Since there are no tariffs and transport costs are identical, each country will face the same prices. Therefore, from the assumption that preferences are identical and homothetic, we have that $\eta_p^i(x) = \eta_p^l(x)$, $\eta^i(x, y) = \eta^l(x, y)$, and $\eta^i_Y(x) = 1$ for all i, l, x, y. It follows immediately, therefore, that $\alpha^i_{jk} = \alpha^l_{jk}$ and $\beta^i_{jY} = 1$ for all i, j, k, l. Similarly, one may show that $g^i_{jk} = g^l_{jk}$ for all i, j, k, l. Finally, let $\theta_j = w_jL_j/\Sigma_k w_kL_k$ denote the proportion of total world income generated by country j. Then, in equilibrium, we must have $\lambda^j_i = \theta^i_j = \theta_j$ for all i, j, from which it follows, by definition, that $\varepsilon_{jk} = \bar{\alpha}^i_{jk} + g^i_{jk}$ for all i, j, k. Referring back to relation (3.20) and noting that the inequality must be strict for any subset of countries, we may state the following proposition.

PROPOSITION 1: *If $a^i_j = a^i_k$ for all i, j, k and if the preferences for goods are homothetic and identical across all countries, then $\varepsilon_{jk} = \bar{\alpha}^i_{jk} + g^i_{jk}$. From this it follows that $[\theta]E$ and hence $E^{-1}[\theta]^{-1}$ are both negative definite.*

The assumptions of Proposition 1 are particularly useful when we wish to focus on the implications of differences in comparative advantage. In those instances when we wish to emphasize the role of transport costs or to allow for differences in preferences across countries, however, these assumptions will not be appropriate. Nevertheless, strong results can still often be obtained if we introduce conditions which are sufficient to guarantee that the matrix of demand elasticities E satisfies the gross substitute condition. Under this assumption, an increase in the wage of any country must act to reduce the demand for labor in that country and increase (or at least not decrease) the demand for labor in any other country. Mathematically, this requires that $\varepsilon_{jj} < 0$ and $\varepsilon_{ij} \geqslant 0$. The essential implication of the gross substitute assumption for comparative statics exercises is summarized in Proposition 2 (see Hawkins and Simons [**6**], also McKenzie [**9**]).

PROPOSITION 2: *If E is a gross substitute matrix, then $E^{-1} < 0$ (i.e., each element of E^{-1} is negative).*

Since this condition is imposed directly on the derived demand for labor, the kind of restrictions on the original Ricardian model which are required to satisfy it may not be immediately obvious. One example of a set of restrictions which do satisfy this assumption is the special case analyzed by D-F-S. They assumed that each country has a unitary income elasticity for each good, $\eta_Y(x)=1$, a unitary own price elasticity, $\eta_P(x)=-1$, and a zero cross price elasticity, $\eta(x, y)=0$. Translating these properties into the elasticities of the derived demand for labor, it can be shown that not only does the aggregate demand for labor satisfy the gross substitute condition, but the demand of each individual country does as well.

The D-F-S assumptions might be relaxed in a number of ways and still generate the gross substitute assumption. Here I will simply note two important biases which may work in opposite directions. As we noted in the previous section, the matrix of specialization elasticities must always satisfy the gross substitute condition. If the specialization effect is large enough, therefore, it may be reasonable to suppose that it dominates any opposing price effect working through a change in the demand for final goods. However, since the endowments of labor are so skewed (i.e., each country owns only its own labor), income effects are likely to be significant. Consequently, if there is a large asymmetry in the demand functions of different countries, strong income effects may counter and even dominate the bias which results from the specialization effect.

Weaker than either the assumption of homothetic preferences or the gross substitute condition is a condition on the matrix of demand elasticities commonly referred to as *Hicksian stability*. It is essentially an n-good analogue of the Marshall-Lerner conditions. Starting at equilibrium, it requires that an increase in the wage rate of any country result in an excess supply of labor in that country after the wage rates of any subset of the other countries have adjusted to equate the supply and demand for the labor of that subset of countries. Mathematically, what is required is that all of the principal minors of $-E$ be positive (Hicks [**7**, AppendixV]). Since this condition is required for a matrix to be negative definite, it follows from Proposition 2 that Hicksian stability is implied by the assumption of identical homothetic preferences and zero transport costs. It is also true that any gross substitute matrix is Hicksian stable (see McKenzie [**9**], also Arrow and Hahn [**1**]). Proposition 3 summarizes the critical implication of this assumption for our purposes.

PROPOSITION 3: *If E is Hicksian stable, then for any $v \in R^I_+$, both Ev and $E^{-1}v$ contain at least one negative element* (*Gale and Nikaido* [**5**], *also Arrow and Hahn* [**1**]).

These three assumptions will alternatively be employed to derive most of the comparative statics results which follow. Since the implications of each of these assumptions are already well known, however (see, for example, Mandell [**11**, Ch. 3]), I will concentrate primarily on those results, which, because of the special structure of the Ricardian model, are unusually striking. I note in passing that a sufficient condition for the uniqueness of the equilibrium is that the economy be Hicksian stable at all wage vectors.[10]

6. GROWTH AND TECHNICAL CHANGE

We turn now to an illustration of how the model may be applied to a number of issues in international trade. In this section, we will investigate how the relative wage rates and the pattern of trade are affected by either a change in the supply of labor or a change in the technology. For the most part, I will confine my attention to the case where the technical improvement in each country is uniform for all the goods it produces. Given this restriction, I show that the analysis of technical change becomes essentially identical to the analysis of changes in the supply of labor. The section will conclude with a brief discussion of some of the welfare implications of the diffusion of technical knowledge.

We begin by differentiating the equilibrium condition (5.1) to obtain

$$\hat{f}_j - \hat{L}_j = 0 \qquad \text{for} \qquad j = 1, \ldots, I. \tag{6.1}$$

Equation (6.1) simply states that, to maintain equilibrium, any change in the supply of a country's labor must be accompanied by a corresponding increase in the demand for that labor. For our present purposes, we may assume that the change in the supply of labor in any country is exogenous. We will also assume that technical change is completely uniform within each country. That is, the percentage change in the labor of any good is constant and independent of its destination. Therefore, we may write $\hat{a}^i_j = \hat{a}_j$ for all i, j. Then, holding tariff rates constant, we have upon substituting equation (3.14) into (5.2) and (5.2) into (6.1) and using the definition of ε_{jk}:

$$\hat{f}_j - \hat{L}_j = \sum_k \varepsilon_{jk}(\hat{w}_k + \hat{a}_k) + (\hat{a}_j - \hat{L}_j) - \sum \lambda^k_j \beta^k_{jY}(\hat{a}_k - \hat{L}_k) = 0. \tag{6.2}$$

A change in the supply of labor in any country k not only affects the excess demand for the labor of its own country, but it also generates an income effect on that country's demand for the labor of every other country. This is captured in the terms $\lambda^k_j \beta^k_{jY}$.

The relationship between the impact of a uniform technical change and the impact of a corresponding change in the supply of labor is also apparent from equation (6.2). Suppose that the immediate response of the wage rate in each country is to adjust so that output prices remain unchanged, i.e., $\hat{w}_k + \hat{a}_k = 0$ for all k. Then it is clear that the effect of the technical change on the demand for labor is exactly what it would be if instead the supply of labor had changed in corresponding proportions. It follows immediately, therefore, that the relative price of goods and the pattern of specialization respond to a change in the technology exactly as they do to a change in the supply of labor. In any case, given the initial impact on the excess demand for labor, wages must then adjust to bring the economy back into equilibrium.

[10] See Gale and Nikaido [**5**]. Arrow and Hahn [**1**, p. 215] show that to establish uniqueness, it is sufficient to show only that the economy is Hicksian stable at all equilibria.

In translating equation (6.2) into matrix notation, it will be useful to "normalize" technical change by defining $\hat{a}_j^* = (\hat{a}_j - \hat{a}_I)$ to be the difference in the percentage change in the labor requirements per unit of output between country j and country I. Then let $\hat{L} = (\hat{L}_1, \ldots, \hat{L}_{I-1})$, $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_{I-1})$, and $\hat{a}^* = (\hat{a}_1^*, \ldots, \hat{a}_{I-1}^*)$. Let $[\lambda\beta]$ represent the $(I-1)\times(I-1)$ matrix whose element in the jth row of the kth column is $\lambda_j^k \beta_{jY}^k$ and let $(\lambda^I \beta_Y^I) = (\lambda_1^I \beta_{1Y}^I, \ldots, \lambda_{I-1}^I \beta_{I-1,Y}^I)$. Then using Walras Law to eliminate the equation for f_I and normalizing $w_I = 1$, we have

$$(6.3) \qquad E(\hat{w} + \hat{a}^*) = (I - [\lambda\beta])(\hat{L} - \hat{a}) + \lambda^I \beta_Y^I (\hat{a}_I - \hat{L}_I).$$

We will concentrate first on the impact of a change in the supply of labor. Suppose the labor in only one country rises, the others remaining unchanged, and we wish to determine the impact on the relative wage rates. Since the choice of numeraire is arbitrary, we may suppose $\hat{L}_I = 1$ and $\hat{L}_j = 0$ for $j < I$. Then, inverting (6.3), we obtain

$$(6.4) \qquad \hat{w} = -E^{-1}(\lambda^I \beta_Y^I).$$

Suppose that $\beta_{jY}^I \geqslant 0$ for all j. This implies that, at constant wages, an increase in L_I increases the excess demand for the labor of all countries except I. Then if E satisfies the gross substitute property, Proposition 2 implies that $\hat{w}_k \geqslant 0$ for all $k \neq I$. In a world of gross substitutes for labor and positive income elasticities, therefore, an increase in the labor supply of any country must depress its wage relative to the wage of any other country and, consequently, relative to the price of any final product. A weaker result can be obtained if the gross substitute assumption is replaced by Hicksian stability. From Proposition 3 it follows that if E is Hicksian stable, an increase in L_I, holding all other L_j constant, must result in an increase in the relative wage of at least one other country.

If we assume identical transport costs and identical homothetic preferences across countries, this latter result takes on a particularly sharp form. Recall that $\theta_i = w_i L_i / \Sigma_j w_j L_J$ denotes the fraction of world income generated by country i. Then in this case we have $\lambda_j^I = \theta_I$ and $\beta_{jY}^I = 1$ for all j. Let e denote the $(I-1)$ vector $(1, \ldots, 1)$, let θ represent the vector $(\theta_1, \ldots, \theta_{I-1})$, and let $[\theta]$ represent the $(I-1)\times(I-1)$ diagonal matrix whose jth element is θ_j. Then from (6.4) we have

$$(6.5) \qquad \theta\hat{w} = -\theta E^{-1} e\theta_I = -\theta E^{-1}[\theta]^{-1}[\theta]e\theta_I$$
$$= -\theta E^{-1}[\theta]^{-1}\theta\theta_I > 0.$$

The final inequality follows from Proposition 1. In the case of identical transport costs and identical homothetic preferences, the average wage in other countries weighted by their shares of world income must rise relative to the wage of country I. In the two country case, (6.5) reduces to

$$(6.6) \qquad \hat{w}_1 = \frac{-\theta_I}{\varepsilon_{11}} > 0$$

which is the result obtained by D-F-S.

Finally, let us consider the welfare implications of a growth in the labor supply in country I. Holding the technology and tariff rates constant, equation (4.2) states that the welfare of country i will rise if and only if

$$\hat{V}^i = \sum_j \theta^i_j(\hat{w}_i - \hat{w}_j) + \hat{L}_i > 0. \tag{6.7}$$

For those countries whose labor supply does not change, the change in their welfare depends only on the weighted sum of the changes in its wage relative to the wages of all other countries. It follows immediately from our previous discussion, therefore, that as long as the economy is Hicksian stable and income elasticities are positive, the welfare of at least one country other than country I must rise in response to an increase in the labor supply in country I. In a two country model, this implies that a growth in the supply of labor in one country always increases the welfare of the other.

For country I, the implications are more ambiguous. If E satisfies the gross substitute condition, then the wage of country I must fall relative to the wage of every other country. Therefore, individual workers in country I must be made worse off. If transport costs are identical and demand is identical and homothetic across countries, then the same conclusion follows from equation (6.5). For the country as a whole, however, the increase in the supply of labor also generates more income. Whether or not the welfare of the entire country falls, therefore, depends on whether or not the deterioration in the terms of trade is sufficient to offset the gain in income. Substituting (6.4) into (6.7) and setting $\hat{L}_I = 1$, we may conclude that a growth in the supply of labor in country I will be "immiserizing" if and only if

$$\hat{V}^I = 1 - \sum_k \theta^I_k \hat{w}_k \equiv 1 - \theta^I \hat{w} = 1 + \theta^I E^{-1}(\lambda^I \beta^I_Y) < 0. \tag{6.8}$$

In the two country case this reduces to the condition $1 + (\theta^I_1/\varepsilon_{11})\lambda^I_1\beta^I_{1Y} < 0$, or in its more familiar forms (setting $I = 2$ and using the definitions of ε^i_{jk} and ε_{jk}):

$$\frac{-\varepsilon_{11}}{\lambda^I_1} = (\varepsilon^1_{21} - \varepsilon^2_{11} - 1) < \theta^2_1\beta^2_{1Y} \quad \text{or} \quad \varepsilon^1_{21} + (\bar{\alpha}^2_{21} + g^2_{21}) < 1 \tag{6.9}$$

(see Bhagwati [**2**]).

We consider next the implications when country I experiences a technical improvement, $\hat{a}_I = -1$, leaving the technology of all other countries unchanged. Then letting $e = (1, \ldots, 1)$, we have from equation (6.3),

$$\hat{w} - e = -E^{-1}(\lambda^I \beta^I_Y). \tag{6.10}$$

The term $(\hat{w}_j - 1)$ represents the change in the wage rate of country j relative to the price of goods sold by country I. Comparing equation (6.10) with equation (6.4), we see that the effect of a technical improvement in country I on these terms is equivalent to a change in the supply of labor in country I, holding the technology fixed. Therefore, we may conclude from the results just derived that: (i) if E is

Hicksian stable, the price of goods produced in country I must fall relative to the wage of at least one other country; (ii) if E satisfies the gross substitution condition, then the price of goods produced in country I must fall relative to the wage rates of all other countries. From this it also follows, that as long as E is Hicksian stable, at least one other country will benefit from a technical improvement in country I.

Now consider the change in the welfare of country I. Holding the supply of labor, the tariff rates, and the technology of all countries constant, we have from equation (4.2) that a one per cent improvement in the technology of country I will increase its welfare if and only if

$$(6.11) \qquad \hat{V}^I = -\sum_j \theta^i_j \hat{w}_j + \theta^I_I = -\sum_{j \neq I} \theta^i_j(\hat{w}_j - 1) + 1 > 0.$$

Substituting into this expression equation (6.10), we find that this is exactly the same criteria as we obtained from the welfare implications of a change in the labor supply of country I. With equation (6.3), we have already established that a change in the technology of different countries has precisely the same effect on the equilibrium relative price of *goods* as does a corresponding change in the labor supply in the different countries. It follows immediately, therefore, that the welfare implications will also be identical. In both cases, the change in the return to an *efficiency* unit of labor is the same. The only difference is that in one instance we increase the quantity of labor, while in the other we increase its productivity.

Let me conclude this section with a brief discussion of the welfare implications of transmitting technical knowledge from one country to another. In addition to analyzing the case where technical change is uniform in each country, D-F-S also examine the case where all technical progress is achieved solely through the transfer of technology from the most efficient producers to the less efficient producers of any good. In the context of their special two country model with identical log linear demand, they demonstrate that as this process occurs, the terms of trade will consistently move against the high wage country. In more general models this need no longer be true. However, they do obtain one striking result which is quite general. When the transfer of technology to the less efficient producers becomes complete, so that all differences in the technology among countries are eliminated, the welfare of that country which initially had the highest wage must have decreased. This result does not depend on the number of countries or any special assumptions on the demand for goods.

The key is to look at how the prices of goods change relative to the wage of the high wage country. Suppose country I is initially the high wage country and normalize wages and prices so that $w_I = 1$. Let $a(x)$ be the labor required to produce good x in the most efficient country and let w_x be the initial wage rate in that country. Then from relation (2.1), we have that the initial price of good x must be less than or equal to $w_x a(x)$. After the diffusion of technical knowledge has become complete, however, the market for labor will clear only if all wage rates rise to equal $w_I = 1$. Since every country can now produce good x with $a(x)$ units of labor, the relation between $p(x)$ and $\bar{p}(x)$, the final price of the good, must satisfy:

$$(6.13)\qquad p(x)\leqslant w_x a(x)\leqslant a(x)=\bar{p}(x).$$

We may conclude, therefore, that the complete diffusion of technical knowledge cannot result in a decrease of any price relative to the wage in country I. In fact, the relative price of any good must strictly increase for which either: (a) the initial price was less than the initial cost of production in that country with the most efficient technology, or (b) initially, country I was not the most efficient producer of that good. Consequently, the welfare of the high wage country must fall. A similar argument can be used to show that the diffusion of technical knowledge must result in an increase in the welfare of the country which initially had the lowest relative wage.

7. TARIFF THEORY

In this section, I use the model to illustrate some implications for the theory of tariffs. The analysis will proceed under several special restrictions. I assume that all tariff rates are initially equal to zero and that all changes in the tariff rates of any country are uniform with respect to all goods imported from the same country. Also, in order to focus on some particularly sharp comparative statics results, I will only analyze the case where all countries have identical homothetic demands and zero transport costs. The problems which emerge when some of these assumptions are relaxed are discussed briefly at the end of this section.

Using equation (3.14) and the definition of ε^i_{jk} we may express the effect of a change in tariff rates, tariff revenue and wages on country i's demand for the labor of country j as

$$(7.1)\qquad \hat{f}^i_j=\sum_k \varepsilon^i_{jk}\hat{w}_k+\beta^i_{jY}\frac{dT^i}{Y^i}+\sum_k(\alpha^i_{jk}+g^i_{jk})\hat{t}^i_k.$$

The tariff revenue, of course, is endogenous. It depends on country i's tariff rates and its demand for the goods produced in each country:

$$(7.2)\qquad T^i=\sum_k(t^i_k-1)\int_{X^i_k}p^i(x)c^i(x)\,dx=\sum_k(t^i_k-1)w_k f^i_k.$$

Assuming $t^i_k=1$ for all k, we obtain upon differentiating (7.2):

$$(7.3)\qquad \frac{dT^i}{Y^i}=\sum_k \theta^k_k\hat{t}^i_k.$$

Substituting (7.3) into (7.1) and using the definition of $\bar{\alpha}^i_{jk}$ then yields

$$(7.4)\qquad \hat{f}^i_j=\sum_k \varepsilon^i_{jk}\hat{w}_k+\sum(\bar{\alpha}^i_{jk}+g^i_{jk})\hat{t}^i_k.$$

When country i imposes a tariff on the goods produced in country k, the effect, before there is any adjustment in the wage rates, is to raise the price that consumers in country i must pay for those goods. By itself, this would also lower the welfare of country i. If the tariff proceeds are rebated back to the consumers in country i, however, real income is restored to its initial level and all that is left is a pure substitution effect resulting from a change in the relative price of goods produced in country k. This is captured in the term $(\bar{\alpha}^i_{jk} + g^i_{jk})$.

If we assume that the derived demand for labor is identical and homothetic across countries, we may write $\varepsilon_{jk} = \varepsilon^i_{jk} = \bar{\alpha}^i_{jk} + g^i_{jk}$ and $\lambda^i_j = \theta_i$ for all i, j, k. Then, using equation (5.2) to sum equation (7.4) over all i, we obtain as a condition for the equilibrium changes in w:

$$(7.5) \qquad \hat{f}_j = \sum_k \varepsilon_{jk}\hat{w}_k + \sum_k \sum_i \theta_i \varepsilon_{jk} \hat{t}^i_k = 0.$$

Again we may let I's labor be numeraire, and use Walras Law to eliminate the equation for $\hat{f}_I$. Similarly, we may use the homogeneity of the pure substitution effect to normalize tariff changes so that $\hat{t}^i_I = 0$ for each country i.[11] Then letting $[\hat{t}]$ represent the $(I-1)\times(I-1)$ matrix whose element in the jth row of the ith column is $\hat{t}^i_j$, and again letting $\theta = (\theta_1, \ldots, \theta_{I-1})$, equation (7.5) may be expressed as

$$(7.6) \qquad E\hat{w} + E[\hat{t}]\theta = 0 \quad \text{or} \quad \hat{w} = -[\hat{t}]\theta.$$

Row by row, equation (7.6) then reduces to:

$$(7.7) \qquad \hat{w}_j = -\sum_i \theta_i \hat{t}^i_j.$$

What this says is that a one per cent increase in the tariff rate of country i levied on goods produced in country j will generate a percentage decrease in the wage of country j relative to every other country equal to the fraction of world expenditure generated by country i. The relative wage of every other country, including country i, will remain unchanged. This result is a consequence of the fact that country i's compensated derived demand function for labor is just the fraction θ_i times the aggregate derived demand function for the entire economy. When country i levies a one per cent tariff on goods produced in country j, therefore, the initial impact on the aggregate demand for labor is exactly θ_i of what it would be if instead the wage of country j had actually risen by one per cent. Consequently, if country j's relative wage is lowered by θ_i per cent, the initial impact on the derived demand for labor will be exactly offset and equilibrium will be restored.

To determine welfare implications of tariff rate changes, under these assumptions, we may substitute (7.3) into (4.2) to obtain

$$(7.8) \qquad \hat{V}^i = -\sum_k \theta_k(\hat{w}_i - \hat{w}_k).$$

Since all tariffs are initially zero, changes in the welfare level of each country are determined completely by the changes in the terms of trade. For the case of

[11] This is the analogue to the familiar result that the effect of a tax on one good is the same as the effect of an equal subsidy on all others.

identical homothetic preferences, we then obtain upon substituting (7.7) into (7.8):

$$\hat{V}^i = \sum_k \theta_k \sum_j \theta_j (\hat{t}^j_k - \hat{t}^j_i). \tag{7.9}$$

Country i benefits from the tariff policy of country j if the average tariff levied by country j weighted by the share of world income of each country is greater than the tariff it levies on country i. The relative importance of the tariff policy of each country is proportional to that country's share of world income.

As a special case, consider again the implications of a one per cent tariff by country j on country I. Then (7.9) implies

$$\begin{aligned} \hat{V}^i &= \theta_I \theta_j \quad \text{for} \quad i \neq I; \\ \hat{V}^I &= -\theta_j(1-\theta_I). \end{aligned} \tag{7.10}$$

The benefit to any country, other than country I, depends only on the size of country I and the size of the country levying the tariff. Otherwise the source of the tariff is unimportant. The loss to country I depends only on the size of the country levying the tariff and the size of the countries which are not taxed.

It should be clear that the basic method of analysis in this section does not depend on the assumption that preferences are identical and homothetic or even that the tariff changes are uniform. For some problems, we might also wish to relax the assumption that all tariff rates are *initially* zero. This can be also accommodated, but only at some cost. As long as we assume that within each country the tariff rates are equal on all goods imported from the same country, the basic analysis remains unchanged. As in any pure exchange mode, however, we must recognize that once tax distortions have been introduced, the change in the welfare of a country depends not only on the change in the terms of trade but also in the resulting change in its demand. This in turn generates a secondary income effect on the country's demand. It is not difficult to incorporate these feedback effects into the derived demand for labor, although once we introduce more than two countries, it is no longer obvious how the qualitative properties of the demand for labor are affected even under very restrictive assumptions.

If we wish to allow for different tariff rates on different goods produced by the same country, however, the analysis becomes more complicated. The labor of each country can only be treated as a composite good as long as the tariff rates on all of the goods it produces are equal. Consequently, if we assume that initially there are different tariff rates on the goods produced in the same country, we can no longer analyze the model as if it were a model of pure exchange. If, for instance, we allow for n different tariff rates, we must analyze the model as if the country produced n different goods. This is not to say that the model becomes untractable, but the basic analogy to the pure exchange model is no longer valid.

8. CONCLUSIONS

The results obtained in the last three sections are by no means exhaustive. They serve primarily to illustrate how the relation between the Ricardian model and a pure exchange model can be exploited to construct a simple framework for analyzing many of the issues in international trade. Admittedly, the formal demonstration of how a general Ricardian model may be reduced to a model of pure exchange is rather tedious. For the most part, however, this is a consequence of the fact that we are working with a very general formulation of the model. When more structure is introduced, as in D-F-S, some of the cumbersome notation can be avoided, and the demonstration becomes much more transparent. In any case, it should be emphasized that there are only two basic principles at work. Except for the specialization effect, all of the properties of the derived demand for labor follow from little more than a statement of the Hicks' Composite Commodity Theorem. This is a direct consequence of the fact that output prices are strictly tied to the wage rates. With respect to the specialization effect, all of its properties follow from the fact that each country purchases its goods from the least cost producer.

This model, of course, may not be appropriate for examining all of the issues in international trade, particularly those issues which focus on the distribution of income, factor intensities, or relative price changes within a country. For other problems, however, it may be convenient to work with a model in which the structure of individual countries is relatively uncomplicated, in order to address more complicated questions about the interactions among countries. For instance, I believe the model might be fruitfully applied to a more detailed examination of the implications of technical change. It has the advantage that relative technical improvement can be unambiguously defined and avoids such complications as relative factor intensities. Moreover, the comparative statics analysis of this model may be extended more easily to three or more countries than in models with more complicated production structures. Similarly, some of the results in the section on tariffs might be extended to re-examine the theory of customs unions where the very nature of the problem requires that the model incorporate at least three countries. Finally, I should also note that it is possible to introduce several factors of production without affecting the analysis as long as we retain the constant returns to scale assumption and assume that the shape of the isoquants are identical for each good within the same country. This permits us to analyze the model as if each good were produced by the same composite factor of production. In [**12**], I used this extended model to re-examine some of the implications of tariff theory for the distribution of income when some of the factors are mobile.

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APPENDIX A

In this appendix, I establish some of the properties of the specialization effect to which I refer in the text. Assume that Assumptions (A1) and (A2) of Section 3.2 are satisfied, and recall that $X_j^i(w)=\{x\in X\colon w_j a_j^i(x)\leqslant w_k a_k^i(x),\quad k=1,\dots,I\}$. As in the text define $B_{jk}^i(w)=\{x\in X_j^i(w)\cap X_k^i(w)\colon d(a_j^i(x)/a_k^i(x))/dx\leqslant 0\}$ to be the set of commodities which are left-hand boundary points for X_j^i and right-hand boundary points for X_k^i, and define $A_{jk}^i=$

$\{x \in X: d(a_j^i(x)/a_k^i(x))/dx = 0\}$. Assumption (A1) states that A_{jk}^i contains only a finite number of commodities for each i, j, k. Therefore, we may prove:

LEMMA 1: *$B_{jk}^i(w)$ is a finite set for each $w \in R_+^I$.*[12]

PROOF: Let $A_{jk}^i = \{x_1, \ldots, x_N\}$ where $x_n > x_{n-1}$ for $1 < n \leqslant N$, and suppose $y_1, y_2 \in B_{jk}^i$ with $y_1 < y_2$. Then, by the definition of B_{jk}^i,

$$\frac{a_j^i(y_1)}{a_k^i(y_1)} = \frac{a_j^i(y_2)}{a_k^i(y_2)},$$

and therefore, by the Mean Value Theorem, there is an $x \in A_{jk}^i \cap (y_1, y_2)$. From this it follows that for any x_n, $1 < n \leqslant N$, there can be at most one $y \in B_{jk}^i$ such that $x_{n-1} < y < x_n$. Since A_{jk}^i is finite, the Lemma is proved. *Q.E.D.*

Two important properties follow from Lemma 1. First, it establishes that for any w, the set of commodities which is produced at least cost in more than one country is negligible. Therefore, $X_j^i(w)$ can be taken as the set of commodities purchased by country i from country j. Secondly, it establishes that for every w, $X_j^i(w)$ can be written as a finite number of intervals (including perhaps some single points).

We may also use Lemma 1 to establish the continuity of derived demand for labor with respect to changes in w. Recall that country i's derived demand for the labor of country j can be expressed as $f_j^i = \int_{X_j^i} a_j^i(x)c^i(x)\,dx$. Let μ be the Lebesque measure on X. Then we may state:

THEOREM 1: *f_j^i is a continuous function of w.*

PROOF: We have assumed in the text that $c^i(x)$ changes differentiably with respect to changes in prices and income and hence from equations (3.4) and (3.6) with respect to wage rates as well. Consequently, all that remains to show is that if we hold the consumption function c^i fixed, the integral $\int_{X_j^i} a_j^i(x)c^i(x)\,dx$ changes continuously with respect to changes in w. As noted in the text, Assumption (A2) implies that $c^i(x)$ is a continuous function x. It is sufficient, therefore, to show that $\mu(X_j^i(w^t)\Delta X_j^i(w^0)) \to 0$[13] for any convergent sequence, $w^t \to w^0$. Let $B_j^i(w) = \bigcup_{k\neq j}(B_{jk}^i \cup B_{kj}^i)$, and for any $\varepsilon > 0$, define $Z_j^i(w^0, \varepsilon) = \{x \in X: |x - y| \geqslant \varepsilon$ for all $y \in B_j^i(w^0)\}$. Since for $k \neq j$, $w_j^0 a_j^i(x) < w_k^0 a_k^i(x)$ for all $x \in Z_j^i(w^0, \varepsilon) \cap X_j^i(w^0)$, there is a $\delta > 0$ such that $(w_j^0 + \delta)a_j^i(x) < (w_k^0 - \delta)a_k^i(x)$ for all $x \in Z_j^i(w^0, \varepsilon) \cap X_j^i(w^0)$. Choose t' sufficient large so that $|w_j^t - w_j^0| < \delta$ for $j = 1, \ldots, I$, and all $t > t'$. Then, by the definition of $X_j^i(w^0)$, $\mu(X_j^i(w^0)\Delta X_j^i(w^t)) < 2\varepsilon \# B_j^i(w)$. Since ε can be chosen arbitrarily small, the theorem then follows from Lemma 1 and the definition of $B^i(w)$. *Q.E.D.*

In the text, we showed that the specialization effect would not only be continuous, but would also be differentiable if the following two conditions were satisfied: (i) $B_{jk}^i(x) \cap B_{jl}^i(w) = \phi$ for all distinct j, k, and l; (ii) if $x \in B_{jk}^i(w)$, then $x \notin A_{jk}^i$. To see why these two restrictions are necessary, consider the example illustrated in Figure 2. The cost of production curves for countries j, k, and l are drawn so that X_j^i is equal to the interval from 0 to x_2 and X_l^i is equal to the interval from x_2 to 1. The set X_k^i, however, is composed of only two isolated points, x_1 and x_2. The problems which arise in this example can be illustrated by considering a small change in w_j and examining the impact on the boundary points of X_j^i.

Consider first the effect on x_2. Here there are three least cost producers of x_2. If w_j increases, then the right-hand boundary of X_j^i moves to the left to an extent determined by the new intersection of the $w_j a_j^i(x)$ curve with the $w_k a_k^i(x)$ curve. In this case, therefore, equation (4.10) implies $dx_2 = \mu_{jk}^i(x_2)\hat{w}_j$. If w_j decreases, however, the boundary of X_j^i moves to the right to a point where the new $w_j a_j^i(x)$ curve intersects the $w_l a_l^i(x)$ curve. Consequently, in this case, we have $dx_2 = \mu_{jl}^i(x_2)\hat{w}_j$. Since it is evident that

$$\frac{da_k^i(x)/dx}{a_k^i(x)} \neq \frac{da_l^i(x)/dx}{a_l^i(x)},$$

it follows by definition that $\mu_{jk}^i \neq \mu_{jl}^i$, and hence, that the right- and left-hand derivatives of x_2 with respect to w_j are not equal.

At the point x_1, the problem is even more serious. This corresponds to the case where $[d(a_j^i(x)/a_k^i(x))/dx] = 0$. If w_j decreases, then the $w_j a_j^i$ curve falls below the $w_k a_k^i$ curve and the set X_j^i

[12] $R_+^I = \{w: w_i > 0$ for $i = 1, \ldots, I\}$.

[13] $X_1 \Delta X_2 \equiv (X_1 - X_2) \cup (X_2 - X_1)$.

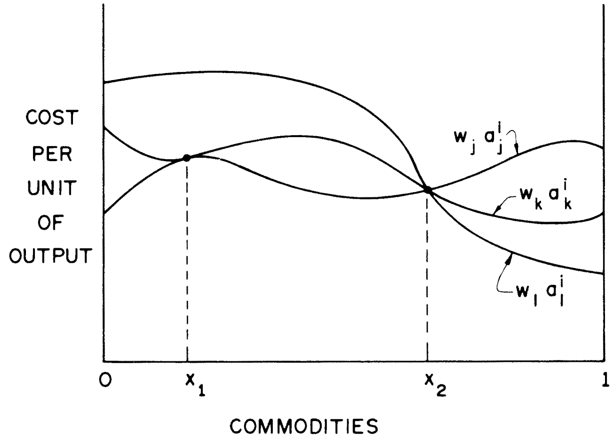

FIGURE 2

is unaffected. If w_j increases, however, an interval around x_1 will emerge in which the cost of production in country j is strictly greater than it is in country k. Consequently, two new boundary points X_j^i appear. Furthermore, because the slopes of the two cost curves are equal at this point, it follows from the definition of μ_{jk}^i, that at these wage rates the rate of change in the boundary points of X_j^i with respect to a change in w_j becomes infinite.

Fortunately, the next lemma establishes that the set of wage rates at which either of these problems can occur is a closed set of Lebesque measure zero. Since we have already established that the derived demand for labor is continuous, we may suppose that it is very unlikely that either of the conditions described above will be violated in equilibrium.[14] Define $W = \{w \in R_+^I: \text{(i) } B_{jk}^i(w) \cap B_{jl}^i(w) = \phi$ for all j, k, l distinct; (ii) $w_j a_j^i(x) = w_k a_k^i(x)$ implies $x \notin A_{jk}^i \cup \{0, 1\}\}$.

LEMMA 2: *The set $R_+^I - W$ is a closed set of Lebesque measure zero as a subset of R_+^I.*

PROOF: For each $w \in R_+^I$ and each $m \leqslant I$, let $\bar{w}^m = (w_1, \ldots, w_m)$; and define F_m as follows:

$$F_1 = \phi.$$

$$F_2 = \{\bar{w}^2: w_1 a_1^i(x) = w_2 a_2^i(x) \text{ for some } i = 1, \ldots, I, \text{ and } x \in A_{1,2}^i\}.$$

For $m > 2$:

$$F_m = \{\bar{w}^m: w_j a_j^i(x) = w_k a_k^i(x) \text{ for some } i = 1, \ldots, I, j, k \leqslant m, j \neq k \text{ and } x \in A_{jk}^i\}$$
$$\cup \{\bar{w}^m: w_j a_j^i(x) = w_k a_k^i(x) = w_l a_l^i(x) \text{ for some } i \in I, j, k, l \leqslant m \text{ and } j \neq k \neq l\}.$$

Let

$$\Phi^m(\bar{w}^m) = \begin{cases} 1 & \text{if } \bar{w}^m \in F_m, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, $F_I = R_+^I - W$. Let μ^I be the Lebesque measure on R^I. Then

$$\mu^I(F_I) = \int_{R_+^I} \Phi^I(w_1, \ldots, w_I)\, dw_1, \ldots, dw_I.$$

Now suppose $\bar{w}^{m-1} \notin F_{m-1}$. Then, by Lemma 1 and Assumption (A1), there are at most a finite number of w_m such that $(\bar{w}^{m-1}, w_m) \in F_m$. If $\bar{w}^{m-1} \in F_{m-1}$, then $(\bar{w}^{m-1}, w_m) \in F_m$ for all w_m. Therefore,

$$\mu^I(F_I) = \int_{R_+^{I-1}} \Phi^{I-1}(w_1, \ldots, w_{I-1})\, dw_1, \ldots, dw_{I-1},$$

[14] Although I have not done so, I expect that an argument similar to the one employed by Debreu [3] can also be used here to make this statement rigorous.

and by induction,

$$\mu^I(F_I)=\int_{R_1}\Phi^1(w_1)\,dw_1=0.$$

Closure follows immediately from (A1) and the definition of W. *Q.E.D.*

Finally, let us turn to some of the properties of the matrix of specialization elasticities. It follows immediately from the definition of g^i_{jk} that

(A.1) $\sum_k g^i_{jk}=0$ for all j.

Equation (A.1) is simply a statement that the pattern of specialization is homogeneous of degree zero in wage rates.

Define $b^i(x)=p^i(x)c^i(x)/Y^i$ to be the proportion of country i's income which is spent on good x. Then it follows from equation (3.2) (assuming $t^i_j(x)=1$ for all x) that $b^i(x)=\theta^i_j d^i_j(x)$ for all $x\in X^i_j$. Also, it may be readily verified that by definition $\mu^i_{jk}(x)+\mu^i_{kj}(x)=0$ for all $x\in B^i_{jk}$. Consequently, we may show that

(A.2) $\sum_j \theta^i_j g^i_{jk}=0$ for all k,

and

(A.3) $\theta^i_j g^i_{jk}=\theta^i_k g^i_{kj}$ for all j, k.

As noted in the text, the specialization elasticities also satisfy $g^i_{jj}<0$ and $g^i_{jk}\geqslant 0$ for all $k\neq j$. Then it follows from equation (A.2) that any submatrix of the matrix $[\theta^i_j g^i_{jk}]$ with the same row and column deleted has a quasi-dominant diagonal. Therefore, from equation (A.2) we may conclude (see McKenzie [**10**, p. 60]) that

(A.4) $\sum_j\sum_k v_j\theta^i_j g^i_{jk}v_k\leqslant 0$ for all $v\in R^I$.

Furthermore the inequality is strict if $v_j=0$ for some j and $v\neq 0$.

Equation (A.2) is essentially a statement of the budget constraint. It requires that the value of the labor demanded by country i will not change as a result in the pattern of specialization. Equation (A.4) states that the specialization matrix is negative semi-definite and equation (A.3) states that the matrix is symmetric. The implication of all four of these equations taken together is that the specialization effect acts just like a pure substitution effect on the derived demand for labor.

APPENDIX B

In this appendix I establish some of the properties of the output elasticities referred to in the text. Formally, our assumptions on the demand for final goods can be summarized as follows. Let (X,χ,μ) be the Lebesgue measure space of commodities and for all $p^i\in L^\infty_+$, $Y^i\in R_+$, the demand, $c^i(p^i,Y^i)\in L^1_+$. Also assume, as in the text, for each $p^i\in L^\infty_+$, $Y^i\in R_+$, $c^i(p^i,Y^i)$ possesses a Frechet differential and that the first order changes in $c^i(x;p^i,Y^i)$ satisfy, for almost all $x\in X$:[15]

$$dc^i(x;p^i,Y^i)=c^i_Y(x;p^i,Y^i)\,dY^i+c^i_p(x;p^i,Y^i)\,dp^i(x)+\int_X c^i_q(x,y;p^i,Y^i)\,dp^i(y)\,dy.$$

Also assume that the budget constraint (3.3)

$$\int_X c^i(x;p^i,Y^i)p^i(x)\,dx=Y^i$$

is satisfied for all p^i, Y^i and that $c^i_Y(\cdot\,;p^i,Y^i)$, $c^i_p(\cdot\,;p^i,Y^i)$, and $c^i_q(\cdot\,;p^i,Y^i)$ are continuous in

[15] Since p^i and c^i refer to equivalence classes of functions in L^∞ and L^1 respectively, any statements about the value of these functions should be understood to hold almost everywhere.

p^i, Y^i [16] and measurable in x and y. Then the elasticities used in the text may be defined as (suppressing the p^i, Y^i arguments)

$$\eta^i_Y(x)=\frac{Y^i}{c^i(x)}c^i_Y(x); \qquad \eta^i_p(x)=\frac{p^i(x)}{c^i(x)}c^i_p(x),$$

and

$$\eta(x,y)=\frac{p^i(y)}{c^i(x)}c^i_q(x,y).$$

If we suppose that the final demand for goods is homogeneous of degree zero in prices and income, then from our expression for the change in $c^i(x)$, we have

$$\eta^i_Y(x)+\eta^i_p(x)+\int_X \eta^i(x,y)\,dy=0. \tag{B.1}$$

As in Appendix A, let $b^i(x)=p^i(x)c^i(x)/Y^i$ denote the proportion of income spent on good x. Then differentiating (3.3) with respect to Y^i yields

$$\int_X b^i(x)\eta^i_Y(x)\,dx=0, \tag{B.2}$$

and differentiating (3.3) with respect to uniform changes in arbitrarily small intervals of prices implies:

$$\int_X b^i(x)\eta^i(x,y)\,dx+b^i(y)\eta^i_p(y)=-b^i(y). \tag{B.3}$$

Recall that $b^i(x)=\theta^i_j d^i_j(x)$ for $x\in X^i_j$. Then we may use the definition of α^i_{jk} and β^i_{jY} to convert equations (B.1) to (B.3) into restrictions on the elasticities of the derived demand for labor: Equation (B.1) becomes

$$\beta^i_{jY}+\sum_k \alpha^i_{jk}=0, \tag{B.4}$$

and equations (B.2) and (B.3) become

$$\sum_j \theta^i_j\beta^i_{jY}=1, \quad \text{and} \tag{B.5}$$

$$\sum_j \theta^i_j\alpha^i_{jk}=-\theta^i_k. \tag{B.6}$$

Also, since the compensated elasticity of demand is defined to be $\bar{\alpha}^i_{jk}=\alpha^i_{jk}+\theta^i_k\beta^i_{jY}$, we have upon substitution into (B.6) and using (B.5), that

$$\sum_j \theta^i_j\bar{\alpha}^i_{jk}=0. \tag{B.7}$$

Finally, we may prove:

THEOREM B1: *Suppose there is a utility function $U\colon L^1_+\to R$ such that for all $(p^i, Y^i)\in L^\infty_+\times R_+$, $\int_X p^i(x)c(x)\,dx\leqslant Y$ implies $U(c^i(p^i,Y^i))>U(c)$ for $c\neq c^i(p^i,Y^i)$. Then the following two conditions must be satisfied*:

$$\theta^i_j\bar{\alpha}^i_{jk}=\theta^i_k\bar{\alpha}^i_{kj}; \tag{B.8}$$

$$\sum_j\sum_k v_j\theta^i_j\bar{\alpha}^i_{jk}v_k\leqslant 0 \quad \text{for all } v\in R^I. \tag{B.9}$$

PROOF: Fix $w^0\in R^I_+$ and fix $X^i_j=X^i_j(w^0)$ for all j. As in (2.2), for each $w\in R^I_+$, define $p^i(x;w)=w_j a^i_j(x)$ for $x\in X^i_j$. Define $e^i\colon R^I_+\times R_+\to R^I$ by

$$e^i_j(w,Y^i)=\int_{X^i_j(w^0)} a^i_j(x)c^i(x;p^i(w),Y^i).$$

Let

$$\bar{E}=\{e\in R^I_+ : e=e^i(w,Y^i) \quad \text{for } (w,Y^i)\in R^I_+\times R_+\}.$$

[16] With respect to the strong topologies.

Then for each $e \in \bar{E}$, there is a unique $c \in L_+^1$ such that $c = c^i(p^i(w^i Y^i))$ for some $(w^i, Y^i) \in R_+^I \times R_+$. To see this, note that if $e^i(w^1, Y^{i1}) = e^i(w^2, Y^{i2})$, then

$$\sum_j (w_j^1 - w_j^2)(e_j^i(w^1, Y^{i1}) - e_j^i(w^2, Y^{i2})) = 0,$$

and therefore

$$\int_X [p^i(x; w^1) - p^i(x; w^2)][c^i(x; p^i(w^1), Y^{i1}) - c^i(x; p^i(w^2), Y^{i2})]\, dx = 0.$$

From this it follows that unless $c^i(p^i(w^1), Y^{i1}) = c^i(p^i(w^2), Y^{i2})$, the hypothesis of the Theorem will be violated.

Therefore, we may define a utility function over $\bar{E}$ by $u^i(e^i(w, Y^i)) = U(c^i(p^i(w), Y^i))$. It may be readily verified that for all w, Y^i: (i) if $we \leqslant Y^i$ and $e \neq e^i(w, Y^i)$, then $u^i(e^i(w, Y^i)) > u^i(e)$; (ii) $w^i e^i(w, Y^i) = Y^i$; and (iii) $e^i(w, Y^i)$ is continuously differentiable. Now fix $Y^{i0} = w_i^0 L_i$ and define

$$s_{jk}^i = \frac{\partial e_j^i(w^0, Y^{i0})}{\partial w_k} + e_k^i(w^0, Y^{i0}) \frac{\partial e_j^i(w^0, Y^{i0})}{\partial Y^i},$$

and let S be the matrix of $[s_{jk}^i]$. Then by Theorem 1 in [**8**], the following conditions are satisfied: (a) $s_{jk}^i = s_{kj}^i$ for all k, j; (b) for any $v \in R_+^I$, $vSv \leqslant 0$.[17] But by definition $e^i(w^0, Y^{i0}) = f^i(w^0)$ from which it follows by the definition of α_{jk}^i and β_{jY}^i, that $\bar{\alpha}_{jk}^i(w^0) = (w_k/f_j^i(w^0))s_{jk}^i$. Relations (B8) and (B9) then follow upon substitution of $\bar{\alpha}_{jk}^i$ into (a) and (b). *Q.E.D.*

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[17] See McKenzie [**9**] for an earlier derivation of this result under similar assumptions.