Market Clearing, Utility Functions, and Securities Prices

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Abstract

We prove the existence of equilibrium in a continuous-time finance model; our results include the case of dynamically incomplete markets as well as dynamically complete markets. In addition, we derive explicitly the stochastic process describing securities prices. The price process depends on the risk-aversion characteristics of the utility function, as well as on the presence of additional sources of wealth (including endowments and other securities). With a single stock, zero endowment in the terminal period, and Constant Relative Risk Aversion (CRRA) utility, the price process is geometric Brownian motion; in essentially any other situation, the price process is *not* geometric Brownian motion.

1 Introduction

Virtually all of the work in continuous-time finance assumes that the prices of securities follow an exogenously specified process, usually either a geometric Brownian motion or some variant of it. But prices of securities are in fact determined day by day and minute by minute by the balancing of supply and demand. A complete model of continuous-time trading requires the derivation of the pricing process as an *equilibrium* determined by more primitive data of the economy, in particular the agents' information, utility functions, and endowments.

To date, the existence of equilibrium in continuous-time finance models has only been established in quite limited special cases. Cox, Ingersoll and Ross [11] provide a very widely used characterization (via a partial differential equation) of equilibrium prices in a general equilibrium model, with dynamically complete markets and multiple (identical) agents, stochastic production and multiple Brownian motions. Since they assume (Assumption A10) that there is a unique function and control satisfying the Bellman equation and a stated regularity condition, their paper does not establish primitive assumptions on the economy sufficient to guarantee the existence of equilibrium. Bick [9] considers a model which is similar to ours in some respects: there is a stock (in positive net supply) which pays a stochastic return in the terminal period T, and zero in periods $t < T$; and a bond (in zero net supply) which pays a deterministic return in the terminal period T and zero in periods $t < T$. He finds necessary and sufficient conditions for a pricing process to be the equilibrium pricing process for a representative agent economy; the conditions are relatively complicated. He and Leland [25] assume the existence of a bond with an exogenously specified interest rate; they derive a necessary and sufficient condition for a pricing process to be an equilibrium pricing process for a stock for a representative agent economy. However, their condition is the existence of a solution of a particular nonlinear Partial Differential Equation. They do not provide any conditions on the primitives of the economy to ensure that a solution of the PDE exists, and hence no conditions on primitives ensuring equilibrium exists. The models in all four papers assume dynamically complete markets, and the proofs depend crucially on that assumption.

In this paper, we establish the existence of equilibrium in a General Equilibrium Incomplete (GEI) markets continuous-time finance model. Generality is not our goal in this paper, and accordingly the model is special in some respects: a single representative agent, with an additively separable, time-independent utility function, and an endowment process which is constant except in the terminal period; to avoid having to handle genericity considerations on existence, we impose a short-sale constraint. The endowment in the terminal period is a function of the terminal value of a d-dimensional Brownian motion β. There are J stocks (with net supply one); stock j pays off only in the terminal period T, when its payoff is $e^{\beta_j(T)}$. There is also a "bond" (with zero net supply) which pays one unit of consumption at date T. [1] Our existence theorem and pricing formula applies equally well to the case in which markets are potentially dynamically complete (i.e., $J = d$) and the case in which markets are necessarily dynamically incomplete ($J < d$).

The principal contribution of this paper is to show that it is possible to explicitly calculate the GEI equilibrium price process in terms of the form of the agent's utility function, and in particular her attitude toward risk; her endowment in the terminal period; and the realizations of the other sources of uncertainty in the economy. With Constant Relative Risk Aversion (CRRA) utility and no endowment in the terminal period, and exactly one source of uncertainty and an associated stock (i.e. $J = d = 1$), equilibrium prices follow a geometric Brownian motion. However, if any of these conditions fail, the stock price process is *not* geometric Brownian motion. For example, with Constant Absolute Risk Aversion (CARA) utility, the distribution of prices at any fixed time $t \in (0, T)$ is *not* log normal.[2] It follows that models that assume that prices follow a geometric Brownian motion and that agents have CARA utility are internally inconsistent.

Even if the agent has CRRA utility, the pricing process will fail to be geometric Brownian motion if the endowment in the terminal period is non-zero. Even if the agent has CRRA utility and zero endowment in the terminal period, the price of each stock will fail to be geometric Brownian motion if

[1] This is best thought of as a long-term zero coupon bond. While the bond pays off with certainty one unit of consumption at the terminal date, its price at dates before t will fluctuate in the equilibrium pricing process. It is thus not risk-free when viewed as a vehicle for transferring consumption across time.

[2] Our derivation of the pricing formula begins with the first order conditions, which are also considered by Bick [9]. However, we use the first order conditions to derive an explicit pricing formula, involving expectations of marginal utility with respect to normal random variables; Bick has no similar formula.

there is more than one stock.

There is a clear intuition why the pricing process fails to be geometric Brownian motion if utility is not CRRA, terminal endowment is non-zero, or there is more than one stock. The agent's tolerance for risk depends on her wealth level, which is affected by the endowment in the terminal period and the performance of all the securities. These wealth effects generate a distortion in the stock price. If the price of a stock is geometric Brownian motion, the volatility is always proportional to the value of the stock; if the stock is the only source of the agent's terminal period wealth, her willingness to hold the stock will be independent of the realization of the stock price. However, if her utility is not CRRA, or there are other sources of terminal period wealth, her willingness to hold the stock will depend on the realization of all the stock prices as well as any information that has been revealed about the terminal period endowment; the stock price must be adjusted to induce her to continue to hold the same amount of the stock. Thus, stocks cannot be priced in isolation from other assets (such as housing) held by agents or from agents' income streams. Our model provides a means to address the relationship among stock prices, other assets, and individual income streams.

At this point, it may be helpful to discuss the relationship of our results to the martingale method of rationalizing arbitrage-free pricing systems. This method was initiated by Harrison and Kreps [24] in the complete markets case, and was subsequently extended to incomplete markets by Back [8], Duffie and Skiadis [23] and others. The essential idea is well-known in finance: if a system of prices does not admit arbitrage, it is a vector martingale with respect to an equivalent probability measure, i.e. a probability which is mutually absolutely continuous with respect to the true objective probability measure. Given a martingale pricing process, one can define a state-dependent felicity function which makes the pricing process an equilibrium for an economy with a representative agent who maximizes expected felicity. This is often understood as saying that "any arbitrage-free pricing system is an equilibrium."

However, in this paper (and the joint work with Anderson cited below), we assert that equilibrium imposes more structure on finance models than that implied by the absence of arbitrage alone. Suppose, for example, we require that the single agent's utility function be the expected utility generated by some state-*independent* felicity function. In that case, the argument just cited that any arbitrage-free pricing system can be justified as an equilibrium will not hold; state-dependence is essential to the proof. Of course, one might object that state-dependence is commonly observed in practice. It is difficult to argue against this objection, but the implication of this objection is not that one should consider a pricing process justified if it can be supported as equilibrium with respect to a felicity function whose state-dependence is carefully chosen to match the peculiarities of the pricing process. The implication is that the state-dependence should be specified as part of the model, and the pricing process should be required to be an equilibrium with respect to the exogenously given state-dependent utility function. Allowing arbitrary state-dependence is a virtue in a result of the form "for all state-dependent felicity functions, ... ;" is is not a virtue in a result of the form "there exists a state-dependent felicity function ... ."

One of the reasons that state-independence appears natural in some models is that the models are partial equilibrium. If a significant portion of household wealth is held in housing, a model that includes stocks but not housing is a partial equilibrium model. Since changes in the value of housing induce wealth effects that alter individuals' willingness to hold stocks, changes in housing values seem, in a stock-only model, to be instances of state-dependent felicity. But in a general equilibrium model which includes both stocks and housing, the state-dependence disappears. In particular, we argue that the relationship between stock pricing and housing can only be properly studied in a general equilibrium model which includes both. More generally, in this paper and the joint companion papers with Anderson, we take the position that all assets and securities should be included in the model, and that felicity functions (and in particular any state-dependence of felicity functions) should be taken as exogenously specified.

This approach has real economic and financial consequences. Our model makes the following specific predictions:

1. In Example 2.9, we consider a situation with zero endowments in the terminal period, and more than one stock. Recall that the terminal dividends of the stocks are given by the terminal values of exponential Brownian motions $e^{\beta_1(T,\omega)}, \ldots, e^{\beta_J(T,\omega)}$. An arbitrage-free pricing formula is given by $p_{A_j}(t,\omega)/p_B(t,\omega) = e^{\beta_j(t,\omega)}$ (the relative price of the j^{th} stock in terms of the bond is the exponential Brownian motion that determines the stock's terminal dividend). However, it is easy to see that this pricing process is *not* an equilibrium with respect to *any*

strictly concave state-independent felicity function. Notice that the given formula for p_{A_j}/p_B is measurable in the filtration generated by β_j, the Brownian motion that generates A_j's dividends. But note that changes in the value of one stock necessarily induce wealth effects that alter the willingness to hold (and hence the equilibrium price) of every other stock. Thus, at equilibrium, the price of a given stock cannot be measurable with respect to the filtration generated by the exponential Brownian motion that drives that stock's dividends; the price of each stock is only measurable in the filtration that drives the dividends of *all* the assets. This observation leads to a specific prediction; the correlation of the returns of two stocks will be different from (and under fairly general conditions less than) the correlation of their underlying dividend streams.

2. The pricing process exhibits stochastic drift and volatility. Since our pricing process is a martingale on the filtration generated by the underlying vector Brownian motion, the Kunita-Watanabe Theorem guarantees that it is a diffusion process with respect to the vector Brownian motion. It is not a geometric Brownian motion because the instantaneous volatility of the diffusion process is stochastic, even though the process that drives the terminal period dividend is geometric Brownian motion (i.e. has deterministic, time-invariant instantaneous volatility). Stochastic instantaneous volatility has been introduced exogenously into finance models in order to better fit actual stock market pricing data; to our knowledge, this is the first time that stochastic instantaneous volatility has been shown to arise endogenously.

3. The presence of non-stock wealth (such as housing or human capital) effects stock pricing in a precisely-defined way.

4. The log of the stock price does not exhibit independent increments.

We note that, while researchers have built finance models which *assume* some of these features in order to provide a better fit to actual stock market data, we are unaware of any finance models which have *derived* these features from economic primitives.

Another important feature of our model is that the interest rate, i.e. the rate of return on the bond, is determined endogenously, rather than being fixed exogenously.

Geometric Brownian motion gives a reasonable but not perfect fit to actual stock market data. The hypothesis that stock prices are geometric Brownian motion is rejected econometrically; see Campbell, Lo and McKinley [10]. Our pricing process provides a class of processes that can be empirically tested to see if they better fit actual stock market data.

This paper is the first of several papers in which Robert Anderson and I intend to address the foundations of Finance from the standpoint of GEI equilibrium. I list and briefly discuss them.

1. We are developing (Anderson and Raimondo [5]) parametric and non-parametric tests for the pricing processes that arise as GEI equilibria.

2. As noted above, in order to focus attention on the form of the pricing processes, we have restricted attention in this paper to a model which is special in a number of ways. We anticipate that the methods of this paper will permit the proof of existence of equilibrium in a much more general continuous-time finance model, including multiple agents, more general securities payouts, no constraint on short sales, and more general utility functions (Anderson and Raimondo [6]).[3]

3. If the pricing process is not geometric Brownian motion, then the Black-Scholes option pricing formula needs to be modified. We address this question in detail in Anderson and Raimondo [7].

We now turn our attention to some methodological aspects of our research. Our goal, ultimately, is to prove the existence of equilibrium and characterize the pricing process in a general multi-agent model, with dynamically incomplete markets. To date, there are no theorems concerning existence of equilibrium with many agents and dynamically incomplete markets. The barrier to obtaining such results has been the absence of a suitable generalization of the fixed-point methods used by Duffie and Shafer [21, 22] in the case of discrete time and state spaces. As we shall discuss below, nonstandard analysis provides an alternative strategy for proving existence. In this paper, we show that it can be used to prove existence in a single-agent model in continuous-time. The existence result in this paper can be

[3] The extension to time-dependent additively separable preferences should pose no problems. The extension to non-additively separable preferences is likely to prove more difficult, because of the problems identified in Hindy and Huang [26, 27] and Hindy, Huang and Kreps [28].

obtained by other methods; for example, given the model and a conjecture of the pricing formula, one can show that it is an equilibrium price using gradient methods and the fact that this is a representative agent model; see, for example, Duffie and Skiadas [23]. Our purpose in this paper, then, is to establish that the nonstandard machinery can be used to prove existence results in a continuous-time model. In another paper (Raimondo [44]), we show that nonstandard methods can be used to prove existence in a multi-agent model in discrete time, the first proof of such a result in a reasonably general setting. In Anderson and Raimondo [6], we provide a specific conjecture for existence of equilibrium in a multi-agent model in continuous time with dynamically incomplete markets. That paper also provides a detailed outline of a proof of the conjecture.

Previous research has assumed that markets are dynamically complete so that the optimal portfolio holdings can be calculated using the Bellman equation, and then conditions for equilibrium can be determined; even then, the conditions for existence have involved endogenous assumptions, such as the existence of a well-behaved solution of a partial differential equation derived from the primitives of the model.

However, there is a well-developed theory of existence of equilibrium in *finite* General Equilibrium Incomplete Markets (GEI) models. Our approach avoids the need to compute the Bellman equation (and thus the need for dynamically complete markets) or to generalize the Duffie-Shafer [21, 22] fixed point argument to infinite-dimensional economies. Nonstandard analysis provides powerful tools to move from discrete to continuous time, and from discrete distributions like the binomial to continuous distributions like the normal; in particular, it provides the ability to transfer computations back and forth between the discrete and continuous settings. Thus, we invoke the Duffie-Shafer existence result for the discrete case and use nonstandard analysis to extend it to the continuous setting.

Anderson [1] provided a construction for Brownian motion and Brownian stochastic integration using nonstandard analysis. In nonstandard analysis, hyperfinite objects are infinite objects which nonetheless possess all the formal properties of finite objects. Anderson's Brownian motion is a hyperfinite random walk which, using a measure-theoretic construction called Loeb measure, can simultaneously be viewed as being a standard Brownian motion in the usual sense of probability theory. While the standard stochastic integral is motivated by the idea of a Stieltjes integral, the actual standard definition of the stochastic integral is of necessity rather indirect because almost every path of Brownian motion is of unbounded variation, and Stieltjes integrals are only defined with respect to paths of bounded variation. However, a hyperfinite random walk is of hyperfinite variation, and hence a Stieltjes integral with respect to it makes perfect sense. Anderson showed that the standard stochastic integral can be obtained readily from this hyperfinite Stieltjes integral.

Anderson's construction of Brownian motion has been used to answer a number of questions in stochastic processes. For the present paper, the most important generalizations of it are the work of Keisler [31] on stochastic differential equations with respect to Brownian motion, and work by Hoover and Perkins [29] and Lindström [34, 35, 36, 37] on stochastic integration with respect to more general martingales. The nonstandard theory of stochastic integration has previously been applied to option pricing in Cutland, Kopp and Willinger [13, 14, 15, 16, 17, 18, 19]. Those papers primarily concern convergence of discrete versions of options to continuous-time versions, and their methods can likely be used to establish convergence results for the option pricing formulas developed in Anderson and Raimondo [7]. Nonstandard analysis has also previously been applied to finance in Khan and Sun [32, 33] to relate the Capital Asset Pricing Model and Arbitrage Pricing Theory in a Single-Period Setting.

Our starting point is a continuous-time model. We use a hyperfinite discretization procedure to construct a model with a hyperfinite number of trading dates, a simple binary tree with a hyperfinite number of nodes. In this setting, the existence of equilibrium follows immediately from Robinson's Transfer Principle and results on existence of equilibrium in GEI models with a finite number of dates and states.[4] We show that the short-sale constraint does not bind at equilibrium. Consequently, as in Magill and Quinzii, we can use the first order conditions to characterize the equilibrium prices. The Central Limit Theorem then allows us to explicitly describe the prices as integrals with respect to a normal distribution. Then, as in the work of Anderson, Keisler, Hoover and Perkins, and Lindström cited above, we use

[4]Magill and Quinzii [40] is an excellent reference on GEI models. Because we assume a short-sales constraint, we can establish existence of a hyperfinite equilibrium using Radner's Theorem [43]. Anderson and I will attempt to eliminate the short-sales constraint when we turn to more general models by using the generic existence result of Duffie and Shafer [21, 22].

the Loeb measure construction to produce an equilibrium of the original continuous-time model; the pricing process generated by that continuous-time equilibrium can be readily determined from the pricing formula in the hyperfinite economy.

While nonstandard analysis plays a central role in the proof, we emphasize that the statement of the theorem is completely standard and can be understood without any knowledge of nonstandard analysis. The equilibrium is an equilibrium of the standard continuous-time model.

2 The Model

In this Section we define the continuous-time model.

1. Trade and consumption occur over a compact time interval $[0, T]$, endowed with a measure λ which agrees with Lebesgue measure on $[0, T)$ and such that $\lambda(\{T\}) = 1$.

2. The information structure is represented by a filtration $\{\mathcal{F}_t : t \in [0, T]\}$ on a probability space $(\Omega, \mathcal{F}, \mu)$. There is a standard d-dimensional Brownian motion $\beta = (\beta_1, \ldots, \beta_d)$ such that β_i is independent of β_j if $i \neq j$ and such that the variance of $\beta_i(t, \cdot)$ is t and $\beta_i(t, \cdot) = E(\beta_i(T, \cdot)|\mathcal{F}_t)$.

3. There is exactly one representative agent. The endowment of the agent satisfies
$$e(t, \omega) = 1 \text{ for all } (t, \omega) \in [0, t] \times \Omega$$
The endowment $e(T, \omega)$ in period T satisfies
$$e(T, \omega) = \rho(\beta_1(T, \omega), \ldots, \beta_d(T, \omega))$$
where $\rho : \mathbf{R}^d \to \mathbf{R}$ is continuous and satisfies
$$0 \leq \rho(x) \leq r + e^{r|x|}$$
for some $r \in \mathbf{R}_+$. The endowment in period $t \in [0, T)$ is interpreted as a rate of flow of endowment, while the endowment in period T is interpreted as a stock or lump. Given a measurable consumption function $c : [0, T] \times \Omega \to \mathbf{R}$, the utility function of the agent is
$$U(c) = E_\mu \left[\int_0^T \varphi_1(c_t) dt + \varphi_2(c_T) \right]$$

where the twice differentiable functions $\varphi_i : \mathbf{R}_{++} \to \mathbf{R}$ $(i = 1, 2)$ satisfy

$$\begin{cases} \varphi_i'(c) & > \ 0 \text{ for } i = 1,2 \\ \varphi_i''(c) & < \ 0 \text{ for } i = 1,2 \\ \varphi_i(z) & \quad \text{is bounded below} \end{cases}$$

Examples of utility functions satisfying the conditions on φ_i are the CARA utilities $\varphi_i(z) = \gamma e^{\alpha z}$ for $\alpha, \gamma < 0$ and the CRRA utilities $\varphi_i(z) = \gamma x^\alpha$ $(0 < \alpha < 1,\ \gamma > 0)$. The assumption that φ_i is bounded below is used at only point in the proof; we conjecture that it can be weakened to $\varphi_2'(z) = O\left(\frac{1}{z^r}\right)$ as $z \to 0$ for some $r \in \mathbf{R}$. If so, the CRRA utility function $\varphi_i(z) = \gamma \ln z$ $(\gamma > 0)$ and the CARA utility $\varphi_i(z) = \gamma x^\alpha$ $(\alpha < 0,\ \gamma < 0)$ would be covered by the theorem.

4. There are $J + 1$ tradable assets, with $0 \le J \le d$: J stocks $A_1, \ldots, A_J$ which pay off[5]

$$A_j(t, \omega) = \begin{cases} 0 & \text{if } t \ne T \\ e^{\beta_j(T,\omega)} & \text{if } t = T \end{cases}$$

and a bond B which pays off

$$B(t, \omega) = \begin{cases} 0 & \text{if } t \ne T \\ 1 & \text{if } t = T. \end{cases}$$

Observe that the payoffs of different stocks are independent. The agent is initially endowed with security holdings $z(0, \omega) = ((1, \ldots, 1), 0)$: one unit of each stock and zero units of the bond. If $J = 0$ (i.e. there are no stocks in the model), we assume that $\rho(x) \ge e^{\alpha \cdot x}$ for some $\alpha \in \mathbf{R}^d$; this will ensure that the income in the terminal period T is not too small.

5. There is a short-sale constraint, i.e. there is some $M > 0$ such that the agent is not permitted to hold less than $-M$ units of either the stock or the bond.

[5] The functional form $A_j(T, \omega) = e^{\beta_j(T,\omega)}$ is not essential. All but one portion of the proof works if $A_j(T, \omega)$ is an arbitrary continuous function of $\beta_j(T, \omega)$ satisfying an exponential growth condition, and that one part works for a large class of functions of β_j, but we have not identified the exact class. Of course, changing the payoff will alter the pricing formula.

6. In order to define the budget set of an agent, we need to have a way of calculating the capital gain the agent receives from a given trading strategy. In other words, we need to impose conditions on prices and strategies that ensure that the stochastic integral of a trading strategy with respect to a price process is defined. The essential requirements are that the trading strategy at time t not depend on information which has not been revealed by time t, and the trading strategy times the variation in the price yields a finite integral. Specifically,

   (a) The σ-algebra of predictable sets, denoted $\mathcal{P}$, is the σ-algebra generated by the sets $\{0\} \times F_0$ and $(s,t] \times F_s$ where $s < t \in \mathbf{R}_+$, $F_0 \in \mathcal{F}_0$, $F_s \in \mathcal{F}_s$ (see Metivier [42]). A stochastic process is said to be predictable if it is measurable with respect to $\mathcal{P}$.

   (b) A security price process is a pair of stochastic processes $p = (p_A, p_B)$, where $p_A = (p_{A_1}, \ldots, p_{A_J})$, and p_{A_j} and p_B are continuous square-integrable martingales with respect to $\{\mathcal{F}_t\}$. p_{A_j} and p_B are priced *cum dividend* at time T. A consumption price process is a stochastic process $p_C(t,\omega)$.

   (c) Given a security price process p, there are unique measures q_{A_j} and q_B on the σ-algebra of predictable sets, which measure the quadratic variation of the components of p; they are generated by

$$\begin{aligned} q_{A_j}((s,t] \times F_s) &= \int_{F_s} (p_{A_j}(t,\omega))^2 - (p_{A_j}(s,\omega))^2 d\mu \\ q_B((s,t] \times F_s) &= \int_{F_s} (p_B(t,\omega))^2 - (p_B(s,\omega))^2 d\mu \end{aligned}$$

   for $s < t$ and $F_s \in \mathcal{F}_s$ and $q(\{0\} \times \Omega) = 0$.

   (d) Given a security price process p, an admissible trading strategy is a pair $(z_A, z_B) : [0,T] \times \Omega \to [-M, \infty) \times [-M, \infty)^d$ such that z_A and z_B are predictable processes and $z_{A_j} \in L^2([0,T] \times \Omega, \mathcal{P}, q_{A_j})$, $z_B \in L^2([0,T] \times \Omega, \mathcal{P}, q_B)$.

7. Given a security price process p and a consumption price process p_C, the budget set is the set of all consumption plans c which satisfy the budget constraint

$$\mathbf{1} \cdot p_A(0) + \int_0^t z dp + \int_0^t p_C(s,\omega)(e(s,\omega) - c(s,\omega)) ds = p(t,\omega) \cdot z(t,\omega)$$

for almost all ω and all $t < T$

$$\mathbf{1}_J \cdot p_A(0) + \int_0^T z dp + \int_0^T p_C(s,\omega)(e(s,\omega) - c(s,\omega))ds$$
$$+(e(T,\omega) + z_A(T,\omega)e^{\beta(T,\omega)} + z_B(T,\omega) - c(T,\omega))p_C(T,\omega)$$
$$= p(T,\omega) \cdot z(T,\omega) \text{ for almost all } \omega$$

for some admissible trading strategy z. We follow standard notation in writing $\mathbf{1} = (1,\ldots,1)$ and

$$\int z dp = \sum_{j=1}^{J} \int z_{A_j} dp_{A_j} + \int z_B dp_B$$

Observe that it is implicit in the definition that $p_C(\cdot,\omega)(e(\cdot,\omega) - c(\cdot,\omega)) \in L^1([0,T])$.

8. Given a price process p, the demand of the agent is a consumption plan and an admissible trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

9. An equilibrium for the economy is a price process p, an admissible trading strategy z and a consumption plan c which lies in the demand set so that the securities and goods markets clear, i.e. for almost all ω

$$\begin{aligned} z_A(t,\omega) &= \mathbf{1} \text{ for all } t \in [0,T] \\ z_B(t,\omega) &= 0 \text{ for all } t \in [0,T] \\ c(t,\omega) &= 1 \text{ for all } t \in [0,T) \\ c(T,\omega) &= e(T,\omega) + \mathbf{1}_J \cdot e^{\beta(T,\omega)} \end{aligned}$$

where $e^{\beta(t,\omega)}$ denotes the vector

$$\left(e^{\beta_1(t,\omega)},\ldots,e^{\beta_d(t,\omega)}\right)$$

and $\mathbf{1}_J = (1,\ldots,1,0,\ldots,0) \in \mathbf{R}^d$ is the vector with J 1's followed by $d - J$ 0's.

Theorem 2.1 *There is a standard probability space* $(\Omega,\mathcal{F},\mu)$*, a filtration* $\mathcal{F}_t$*, and a d-dimensional Brownian motion* $\beta = (\beta_1,\ldots,\beta_d)$ *such that the* *continuous time finance model just described has an equilibrium. The pricing process is given by*

$$\begin{array}{rcl} p_{A_j}(t,\omega) & = & e^{\beta_j(t,\omega)} \int_{-\infty}^{\infty} \varphi_2' \left(F(t,\omega,x)\right) e^{\sqrt{T-t}x_j} d\Phi(x) \\ p_B(t,\omega) & = & \int_{-\infty}^{\infty} \varphi_2' \left(F(t,\omega,x)\right) d\Phi(x) \\ p_C(t,\omega) & = & \varphi_1'(1) \text{ for } t < T \\ p_C(T,\omega) & = & \varphi_2' \left(F(T,\omega,0)\right) \\ \frac{p_{A_j}(t,\omega)}{p_B(t,\omega)} & = & e^{\beta_j(t,\omega)} \frac{\int_{-\infty}^{\infty} \varphi_2'(F(t,\omega,x)) e^{\sqrt{T-t}x_j} d\Phi(x)}{\int_{-\infty}^{\infty} \varphi_2'(F(t,\omega,x)) d\Phi(x)} \end{array} \tag{1}$$

where

$$F(t,\omega,x) = \rho\left(\beta(t,\omega) + \sqrt{T-t}x\right) + \mathbf{1}_J \cdot \left(e^{\beta(t,\omega)+\sqrt{T-t}x}\right)$$

and Φ is the cumulative distribution function of the standard d-dimensional normal.

Remark 2.2 Notice that the price ratio $\frac{p_{A_j}}{p_B}$ is geometric Brownian motion $e^{\beta_j(t,\omega)}$ multiplied by a correction factor; this correction factor is identically one when (and essentially only when) the following conditions are *all* satisfied:

1. the utility function φ_2 exhibits Constant Relative Risk Aversion
2. ρ is identically zero, i.e. the agent has no endowment in the terminal period T
3. $d = 1$, i.e. the uncertainty is described by a single scalar Brownian motion

If any of these conditions is violated the price ratio $\frac{p_{A_j}}{p_B}$ cannot be geometric Brownian motion except possibly under some fluke circumstances that cause the corrections for these three factors to cancel out.

Remark 2.3 It is well known that stochastic differential equations need not have solutions on the probability space and filtration on which they are defined. Roughly speaking, there may not be enough measurable sets to define the solution. As a consequence, existence theorems for solutions of stochastic differential equations take one of two forms: either they assume that the probability space and filtration have certain nice properties to start with and demonstrate the existence of so-called strong solutions, or they work with an arbitrary probability space and filtration and prove the existence of so-called "weak solutions" which can be defined on a richer probability space and filtration. We have chosen to take the former route and assume our probability space and Brownian motion have nice properties to begin with, but our methods will also show that if we begin with an arbitrary probability space and Brownian motion, one can obtain a "weak equilibrium," an equilibrium on a different probability space, filtration, and Brownian motion. Note that any two d-dimensional Brownian motions are essentially the same from the standpoint of probability theory; for example, they have the same finite-dimensional distributions, and they induce the same measure on $C([0,1],\mathbf{R}^d)$. Similarly, every weak equilibrium of the model just described will satisfy the pricing formula given in Equation (1).

Before we turn to the proof, it may be helpful to give several examples of models included in Theorem 2.1 and the pricing processes that prevail in those models.

Example 2.4 Suppose in the setting of Theorem 2.1, we take $J = d = 1$, so the uncertainty in the model is encoded in one scalar Brownian motion β, and there is a stock A whose payoff in the terminal period T is $e^{\beta(T,\cdot)}$. Suppose also that ρ is identically zero, so the agent has no endowment in the terminal period. Markets are dynamically complete. The pricing process is

$$\begin{array}{rcl} p_A(t,\omega) & = & e^{\beta(t,\omega)} \int_{-\infty}^{\infty} \varphi_2' \left(e^{\beta(t,\omega)+\sqrt{T-t}x}\right) e^{\sqrt{T-t}x} d\Phi(x) \\ p_B(t,\omega) & = & \int_{-\infty}^{\infty} \varphi_2' \left(e^{\beta(t,\omega)+\sqrt{T-t}x}\right) d\Phi(x) \\ p_C(t,\omega) & = & \varphi_1'(1) \text{ for } t < T \\ p_C(T,\omega) & = & \varphi_2' \left(e^{\beta(T,\omega)}\right) \\ \frac{p_A(t,\omega)}{p_B(t,\omega)} & = & e^{\beta(t,\omega)} \frac{\int_{-\infty}^{\infty} \varphi_2'\left(e^{\beta(t,\omega)+\sqrt{T-t}x}\right) e^{\sqrt{T-t}x} d\Phi(x)}{\int_{-\infty}^{\infty} \varphi_2'\left(e^{\beta(t,\omega)+\sqrt{T-t}x}\right) d\Phi(x)} \end{array} \tag{2}$$

where Φ is the cumulative distribution function of the standard 1-dimensional normal. If φ_2 is a CRRA utility function, it is an easy calculation that $\frac{p_A(t,\omega)}{p_B(t,\omega)} = e^{\beta(t,\omega)}$. However, if φ_2 is not CRRA, $\frac{p_A}{p_B}$ is not geometric Brownian motion; indeed, with CARA utility, $\frac{p_A(t,\omega)}{p_B(t,\omega)}$ is not log-normal when $0 < t < T$.

Example 2.5 Suppose we modify Example 2.4 by setting ρ to be identically one. The addition of even this deterministic endowment in period T alters the pricing process so it is not geometric Brownian motion, even if φ_2 is CRRA. The pricing process is

$$\begin{array}{rcl} p_A(t,\omega) & = & e^{\beta(t,\omega)} \int_{-\infty}^{\infty} \varphi_2' \left(1 + e^{\beta(t,\omega)+\sqrt{T-t}x}\right) e^{\sqrt{T-t}x} d\Phi(x) \\ p_B(t,\omega) & = & \int_{-\infty}^{\infty} \varphi_2' \left(1 + e^{\beta(t,\omega)+\sqrt{T-t}x}\right) d\Phi(x) \\ p_C(t,\omega) & = & \varphi_1'(1) \text{ for } t < T \\ p_C(T,\omega) & = & \varphi_2' \left(1 + e^{\beta(T,\omega)}\right) \\ \frac{p_A(t,\omega)}{p_B(t,\omega)} & = & e^{\beta(t,\omega)} \frac{\int_{-\infty}^{\infty} \varphi_2'\left(1+e^{\beta(t,\omega)+\sqrt{T-t}x}\right) e^{\sqrt{T-t}x} d\Phi(x)}{\int_{-\infty}^{\infty} \varphi_2'\left(1+e^{\beta(t,\omega)+\sqrt{T-t}x}\right) d\Phi(x)} \end{array} \quad (3)$$

where Φ is the cumulative distribution function of the standard 1-dimensional normal.

Example 2.6 Suppose in the setting of Theorem 2.1, we take $J = 0$ and $d = 1$, so the uncertainty in the model is encoded in one scalar Brownian motion β, but there is no associated stock available to be traded. Suppose also that $\rho(x) = e^x$, so the agent has endowment $e^{\beta(T,\cdot)}$ in the terminal period T. The agent's endowment in period T is the same as the dividend that the stock A in Example 2.4 paid in period T; in particular, the equilibrium consumptions will be the same in the two examples. However, because one cannot trade a stock tied to the Brownian motion, markets are dynamically incomplete.[6] At the equilibrium prices, market clearing requires that the holding z_B of the bond be identically zero. The pricing formula is

$$\begin{array}{rcl} p_B(t,\omega) & = & \int_{-\infty}^{\infty} \varphi_2' \left(e^{\beta(t,\omega)+\sqrt{T-t}x}\right) d\Phi(x) \\ p_C(t,\omega) & = & \varphi_1'(1) \text{ for } t < T \\ p_C(T,\omega) & = & \varphi_2' \left(e^{\beta(T,\omega)}\right) \end{array} \quad (4)$$

where Φ is the cumulative distribution function of the standard 1-dimensional normal.

Example 2.7 Suppose we modify Example 2.4 by eliminating the bond. This does not fall under Theorem 2.1 as stated, but essentially the same

[6] The bond can be used to move consumption across states in the terminal period T, but only by also changing consumption at periods $t < T$. For example, the agent could purchase the bond at time t if $\beta(t,\omega)$ is low, and sell the bond short at time t if $\beta(t,\omega)$ is high. If the agent follows such a trading stategy, consumption in period T is smoothed, at the cost of making consumption at periods $t < T$ more variable.

argument guarantees that equilibrium exists.[7] In Example 2.4, the equilibrium trading strategy prescribes that the agent holds zero units of the bond at all (t,ω); accordingly, the equilibrium consumptions and stock holdings in this example are the same as those in Example 2.4 paid in period T; in particular, the equilibrium consumptions will be the same in the two examples. However, in this example, there is no market for the bond. Markets are dynamically incomplete.[8] At the equilibrium prices, market clearing requires that the holding z_A of the stock be identically one. The pricing formula is

$$\begin{array}{rcl} p_A(t,\omega) &=& e^{\beta(t,\omega)} \int_{-\infty}^{\infty} \varphi_2' \left(e^{\beta(t,\omega)+\sqrt{T-t}x}\right) e^{\sqrt{T-t}x} d\Phi(x) \\ p_C(t,\omega) &=& \varphi_1'(1) \text{ for } t < T \\ p_C(T,\omega) &=& \varphi_2'\left(e^{\beta(T,\omega)}\right) \end{array} \tag{5}$$

where Φ is the cumulative distribution function of the standard 1-dimensional normal.

Example 2.8 Suppose in the setting of Theorem 2.1, we take $J = 1$ and $d = 2$, so the uncertainty in the model is encoded in a two-dimensional Brownian motion $\beta = (\beta_1, \beta_2)$, and there is a single stock A whose payoff in the terminal period T is $e^{\beta_1(T,\cdot)}$. Suppose also that $\rho(x) = e^{x_2}$, so the agent's endowment in period T is $e^{\beta_2(T,\cdot)}$. Since $m < d$, the markets are dynamically incomplete. The pricing process is

$$\begin{array}{rcl} p_A(t,\omega) &=& e^{\beta(t,\omega)} \int_{-\infty}^{\infty} \varphi_2' \left(e^{\beta_1(t,\omega)+\sqrt{T-t}x_1} + e^{\beta_2(t,\omega)+\sqrt{T-t}x_2}\right) e^{\sqrt{T-t}x_1} d\Phi(x) \\ p_B(t,\omega) &=& \int_{-\infty}^{\infty} \varphi_2' \left(e^{\beta_1(t,\omega)+\sqrt{T-t}x_1} + e^{\beta_2(t,\omega)+\sqrt{T-t}x_2}\right) d\Phi(x) \\ p_C(t,\omega) &=& \varphi_1'(1) \text{ for } t < T \\ p_C(T,\omega) &=& \varphi_2'\left(e^{\beta_1(T,\omega)} + e^{\beta_2(T,\omega)}\right) \\ \frac{p_A(t,\omega)}{p_B(t,\omega)} &=& e^{\beta(t,\omega)} \frac{\int_{-\infty}^{\infty} \varphi_2'\left(e^{\beta_1(t,\omega)+\sqrt{T-t}x_1} + e^{\beta_2(t,\omega)+\sqrt{T-t}x_2}\right) e^{\sqrt{T-t}x_1} d\Phi(x)}{\int_{-\infty}^{\infty} \varphi_2'\left(e^{\beta_1(t,\omega)+\sqrt{T-t}x_1} + e^{\beta_2(t,\omega)+\sqrt{T-t}x_2}\right) d\Phi(x)} \end{array} \tag{6}$$

[7] The only change, other than notation, is that in the construction of the consumption plan $\hat{c}$ and admissible trading strategy $\hat{y}$, one must buy our sell units of the stock rather than units of the bond.

[8] The stock can be used to move consumption across states in the terminal period T, but only by also changing consumption at periods $t < T$. For example, the agent could purchase additional units of the stock at time t if $\beta(t,\omega)$ is low, and sell some units of the stock at time t if $\beta(t,\omega)$ is high. If the agent follows such a trading stategy, consumption in period T is smoothed, at the cost of making consumption at periods $t < T$ more variable.

where Φ is the cumulative distribution function of the standard 2-dimensional normal. Notice that although the dividend of A (which equals $\frac{p_A(T,\cdot)}{p_B(T,\cdot)}$) is independent of β_2, $p_A(t,\cdot)$ and $\frac{p_A(t,\cdot)}{p_B(t,\cdot)}$ depend on both β_1 and β_2 for $t < T$. Even if φ_2 is CRRA, $\frac{p_A}{p_B}$ is not geometric Brownian motion.

Example 2.9 Suppose in the setting of Theorem 2.1, we take $J = d = 2$, so the uncertainty in the model is encoded in a two-dimensional Brownian motion $\beta = (\beta_1, \beta_2)$; for each β_j, there is a stock A_j whose payoff in the terminal period T is $e^{\beta_j(T,\cdot)}$. Suppose also that ρ is identically zero, so the agent's endowment in period T is zero. The markets are dynamically complete. The pricing process is

$$\begin{array}{rcl} p_{A_j}(t,\omega) & = & e^{\beta_j(t,\omega)} \int_{-\infty}^{\infty} \varphi_2' \left(e^{\beta_1(t,\omega)+\sqrt{T-t}x_1} + e^{\beta_2(t,\omega)+\sqrt{T-t}x_2} \right) e^{\sqrt{T-t}x_j} d\Phi(x) \\ p_B(t,\omega) & = & \int_{-\infty}^{\infty} \varphi_2' \left(e^{\beta_1(t,\omega)+\sqrt{T-t}x_1} + e^{\beta_2(t,\omega)+\sqrt{T-t}x_2} \right) d\Phi(x) \\ p_C(t,\omega) & = & \varphi_1'(1) \text{ for } t < T \\ p_C(T,\omega) & = & \varphi_2' \left(e^{\beta_1(T,\omega)} + e^{\beta_2(T,\omega)} \right) \\ \frac{p_{A_j}(t,\omega)}{p_B(t,\omega)} & = & e^{\beta_j(t,\omega)} \frac{\int_{-\infty}^{\infty} \varphi_2' \left(e^{\beta_1(t,\omega)+\sqrt{T-t}x_1} + e^{\beta_2(t,\omega)+\sqrt{T-t}x_2} \right) e^{\sqrt{T-t}x_j} d\Phi(x)}{\int_{-\infty}^{\infty} \varphi_2' \left(e^{\beta_1(t,\omega)+\sqrt{T-t}x_1} + e^{\beta_2(t,\omega)+\sqrt{T-t}x_2} \right) d\Phi(x)} \end{array} \tag{7}$$

where Φ is the cumulative distribution function of the standard 2-dimensional normal. Notice that although the dividend of A_1 (which equals $\frac{p_{A_1}(T,\cdot)}{p_B(T,\cdot)}$) is independent of β_2, $p_{A_j}(t,\cdot)$ and $\frac{p_{A_j}(t,\cdot)}{p_B(t,\cdot)}$ depend on both β_1 and β_2 for $t < T$. The larger β_1 is, the smaller is p_{A_2}, while typically expect larger β_1 to result in higher p_{A_1}; thus, we would typically expect a negative correlation between p_{A_1} and p_{A_2}, even though they are driven by independent dividend processes. Even though the endowment in period T is zero, and even if φ_2 is CRRA, $\frac{p_A}{p_B}$ is not geometric Brownian motion.

3 Proof

Up to now, all of our definitions and results have been stated without any reference to nonstandard analysis. Our proof makes extensive use of nonstandard analysis, in particular Anderson's construction of Brownian Motion and the Itô Integral ([1]) and Lindström's extension of that construction to stochastic integrals with respect to L^2 martingales [34, 35, 36, 37]. It is beyond the scope of this paper to develop these methods; excellent references are Anderson [3] and Hurd and Loeb [30].

We construct our probability space, filtration and Brownian Motion following Anderson's construction [1]. Specifically, we construct a hyperfinite economy as follows:

1. Choose $n \in {}^*\mathbf{N} \setminus \mathbf{N}$. For $t \in [0,T]$, define $\hat{t} = \frac{[nt]}{n}$; in particular, $\hat{T} = \frac{[nT]}{n}$. Define $\Delta T = \frac{1}{n}$, $\mathcal{T} = \{0, \Delta T, 2\Delta T, \ldots, \hat{T}\}$, $\hat{\Omega} = \left(\{-1,1\}^d\right)^{\mathcal{T}\setminus\{\hat{T}\}}$. If $s \in \mathcal{T} \setminus \{\hat{T}\}$, we write $\omega_s = (\omega_{s1}, \ldots, \omega_{sd})$.

   The internal hyperfinite measure $\hat{\mu}$ on $\hat{\Omega}$ is given by
   $$\hat{\mu}(A) = \frac{|A|}{|\hat{\Omega}|}$$
   for every $A \in \hat{\mathcal{F}}$, the algebra of all internal subsets of $\hat{\Omega}$. For $t \in \mathcal{T}$, $\hat{\mathcal{F}}_t$ is the algebra of all internal subsets of $\hat{\Omega}$ that respect the equivalence relation $\omega \sim_t \omega' \Leftrightarrow \omega_s = \omega_t$ for all $s \leq t$. The internal measure $\hat{\lambda}$ on $\mathcal{T}$ is given by $\hat{\lambda}(\{t\}) = \Delta T$ if $t < \hat{T}$ and $\hat{\lambda}(\{\hat{T}\}) = 1$.

2. For $i = 1, \ldots, m$, define $\hat{\beta}_j : \mathcal{T} \times \hat{\Omega} \to {}^*\mathbf{R}$ by
   $$\hat{\beta}_j(t,\omega) = \sum_{s \leq t,\, s \in \mathcal{T}} \frac{\omega_{si}}{\sqrt{n}} \text{ and } \hat{\beta}(t,\omega) = (\hat{\beta}_1(t,\omega), \ldots, \hat{\beta}_d(t,\omega))$$
   Thus, $\hat{\beta}$ is a d-dimensional hyperfinite random walk.

3. Let $\Omega = \hat{\Omega}$ and let $(\Omega, \mathcal{F}, \mu)$ be the (complete) Loeb measure generated by $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mu})$ (Loeb [38]). Although $(\Omega, \mathcal{F}, \mu)$ is generated by a nonstandard construction, Loeb showed that it is a probability space in the usual standard sense. Let $\mathcal{F}_t$ be the σ-algebra of all elements of $\mathcal{F}$ which respect the equivalence relation $\sim_t$.

4. Let $\beta : [0,T] \times \Omega \to \mathbf{R}^d$ be defined by $\beta(t,\omega) = {}^\circ\left(\hat{\beta}\left(\frac{[nt]}{n}, \omega\right)\right)$. Anderson [1] showed that β is a d-dimensional Brownian motion in the usual standard sense, and that $\beta(t,\cdot) = E(\beta(T,\cdot)|\mathcal{F}_t)$.

5. For all $\omega \in \hat{\Omega}$, define $\hat{e}(t,\omega) = e(t,\omega) = 1$ for all $t \in \mathcal{T}$, $t < \hat{T}$ and $\hat{e}(\hat{T},\omega) = {}^*\rho(\beta(\hat{T},\omega))$.

6. For all $\omega \in \hat{\Omega}$, define $\hat{A}(t,\omega) = A(t,\omega) = \hat{B}(t,\omega) = B(t,\omega) = 0$ for all $t < \hat{T}$, and $\hat{A}(\hat{T},\omega) = e^{\hat{\beta}(\hat{T},\omega)}$ (i.e. $\hat{A}_j(\hat{T},\omega) = e^{\hat{\beta}_j(\hat{T},\omega)}$, $i = 1,\ldots m$) $A(T,\omega) = e^{\beta(T,\omega)}$, $\hat{B}(\hat{T},\omega) = B(T,\omega) = 1$. Note that $A(T,\omega) = {}^\circ\hat{A}(\hat{T},\omega)$ for μ-almost all ω.

7. Given an internal consumption plan $\hat{c}$, the agent's utility is

$$\hat{U}(\hat{c}) = E_{\hat{\mu}}\left(\left(\Delta T \sum_{s\in\mathcal{T},s<\hat{T}} {}^*\varphi_1(\hat{c}(t,\omega))\right) + {}^*\varphi_2(\hat{c}(\hat{T},\omega))\right)$$

8. A security price is an internal function $\hat{p} = (\hat{p}_A,\hat{p}_B) : \mathcal{T}\times\hat{\Omega} \to {}^*\mathbf{R}^m_+ \times {}^*\mathbf{R}_+$. A consumption price is an internal function $\hat{p}_C : \mathcal{T}\times\hat{\Omega} \to {}^*\mathbf{R}_+$.

9. An admissible trading strategy is $\hat{z} = (\hat{z}_A,\hat{z}_B) : \mathcal{T}\times\hat{\Omega} \to {}^*\mathbf{R}^m \times {}^*\mathbf{R}$ which satisfies the short-sale constraint $\hat{z}(t,\omega) \geq ((-M,\ldots,-M),-M)$ for all t,ω and such that $\hat{z}(t,\cdot)$ is $\hat{\mathcal{F}}_t$-measurable.

10. A consumption plan is an internal function $\hat{c} : \mathcal{T}\times\hat{\Omega} \to {}^*\mathbf{R}_+$. The budget set is the set of all consumption plans which satisfy the budget constraint

$$\begin{aligned}
\hat{p}_A(0) \quad &+ \sum_{s\in\mathcal{T},s<t} (\hat{z}(s,\omega)\cdot(\hat{p}(s+\Delta T,\omega) - \hat{p}(s,\omega))) \\
&+ \sum_{s\in\mathcal{T},s<t} \Delta T\,(\hat{p}_C(s,\omega)(\hat{e}(s,\omega) - \hat{c}(s,\omega))) = \hat{z}(t,\omega)\cdot\hat{p}(t,\omega) \\
&\quad \text{for all } t\in\mathcal{T} \text{ and all } \omega\in\hat{\Omega} \\
\hat{p}_A(0) \quad &+ \sum_{s\in\mathcal{T},s<\hat{T}} (\hat{z}(s,\omega)\cdot(\hat{p}(s+\Delta T,\omega) - \hat{p}(s,\omega))) \\
&+ \sum_{s\in\mathcal{T},s<\hat{T}} \Delta T\,(\hat{p}_C(s,\omega)(\hat{e}(s,\omega) - \hat{c}(s,\omega))) \\
&+ \hat{p}_C(\hat{T},\omega)\left(\hat{e}(\hat{T},\omega) + \mathbf{1}_J\cdot\left(z_A(T,\omega)e^{\hat{\beta}(\hat{T},\omega)}\right) + z_B(\hat{T},\omega) - \hat{c}(\hat{T},\omega)\right) \\
&= \hat{z}(T,\omega)\cdot\hat{p}(T,\omega) \\
&\quad \text{for all } \omega\in\hat{\Omega}
\end{aligned}$$

for some admissible trading strategy $\hat{z}$.

11. Given a security price $\hat{p}$ and a consumption price $\hat{p}_C$, the demand of the agent is a consumption plan and an admissible trading strategy which satisfy the budget constraint and such that the consumption plan maximizes utility over the budget set.

12. An equilibrium for the economy is a security price process $\hat{p}$, a consumption price process $\hat{p}_C$, an admissible trading strategy $\hat{z}$ and a consumption plan $\hat{c}$ which lies in the demand set so that the securities and goods markets clear, i.e. for all ω

$$\begin{aligned}
\hat{z}_A(t,\omega) &= \mathbf{1} \text{ for all } t < \hat{T} \\
\hat{z}_B(t,\omega) &= 0 \text{ for all } t < \hat{T} \\
\hat{c}(t,\omega) &= 1 \text{ for all } t < \hat{T} \\
\hat{c}(\hat{T},\omega) &= \hat{e}(\hat{T}) + \mathbf{1}_J \cdot e^{\hat{\beta}(\hat{T},\omega)} = e(T) + \mathbf{1}_J \cdot e^{\hat{\beta}(\hat{T},\omega)}
\end{aligned}$$

Theorem 3.1 *The hyperfinite economy just described has an equilibrium. The pricing process is given by*

$$\begin{aligned}
\hat{p}_{A_j}(t,\omega) &= e^{\hat{\beta}_j(t,\omega)} \int_{{}^*\mathbf{R}} {}^*\varphi_2' \left(\hat{F}(t,\omega,x)\right) e^{\sqrt{\hat{T}-t}x_j} d\hat{\Phi}(x) \\
\hat{p}_B(t,\omega) &= \int_{{}^*\mathbf{R}} {}^*\varphi_2' \left(\hat{F}(t,\omega,x)\right) d\hat{\Phi}(x) \\
\hat{p}_C(t,\omega) &= \varphi_1'(1) \textit{ for } t < T \\
\hat{p}_C(\hat{T},\omega) &= {}^*\varphi_2'(\hat{F}(\hat{T},\omega,0)) \\
\frac{\hat{p}_A(t,\omega)}{\hat{p}_B(t,(\omega))} &= e^{\hat{\beta}(t,\omega)} \frac{\int_{{}^*\mathbf{R}} {}^*\varphi_2'\left(\hat{F}(t,\omega,x)\right) e^{\sqrt{\hat{T}-t}x} d\hat{\Phi}(x)}{\int_{{}^*\mathbf{R}} {}^*\varphi_2'\left(\hat{F}(t,\omega,x)\right) d\hat{\Phi}(x)}
\end{aligned}$$

where

$$\hat{F}(t,\omega,x) = {}^*\rho\left(\hat{\beta}(t,\omega) + \sqrt{\hat{T}-t}x\right) + \mathbf{1}_J \cdot \left(e^{\hat{\beta}(t,\omega)} e^{\sqrt{\hat{T}-t}x}\right)$$

and $\hat{\Phi}$ is the cumulative distribution function of the d-dimensional normalized binomial distribution, each of whose components is distributed as

$$\sqrt{\frac{4\Delta T}{\hat{T}-t}}\, {}^*b\left(\left(\frac{\hat{T}-t}{\Delta T}, \frac{1}{2}\right) - \frac{\hat{T}-t}{2\Delta T}\right)$$

Proof: If we replace "hyperfinite" by "finite" everywhere in the definition of the hyperfinite economy, this is just a finite GEI economy with a short sale constraint. By Radner [43], the finite GEI economy has an equilibrium; by Robinson's Transfer Principle, the hyperfinite economy has an equilibrium. The market clearing condition $\hat{z} = ((1, \dots, 1), 0)$ guarantees that, at equilibrium, the short sale constraint is not binding.[9] As we shall see, this allows us to invoke the first order conditions, as in Magill and Quinzii [40].

The market clearing conditions on $\hat{c}$ guarantee that consumption is positive in every period. Since the security payoffs are nonnegative for all (t, ω) and strictly positive for (T, ω) for all ω, the absence of arbitrage guarantees that $p_A(t, \omega) \gg 0$ and $p_B(t, \omega) > 0$ for all (t, ω).

Since $\hat{c}(t, \omega) > 0$ for all (t, ω), the agent can adjust consumption at time t_0 and state ω_0 by a sufficiently small infinitesimal without violating the nonnegativity constraint at any (t, ω). Specifically, the agent can do either of the following without violating the nonnegativity constraint:

1. The agent can reduce consumption at (t_0, ω_0) by a sufficiently small infinitesimal α, buy $\frac{\alpha \hat{p}_C(t_0,\omega_0)}{\hat{p}_{Aj}(t_0,\omega_0)}$ units of stock A_j (or reduce her short position in the stock by that same number of units) and hold these units in addition to the holdings prescribed by $\hat{z}$, so that consumption is unchanged in periods t with $t_0 < t < \hat{T}$, and consumption in period $\hat{T}$ is increased by $\alpha e^{\beta_j(\hat{T},\omega)}$ for all $\omega \sim_{t_0} \omega_0$.

2. The agent can increase consumption at (t_0, ω_0) by a sufficiently small infinitesimal α, sell $\frac{\alpha \hat{p}_C(t_0,\omega_0)}{\hat{p}_{Aj}(t_0,\omega_0)}$ units of stock A_j (or increase her short position in the stock by that same number of units) and hold this number of units fewer than the holdings prescribed by $\hat{z}_i$, so that consumption is unchanged in periods t with $t_0 < t < \hat{T}$, and consumption in period $\hat{T}$ is decreased by $\alpha e^{\beta_j(\hat{T},\omega)}$ for all $\omega \sim_{t_0} \omega_0$.

The agent can also make analogous changes to her trading strategy for the bond B without violating the nonnegativity constraint. Therefore, the first

[9] This doesn't rule out the possibility that, at the price $\hat{p}$, the agent would prefer a trading strategy $\hat{z}$ which violates the short sale constraint; only that the equilibrium trading strategy is at least as good as any trading strategy which does satisfy the short sale constraint; and the equilbrium trading strategy never hits the short sale constraint. This is enough to invoke the first order conditions for the price.

order conditions and the fact that $\hat{B}(\hat{T},\omega)$ is identically one imply that

$$\begin{array}{rcl} \frac{\hat{p}_A(t,\omega)}{\hat{p}_C(t,\omega)} & = & \frac{E(\varphi_2'(\hat{c}(\hat{T},\cdot))\hat{A}(\hat{T},\cdot)|(t,\omega))}{\varphi_1'(\hat{c}(t,\omega))} \text{ for } t<\hat{T} \\ \frac{\hat{p}_A(\hat{T},\omega)}{\hat{p}_C(\hat{T},\omega)} & = & \frac{E(\varphi_2'(\hat{c}(\hat{T},\cdot))\hat{A}(\hat{T},\cdot)|(\hat{T},\omega))}{\varphi_2'(\hat{c}(\hat{T},\omega))} = \hat{A}(\hat{T},\omega) \\ \frac{\hat{p}_B(t,\omega)}{\hat{p}_C(t,\omega)} & = & \frac{E(\varphi_2'(\hat{c}(\hat{T},\cdot)))|(t,\omega))}{\varphi_1'(\hat{c}(t,\omega))} \text{ for } t<\hat{T} \\ \frac{\hat{p}_B(\hat{T},\omega)}{\hat{p}_C(\hat{T},\omega)} & = & \frac{E(\varphi_2'(\hat{c}(\hat{T},\cdot)))|(\hat{T},\omega))}{\varphi_2'(\hat{c}(\hat{T},\omega))} = 1 \end{array}$$

so normalizing prices by setting $\hat{p}_C(t,\omega) = \varphi_1'(\hat{c}(t,\omega)) = \varphi_1'(1)$ for $t < \hat{T}$ and $\hat{p}_C(\hat{T},\omega) = \varphi_2'(\hat{c}(\hat{T},\omega))$, and imposing the market-clearing condition $\hat{c}(\hat{T},\omega) = \hat{e}(\hat{T},\omega) + \mathbf{1}_J \cdot e^{\hat{\beta}(\hat{T},\omega)}$, we have

$$\begin{array}{rcl} \hat{p}_A(t,\omega) & = & E(\varphi_2'(\hat{e}(\hat{T},\cdot)+\hat{A}(\hat{T},\cdot))\hat{A}(\hat{T},\cdot)|(t,\omega)) \\ \hat{p}_B(t,\omega) & = & E(\varphi_2'(\hat{e}(\hat{T},\cdot)+\hat{A}(\hat{T},\cdot))\hat{B}(\hat{T},\cdot)|(t,\omega)) \\ \frac{\hat{p}_A(t,\omega)}{\hat{p}_B(t,\omega)} & = & \frac{E(\varphi_2'(\hat{e}(\hat{T},\cdot)+\hat{A}(\hat{T},\cdot))\hat{A}(\hat{T},\cdot)|(t,\omega))}{E(\varphi_2'(\hat{e}(\hat{T},\cdot)+\hat{A}(\hat{T},\cdot))\hat{B}(\hat{T},\cdot)|(t,\omega))} \end{array} \tag{8}$$

The result then follows from the observation that the conditional distribution $(\hat{\beta}(\hat{T},\cdot)|(t,\omega))$ is $\hat{\beta}(t,\omega)$ plus the d-dimensional normalized binomial $2\sqrt{\Delta T}{}^*b\left(\left(\frac{\hat{T}-t}{\Delta T},\frac{1}{2}\right)-\frac{\hat{T}-t}{2\Delta T}\right)$, so the conditional distribution $(\hat{A}(t,\cdot)|(t,\omega))$ is $e^{\hat{\beta}(t,\omega)}e^{\sqrt{\hat{T}-t}X}$ where X has the cumulative distribution function $\hat{\Phi}$. ■

Before proceeding, we need the following technical result. The standard version of this result is well known among probabilists, but we have been unable to find a reference in the literature. Accordingly, we will give a full proof of the nonstandard version.

Proposition 3.2 . *Suppose* $a \in \mathrm{ns}({}^*\mathbf{R}^d)$. *Let*

$$f_a(x) = e^{a\cdot x} = e^{a_1x_1+\cdots+a_dx_d}$$

Then $f_a \in SL^1({}^*\mathbf{R}^d, d\hat{\Phi})$.

Proof: First note that if $d=1$,

$$\begin{array}{l} \left(\frac{1}{2^n}\right)\sum_{j=0}^{n}\binom{n}{j}e^{a\left(\frac{2j-n}{\sqrt{n}}\right)} \\ = \left(\frac{e^{a/\sqrt{n}}+e^{-a/\sqrt{n}}}{2}\right)^n \end{array}$$

$$
\begin{aligned}
&= \left(1+\frac{(a/\sqrt{n})^2}{2!}+\frac{(a/\sqrt{n})^4}{4!}+\cdots\right)^n \\
&= \left(1+\frac{a^2}{2n}+O\left(\frac{1}{n^2}\right)\right)^n \\
&= \left(1+\frac{a^2}{2n}\right)^n+O\left(\frac{n}{n^2}+\frac{n^2}{n^4}+\cdots+\frac{n^n}{n^{2n}}\right) \\
&\simeq e^{a^2/2}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{{}^*\mathbf{R}^d} f(x)d\hat{\Phi} \\
&= \left(\frac{1}{2^n}\right)^d \sum_{j_1=0}^{n}\cdots\sum_{j_d=0}^{n} \binom{n}{j_1}\times\cdots\times\binom{n}{j_d} e^{a_1\left(\frac{2j_1-n}{\sqrt{n}}\right)+\cdots+a_d\left(\frac{2j_d-n}{\sqrt{n}}\right)} \\
&= \prod_{k=1}^{d}\left(\frac{1}{2^n}\right)\sum_{j_k=0}^{n}\binom{n}{j_k} e^{a_k\left(\frac{2j_k-n}{\sqrt{n}}\right)} \\
&\simeq \prod_{k=1}^{d} e^{a_k^2/2} \\
&= e^{|a|^2/2}
\end{aligned}
$$

It is well known that if $a \in \mathbf{R}^d$,

$$\int f(x)d\Phi = e^{|a|^2/2}$$

Since the binomial distribution converges in distribution to the normal distribution, Anderson and Rashid [4] shows that $L(d\hat{\Phi})\mathrm{st}^{-1} = d\Phi$, where $L(d\hat{\Phi})$ is the Loeb measure generated by $d\hat{\Phi}$. Then

$$
\begin{aligned}
\int_{{}^*\mathbf{R}^d} f_a d\hat{\Phi} &\simeq e^{|a|^2/2} \\
&\simeq e^{|{}^\circ a|^2/2} \\
&= \int_{\mathbf{R}^d} f_{{}^\circ a} d\Phi \\
&= \int_{\mathbf{R}^d} f_{{}^\circ a} L(d\hat{\Phi}\mathrm{st}^{-1}) \\
&= \int_{{}^*\mathbf{R}^d} {}^\circ f_a L(d\hat{\Phi})
\end{aligned}
$$

which proves that $f_a \in SL^1({}^*\mathbf{R}^d, d\hat{\Phi})$. ■

Theorem 3.3 *Suppose that $\varphi_2'(c) = O(1/c^r)$ as $c \to 0$, for some $r \in \mathbf{R}$. Then for μ-almost all ω, the equilibrium pricing process satisfies*

$$\begin{array}{rcl} {}^\circ\left(\hat{p}_A(t,\omega)\right) & = & e^{\beta({}^\circ t,\omega)} \int_{-\infty}^{\infty} \varphi_2'\left(F({}^\circ t,\omega,x)\right) e^{\sqrt{T-{}^\circ t}x} d\Phi(x) \\ {}^\circ\left(\hat{p}_B(t,\omega)\right) & = & \int_{-\infty}^{\infty} \varphi_2'\left(F({}^\circ t,\omega,x)\right) d\Phi(x) \\ {}^\circ\left(\hat{p}_C(t,\omega)\right) & = & \varphi_1'(1) \\ {}^\circ\left(\hat{p}_C(\hat{T},\omega)\right) & = & \varphi_2'\left(F(T,\omega,0)\right) \\ {}^\circ\left(\frac{\hat{p}_A(t,\omega)}{\hat{p}_B(t,\omega)}\right) & = & e^{\beta({}^\circ t,\omega)} \frac{\int_{-\infty}^{\infty} \varphi_2'(F({}^\circ t,\omega,x)) e^{\sqrt{T-{}^\circ t}x} d\Phi(x)}{\int_{-\infty}^{\infty} \varphi_2'(F({}^\circ t,\omega,x)) d\Phi(x)} \end{array}$$

for all $t \in \mathcal{T}$, where Φ is the cumulative distribution function of the standard d-dimensional normal.

Proof: Anderson [1] showed that, for almost all ω, ${}^\circ\hat{\beta}(t,\omega) = \beta({}^\circ t,\omega) \in \mathbf{R}$ for all $t \in \mathcal{T}$; fix any such ω. Note that, since ρ is continuous, $\hat{F}(t,\omega,x) \simeq F({}^\circ t,\omega,{}^\circ x)$ for all $t \in \mathcal{T}$. Note that $\varphi_2'(c)$ is decreasing and there exists $\gamma \in \mathbf{R}$ such that $\varphi_2'(c) \leq \frac{\gamma}{c^r}$ for all $c \in (0,1]$. Since the d-dimensional binomial distribution $\hat{\Phi}$ converges in distribution to the d-dimensional normal distribution Φ, Anderson and Rashid [4] shows that $L(d\hat{\Phi})\mathrm{st}^{-1} = d\Phi$, where $L(d\hat{\Phi})$ is the Loeb measure generated by $d\hat{\Phi}$. Thus, if $J \geq 1$,

$$\begin{array}{l} \left| {}^*\varphi_2'\left(\hat{F}(t,\omega,x)\right) e^{\sqrt{\hat{T}-t}x} \right| \\ \quad \leq \ \varphi_2'(1) e^{\sqrt{\hat{T}-t}x} + \gamma\left(\mathbf{1}_J \cdot \left(e^{\hat{\beta}(t,\omega)} e^{\sqrt{T}x}\right)\right)^{-r} e^{\sqrt{\hat{T}-t}x} \\ \quad \in \ SL^1(d\hat{\Phi}) \end{array}$$

by Proposition 3.2; the argument in case $J = 0$, using the lower bound $\rho(x) \geq e^{\alpha \cdot x}$ for some $\alpha \in \mathbf{R}^d$, is similar. Therefore,

$$\begin{array}{l} \int_{{}^*\mathbf{R}} {}^*\varphi_2'\left(\hat{F}(t,\omega,x)\right) e^{\sqrt{\hat{T}-t}x} d\hat{\Phi} \\ \quad \simeq \ \int_{{}^*\mathbf{R}} \varphi_2'\left({}^\circ\hat{F}(t,\omega,x)\right) e^{\sqrt{T-{}^\circ t}{}^\circ x} L(d\hat{\Phi}) \\ \quad = \ \int_{\mathbf{R}} \varphi_2'\left(F({}^\circ t,\omega,x)\right) e^{\sqrt{T-{}^\circ t}x} L(d\hat{\Phi})\mathrm{st}^{-1} \\ \quad = \ \int_{\mathbf{R}} \varphi_2'\left(F({}^\circ t,\omega,x)\right) e^{\sqrt{T-{}^\circ t}x} d\Phi \end{array}$$

which proves the formula for ${}^\circ p_A(t,\omega)$. The proof of the formula for ${}^\circ p_B(t,\omega)$ is essentially the same, and slightly easier. ■

Theorem 3.4 *$\hat{p}_A$ and $\hat{p}_B$ are internal almost surely S-continuous SL^2 martingales with respect to the internal filtration $\{\hat{\mathcal{F}}_t\}$. If we define*

$$\begin{aligned} p_A(t,\omega) &= {}^\circ\hat{p}_A(\hat{t},\omega) \\ p_B(t,\omega) &= {}^\circ\hat{p}_B(\hat{t},\omega) \end{aligned}$$

for $t \in [0,T]$, then p_A and p_B are almost surely continuous square integrable martingales with respect to the filtration $\{\mathcal{F}_t\}$.

Proof: Anderson [1] showed that for almost all ω, $\hat{\beta}(\cdot,\omega)$ is S-continuous; for any such ω, $\hat{p}_A(\cdot,\omega)$ and $\hat{p}_B(\cdot,\omega)$ are S-continuous, so $p_A(\cdot,\omega)$ and $p_B(\cdot,\omega)$ are continuous (this is a well known result in nonstandard analysis; see, for example, Proposition 3.5.5 of [3]). By Equation 8, $\hat{p}_A$ and $\hat{p}_B$ are internal martingales with respect to the internal filtration $\{\hat{\mathcal{F}}_t\}$. By Theorem 12(ii) of Anderson [1], p_A and p_B are martingales with respect to the filtration $\{\mathcal{F}_t\}$. $\hat{\beta}(\hat{T},\cdot)$ is distributed as $e^{\sqrt{\hat{T}}X}$ where X has the cumulative distribution function $\hat{\Phi}$, so $(\hat{\beta}(\hat{T},\cdot))^2$ is distributed as $e^{2\sqrt{\hat{T}}X} \in SL^1(d\hat{\Phi})$ by Proposition 3.2, so $\hat{\beta}(\hat{T},\cdot) \in SL^2(\hat{\mu})$. It follows from Theorem 12(ii) of Anderson [1] that $\hat{p}_A(t,\cdot) \in SL^2(\hat{\mu})$ for all $t \in \mathcal{T}$, and p_A is square integrable. The argument that $\hat{p}_B$ is SL^2 and p_B is square integrable is similar. ■

Proof of Theorem 2.1: For $t \in [0,T]$, let

$$\begin{aligned} p_C(t,\omega) &= {}^\circ\hat{p}_C(\hat{t},\omega) = \varphi_1'(1) \text{ for } t < T \\ p_C(T,\omega) &= {}^\circ\hat{p}_C(\hat{T},\omega) = \varphi_2'(e(T) + e^{\beta(T,\omega)}) \\ z_A(t,\omega) &= \mathbf{1} \text{ for all } t \in [0,T] \\ z_B(t,\omega) &= 0 \text{ for all } t \in [0,T] \\ c(t,\omega) &= 1 \text{ for all } t \in [0,T) \\ c(T,\omega) &= F(T,\omega,0) \end{aligned}$$

We will verify that this is an equilibrium for the standard continuous-time economy. By Theorem 3.4, p_A and p_B are continuous square-integrable martingales with respect to $\{\mathcal{F}_t\}$. Notice that ${}^\circ\hat{c}(\hat{T},\omega) = c(T,\omega)$ for almost all ω, and $\hat{c}(\hat{T},\cdot) \in SL^1(d\hat{\mu})$ by Proposition 3.2, so $\hat{U}(\hat{c}) \simeq U(c)$. Observe that $\hat{z}$ is a 2-lifting of z with respect to $\hat{p}$, so by Theorem 17 of Lindström [35], z is an admissible trading strategy with respect to p and, for almost all ω,

$$\int_0^t z dp = {}^\circ \int_0^{\hat{t}} \hat{z} d\hat{p}$$

for all $t \in [0,T]$, so it follows that c is in the budget set.

If this is not an equilibrium, there must be an admissible trading strategy (y_A, y_B) and a consumption plan c' which belongs to the budget set given y such that $U(c') > U(c) = T\varphi_1(1) + \varphi_2(F(T,\omega,0)) > -\infty$.

Anderson [1] showed that c' has a 1-lifting $\vec{c}'$, i.e. $\hat{c}' \in SL^1(\hat{\lambda} \times \hat{\mu})$, $\vec{c}'(t,\cdot)$ is $\hat{\mathcal{F}}_t$ measurable, and $\vec{c}'(t,\omega) = c'({}^\circ t,\omega)$ for $L(\hat{\lambda} \times \hat{\mu})$ almost all (t,ω). Given $\varepsilon \in \mathbf{R}_{++}$, consider $\vec{c}'_\varepsilon(t,\omega) = \min\{\varepsilon, \vec{c}'(t,\omega)\}$. Note that

$$\varphi_1(\varepsilon) \le \varphi_1(\vec{c}'_\varepsilon(t,\omega)) \le \varphi_1(1) + \varphi_1'(1)\vec{c}'(t,\omega)$$

and

$$\varphi_2(\varepsilon) \le \varphi_2(\vec{c}'_\varepsilon(T,\omega)) \le \varphi_2(1) + \varphi_2'(1)\vec{c}'(T,\omega)$$

Thus, ${}^\circ\hat{U}(\vec{c}'_\varepsilon) \to U(c')$ as $\varepsilon \to 0$; thus, we may find $\varepsilon \simeq 0$ such that $\hat{U}(\vec{c}'_\varepsilon) \simeq U(c')$; let $\hat{c}' = \vec{c}'_\varepsilon$, and notice that $\hat{c}'$ is a 1-lifting of c'. Thus, for almost all ω,

$$\int_0^t \hat{p}_C(s,\omega)\hat{c}'(s,\omega)d\hat{\lambda} \simeq \int_0^{{}^\circ t} p_C(s,\omega)c'(s,\omega)d\lambda$$

for all $t \in \mathcal{T}$.

By Lemma 15 of Lindström [35], y has a 2-lifting $\hat{y} = (\hat{y}_A, \hat{y}_B)$ with respect to $\hat{p}$. By Theorem 17 of Lindström [35], for almost all ω

$$\int_0^t \hat{y}d\hat{p} \simeq \int_0^{{}^\circ t} ydp$$

for all $t \in \mathcal{T}$. Thus, if we define $f(\omega)$ to be the maximum over $t \in \mathcal{T}$ of the amount by which the cost of the consumption through time t prescribed by $\hat{c}'$ violates the budget constraint generated by $\hat{y}$, then f is an internal function and $f(\omega) \simeq 0$ for almost all ω. Observe also that

$$g(\omega) = \frac{\max\{p_A(t,\omega) : t \in \mathcal{T}\}}{\min\{p_A(t,\omega) : t \in \mathcal{T}\}}$$

is an internal function of ω which is non-infinitesimal almost surely. It follows that there is an infinitesimal δ and an internal set $\Omega' \subset \Omega$ such that $\hat{\mu}(\Omega') \simeq 1$ and for all $\omega \in \Omega'$, $f(\omega) < \delta$ and $g(\omega) > \sqrt{\delta}$.

Now, we construct a consumption process $\hat{\hat{c}}$ and an admissible trading strategy $\hat{\hat{y}}$ which finances it. $\hat{\hat{y}}_B(t,\omega) = \hat{y}_B(t,\omega)$ for all (t,ω). For each ω, we set $\hat{\hat{c}}(t,\omega) = 0$ and use the income saved to purchase additional units of the bond (or sell less of the bond) than called for by $\hat{y}$, until the first $t = t_1(\omega)$ such that $\hat{p} \cdot (\hat{\hat{y}}(t,\omega) - \hat{y}(t,\omega)) \geq \sqrt{\delta}$. For $t \geq t_1(\omega)$, $\hat{\hat{c}}(t,\omega) = \hat{}(t,\omega)$, and $\hat{\hat{y}}_A(t,\omega)$ is set at whatever level is needed to finance the consumption, given that $\hat{\hat{y}}_B(t,\omega) = \hat{y}_B(t,\omega)$. Continue this until the first $t = t_2(\omega)$ for which this formula would yield $\hat{\hat{y}}_A(t,\omega) \leq \hat{y}_A(t,\omega)$; adjust $\hat{\hat{c}}(t_2(\omega),\omega)$ downward so that it can be financed by $\hat{\hat{y}}_A(t_2(\omega),\omega)$, and set $\hat{\hat{y}}_A(t,\omega) = \hat{y}_A(t,\omega)$ for $t > t_2(\omega)$. Observe that t_1 and t_2 are internal stopping times, so $\hat{\hat{y}}$ is an admissible internal trading strategy. Observe also that $t_1(\omega) \simeq 0$ and $t_2(\omega) > \hat{T}$ for $\omega \in \Omega'$; and $\hat{\hat{c}}(t,\omega) = c(t,\omega)$ almost surely. Since φ_1 and φ_2 are bounded below, ${}^\circ\hat{U}(\hat{\hat{c}}) = {}^\circ\hat{U}(\hat{c}') = U(c') > U(c) \simeq {}^\circ\hat{U}(\hat{c})$, which contradicts the fact that $\hat{p}$, $\hat{p}_C$, $\hat{c}$ and $\hat{z}$ are an internal equilibrium of the hyperfinite economy.■

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