

Asset Price Fluctuations without Aggregate Shocks*

Costas Azariadis[†]

Leo Kaas[‡]

Abstract

We analyze the pricing of a productive asset in a class of dynamic exchange economies with heterogeneous, infinitely-lived agents, and self-enforcing intertemporal trades. Individual incomes fluctuate and are correlated; preferences, dividends and aggregate income are fixed. Most economies in this class have a unique equilibrium that converges to a limit cycle. This equilibrium segments households into groups of autarkic agents who never trade assets, recurrently rationed agents behaving like finitely-lived overlapping generations, and perpetually unrationed agents. In almost all economies, the set of unrationed households changes over time, shifting excess demand correspondences and causing movements in asset returns. Examples suggest that the amplitude of these movements is negatively correlated with the productivity of the asset and with the length of exclusion from intertemporal trading and that returns on productive assets with safe dividends vary more than the return on safe portfolios of contingent claims.

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[†]Department of Economics, University of California, Los Angeles, CA 90095-1477, USA. E-mail: azariadi@econ.ucla.edu

[‡]Department of Economics, University of Vienna, Hohenstaufengasse 9, 1010 Vienna, Austria. E-mail: leo.kaas@univie.ac.at

1 Introduction

Asset price volatility is an enduring feature of financial time series and a striking anomaly for dynamic general equilibrium models of asset pricing with homogeneous consumers. Campbell (2002), for instance, finds that diversified stock portfolios undergo large movements in returns relative to observed changes in the growth rates of dividends and aggregate consumption.¹ The puzzling nature of the co-movement between aggregate consumption growth and the price-dividend ratio has led researchers to non-standard assumptions about the representative household's utility function like strong habit persistence, as in Constantinides (1990) and Campbell and Cochrane (1999) and other forms of extremely poor substitutability of consumption at different date-state events.

Using standard assumptions on preferences, this paper analyzes a class of pure exchange economies, with heterogeneous households and no commitment to intertemporal trades, in which the price of a productive asset fluctuates even though the structure of the economy remains unchanged. This class of economies is the two-state-of-nature counterpart of settings studied in Alvarez and Jermann (2000) and Kehoe and Levine (2001), but with substantially more heterogeneity added. To simplify matters, we abstract completely from aggregate shocks, except for a numerical example at the end of Section 6. We focus instead on environments with correlated individual incomes, constant dividends and constant aggregate income. These environments may be interpreted as suffering from *asymmetric sectoral shocks* which take income away from a few sectors or households and redistribute it to all other sectors or households. As in earlier literature, we assume that consumers cannot credibly commit to deliver future consumption but are dissuaded from actual default by a central credit agency which seizes the assets of defaulters and keeps their names on a black list which denies them credit for a fixed term of $L \geq 0$ periods. The threat of market exclusion is sufficient to prevent everyone from borrowing more than they would be willing to repay given the length of exclusion from intertemporal trading. Longer exclusion lengths strengthen the self-enforcing nature of intertemporal trades, and typically produce outcomes with smaller fluctuations and better welfare properties.

Heterogeneity among consumers leads to a natural pattern of market segmentation.

¹Campbell reports in quarterly US data over the period 1947.2 to 1998.4 annualized standard deviations of 15.6% for real stock returns, 6% for real dividend growth, and 1.1% for the seasonally adjusted growth rate in real consumption of non-durables and services.

Impatient agents with relatively stable incomes and intertemporally substitutable consumption are not allowed to borrow at all and stay out of asset markets forever. All other agents are recurrently rationed, with the possible exception of the most patient consumers.

Each period, asset prices and loan yields are determined by the trading plans of unrationed consumers who buy the productive asset and lend out whatever rationed consumers are allowed to borrow. As the set of unrationed households changes from one period to the next, excess demand correspondences for loans and productive assets shift, causing movements in asset prices. In rough terms, the market applies to future dividends a time-varying discount rate that reflects the marginal rate of intertemporal substitution of consumers who happen to be unrationed each period. Our examples suggest that the size of fluctuations in asset prices is negatively correlated with the productivity of the asset and with the strength of the self-enforcement mechanism.

The time-varying discounting mechanism fails in some important special cases in which asset prices and loan yields remain constant. One case, analyzed in an otherwise general setting by Alvarez and Jermann (2000), is when the *dividend is zero* and all assets exist in zero net supply. Then autarky is an equilibrium; in an economy with heterogeneous agents, autarky is supported by a market yield equal to the lowest marginal rate of substitution among all households. Another case consists of economies in which *either capital is sufficiently productive or the self-enforcement mechanism is strong* (L is large) and all agents share a common, sufficiently low rate of time preference. In this case, the first-best commitment equilibrium turns out to be an equilibrium in the no-commitment economy. A third case are *symmetric economies*, like the one studied in Kehoe and Levine (2001), in which consumers are mirror images of each other. Excess demands for assets in symmetric economies do not vary with time since the set of unrationed consumers has the same tastes and resources every period. If these assumptions fail, a changing set of consumers *must be rationed each period*, and the economy has generically no steady state; it settles down to a limit cycle.

There is a large literature on the role of heterogeneity in asset pricing that may be classified into two groups. One group of papers assumes that investors have homogeneous preferences but are subject to *different idiosyncratic income risks* (Constantinides and Duffie (1996), Heaton and Lucas (1996) and Lustig (2001)). When the aggregate state changes, the wealth distribution of investors shifts which causes the set of constrained and unconstrained agents to vary over time. Using a market

friction like incomplete markets (Constantinides and Duffie), trading costs (Heaton and Lucas) or endogenous borrowing constraints (Lustig), these papers study symmetric economies in which the income distribution only depends on the aggregate state, but is invariant if aggregates are constant. They reach this conclusion either by assuming a two-agent framework in which individual incomes are perfectly negatively correlated (Heaton and Lucas) or by invoking the law of large numbers for a continuum of investors whose income shocks are cross-sectionally uncorrelated (Constantinides and Duffie, Lustig). With any of these assumptions, asset prices cannot fluctuate without movements in aggregate output and dividends, as they do in our paper.

Another group of papers deals with *heterogeneity in preferences* (Krusell and Smith (1998), Kiyotaki and Moore (1997), Sandroni (1999)). Krusell and Smith use borrowing constraints to keep agents from perfectly insuring themselves against idiosyncratic risk, and they allow for a small amount of heterogeneity in discount factors to match the equilibrium wealth distribution to the data. The law of large numbers applies here as well and leads to a unique invariant wealth distribution in every aggregate state, so that our mechanism is absent from their model. Kiyotaki and Moore use heterogeneity in preferences and technology to segment households into groups of borrowing and lending agents. The interaction between credit limits and asset prices generates damped oscillations around the unique steady state equilibrium. In contrast, our economy has generally no steady state, but the unique asymptotic equilibrium is a cycle. Perhaps closest to ours is the idea of Sandroni that asset price fluctuations are induced by time-varying characteristics of traders. In his model, highly impatient agents endowed with capital shares enter the economy stochastically, causing capital prices to fluctuate. Sandroni, however, assumes complete markets and commitment to intertemporal trade; his mechanism would not generate asset price cycles without this exogenous arrival process. By contrast, in our economy of limited commitment, the set of constrained and unconstrained agents changes endogenously, as do asset demand and supply. Furthermore, as we show in Section 6, preference heterogeneity is not decisive for the operation of our mechanism. Heterogeneity of endowments works just as well.

In the remainder of this paper we set up in Section 2 the general environment for a deterministic class of economies, and we examine in Section 3 the conditions under which the commitment allocation and the autarkic allocation are equilibria for an arbitrary length of exclusion $L \geq 0$. Section 4 deals with symmetric economies and establishes the existence of a unique and determinate steady state. Section 5

shows that the unique asymptotic equilibrium in asymmetric economies is a limit cycle. Section 6 extends our results to uncertainty in a Markovian economy with two income states and complete markets. Section 7 concludes with a succinct summary of results, connects these to the literature and discusses potential extensions. Proofs not included in the text are collected in the Appendix.

2 The environment

We study a pure-exchange economy in discrete time $t = 0, 1, \dots$ populated by a continuum of infinitely lived agents. There is a finite set of agent types $i \in I = \{1, \dots, I\}$ and a unit mass of identical agents of type i , so that we can identify type i with agent i . With abuse of notation, I denotes both the set of agent types and its cardinality. The structure of the economy is stationary in the sense that agents' preferences are time-independent and aggregate consumption possibilities are constant.

There is a single non-durable consumption good in each period and a non-depreciating durable asset (capital) that pays its owner $d \geq 0$ units of the consumption good in every period and state. The stock of capital is normalized to unity. Unproductive capital ($d = 0$) can be interpreted as fiat money. The capital price at the end of each period (ex-dividend) is denoted p_t , and the gross interest rate on loans between periods t and $t + 1$ is R_t . The absence of arbitrage opportunities between safe loans and safe capital means that $R_t = (p_{t+1} + d)/p_t$ for all $t \geq 0$.

Agent $i \in I$ has the stationary utility function

$$(1 - \beta_i) \sum_{t=0}^{\infty} \beta_i^t u_i(c_{it}) \tag{1}$$

defined over consumption plans $(c_{it})_{t \geq 0}$. $\beta_i \in (0, 1)$ is agent i 's discount factor. Without loss of generality we assume $\beta_1 \geq \beta_2 \geq \dots \geq \beta_I$, so that agent i is at least as patient as agent j , whenever $1 \leq i < j \leq I$. The period utility functions u_i are assumed to be differentiable, strictly increasing and strictly concave.

Agents face a periodic individual endowment process (y_{it}) which we specify as a deterministic sequence alternating between the two values $y_i^H = y_i(1 + \alpha_i)$ and $y_i^L = y_i(1 - \alpha_i)$. Hence, $y_{it} = y_i^H$ implies that $y_{i,t+1} = y_i^L$, and vice versa. Since endowments of all agents of type i are perfectly correlated, we interpret individual income fluctuations as *sectoral shocks* which give rise to labor income variations of

the households employed in different sectors of the economy, rather than as *strictly idiosyncratic shocks* to each individual household. We assume that these sectoral shocks offset each other so that there are no aggregate fluctuations; the aggregate endowment in every period is a constant, $\sum_{i \in I} y_{it} \equiv \Omega$. Each period the consumption good is in aggregate supply $\Omega_d \equiv \Omega + d$. Though our analysis deals mainly with deterministic fluctuations, we show in Section 6 that our results extend to economies in which the endowment process follows a two-state Markov process and asset markets are complete.

There is a sequence of asset markets in which agents trade capital shares and claims on the consumption good. We denote by a_{it} the asset position of agent i at the end of period t in units of the consumption good. Assuming that agents hold no initial claims on each other and are endowed with x_{i0} units of capital ($\sum_{i \in I} x_{i0} = 1$), households face the sequence of budget constraints

$$\begin{aligned} a_{i0} &= (p_0 + d)x_{i0} + y_{i0} - c_{i0} , \\ a_{it} &= R_{t-1}a_{i,t-1} + y_{it} - c_{it} \quad , \quad t \geq 1 . \end{aligned} \tag{2}$$

We summarize the fundamentals of the economy by the list $\mathcal{E} = (B, A, Y, X_0, U, d)$ where $B = (\beta_i)_{i \in I}$, $A = (\alpha_i)_{i \in I}$, $Y = (y_i)_{i \in I}$, $X_0 = (x_{i0})_{i \in I}$, and $U = (u_i)_{i \in I}$.

A crucial feature of the model is that agents cannot commit to repay loans. In particular, future labor income (endowments) cannot be seized and therefore cannot be used as collateral. We assume, however, that there is a technology which permits the exclusion of defaulting agents from future trading in asset markets. When someone defaults on a loan, a powerful “credit authority” is able to seize the defaulter’s future assets and also to preclude him/her from borrowing. The assumption that future labor income cannot be confiscated after default is based on legal arguments as well as incentive arguments. However, information on creditworthiness of agents can easily be made publicly available, so that defaulters can be banned from borrowing in the future. Also assets like stocks and bonds easily change ownership and may be seized by creditors in case of default.

In contrast to much of the earlier literature, we allow for weaker enforcement mechanisms in which the memory of the credit authority may be finite. In particular, the enforcement mechanism is specified by the length of exclusion $L \in \{0, 1, \dots, \infty\}$ meaning that a defaulting agent is excluded from asset market trades for L periods after default. This framework includes the two polar cases $L = \infty$ and $L = 0$. The case of infinite exclusion $L = \infty$ corresponds to the enforcement mechanism of Kehoe and Levine (1993). In the opposite extreme $L = 0$ there is no enforcement,

implying that agents must be precluded from any borrowing. This model was originally used by Bewley (1980) and Townsend (1980) to study monetary policy and the value of fiat money.²

Any exclusion length L gives rise to endogenous debt constraints $b_{it} \geq 0$ (to be determined in equilibrium) that are just tight enough to prevent all agents from default: on the one hand, no creditor is willing to lend more to a debtor than the debtor is willing to repay, i.e. the debtor's participation constraint has to be satisfied. On the other hand, the debt constraint can only bind if the agent's participation constraint binds (i.e. debt constraints are not too tight). These features are specified in parts (iii) and (iv) of the following equilibrium definition.

Definition of equilibrium:

An *equilibrium without commitment* for the economy $\mathcal{E} = (B, A, Y, X_0, U, d)$ and exclusion length $L \in \mathbb{N}_0 \cup \{\infty\}$ is a list of consumption plans, asset holdings and debt constraints $(c_{it}, a_{it}, b_{it})_{t \geq 0}$, $c_{it}, b_{it} \geq 0$, $i \in I$, and a list of interest rates and capital prices $(R_t, p_t)_{t \geq 0}$ such that

- (i) For all $i \in I$, $(c_{it}, a_{it})_{t \geq 0}$ maximizes (1) subject to (2) and to $a_{it} \geq -b_{it}$ for all $t \geq 0$ at given prices $(R_t, p_t)_{t \geq 0}$.
- (ii) Markets clear, i.e., for all $t \geq 0$

$$\sum_{i \in I} c_{it} = \Omega_d \quad \text{and} \quad \sum_{i \in I} a_{it} = p_t .$$

- (iii) Debt constraints prevent default: for any default date $t \geq 0$ and $i \in I$, the market payoff from $t + 1$ forward is no smaller than the default payoff, that is,

$$V_{i,t+1} \equiv (1 - \beta_i) \sum_{\tau=t+1}^{\infty} \beta_i^{\tau-t-1} u_i(c_{i\tau}) \geq \bar{V}_{i,t+1} , \tag{3}$$

where

$$\begin{aligned} \bar{V}_{i,t+1} &= \max_{\hat{c}_i, \hat{a}_i} (1 - \beta_i) \sum_{\tau=t+1}^{\infty} \beta_i^{\tau-t-1} u_i(\hat{c}_{i\tau}) \\ \text{s.t.} \quad &\hat{c}_{i\tau} = y_{i\tau} , \hat{a}_{i\tau} = 0 \quad , \tau = t + 1, \dots, t + L, \\ &\hat{a}_{i\tau} = R_{\tau-1} \hat{a}_{i,\tau-1} + y_{i\tau} - c_{i\tau} \geq -b_{i\tau} , \tau \geq t + L + 1 . \end{aligned}$$

²In the terminology of Kehoe and Levine (2001), $L = \infty$ corresponds to a “debt constrained economy” and $L = 0$ refers to a “liquidity constrained economy”.

- (iv) Debt constraints are not too tight, i.e., whenever $a_{it} = -b_{it}$ binds in problem (i), the participation constraint (3) is satisfied with equality.

Parts (i) and (ii) of this definition are standard in commitment economies where debt limits b_{it} are defined from the intertemporal budget constraint. Parts (iii) and (iv) specify these limits in environments without commitment. The calculation and enforcement of debt constraints are left to the credit authority which is assumed to possess sufficient knowledge of each agent's tastes, endowments and asset trades.

It is worth pointing out that the debt constraints are the same for an agent who does not default and stays in the market as they are for someone who defaults and rejoins the asset market after L periods of exclusion. The reason is that the returning agent faces the same prices and the same endowment profile as the agent in the market. Therefore, the same debt constraints are imposed to enforce market participation on both agents. In equilibrium, of course, no agent ever defaults since the participation constraints are satisfied (assuming that agents do not default whenever the participation constraint binds).

In the case $L = 0$, debt constraints must be zero for all agents. Indeed, without exclusion from intertemporal trade, any agent with a negative asset position would choose to default ($\bar{V}_{i,t+1} > V_{i,t+1}$ if $a_{it} < 0$). Hence, equilibrium in this economy simplifies to (i) and (ii) with $b_{it} = 0$ for all t and i . If $L = \infty$, the participation constraint (iii) takes again a relatively simple form. Since agents are excluded from borrowing and lending forever, their default payoff equals the utility of autarky. When agent i has low income in period t , the autarky payoff is $\bar{V}_{i,t+1} = (u_i(y_i^H) + \beta_i u_i(y_i^L)) / (1 + \beta_i)$ (with superscripts H, L interchanged when the agent has high income in period t). Kehoe and Levine (1993, 2001) include the participation constraints (3) directly into the consumer's maximization problem (i) leaving debt constraints unspecified. Since it is not easily possible to extend their definition to the case in which the credit authority has finite memory, we prefer to follow the equilibrium definition of Alvarez and Jermann (2000) who specify debt constraints explicitly. This definition does not require a priori that agents satisfy their intertemporal budget constraint. However, if the present value of the endowment stream at market prices is finite, agents satisfy their intertemporal budget constraint and, consequently, there are no asset price bubbles (see e.g. Santos and Woodford (1997)).

3 Commitment and autarkic equilibria

In this section we explore the circumstances under which two specific allocations emerge as equilibria without commitment: one is the first–best commitment equilibrium; the other is an autarkic equilibrium in which there is no asset trading at all. We say that a given allocation is *implementable for the exclusion length L* whenever it coincides with the outcome of an equilibrium without commitment for an exclusion length of L periods.

In a commitment equilibrium, no agent is constrained at any time so that $u'_i(c_{it}) = \beta_i R_t u'_i(c_{i,t+1})$ for all t and $i \in I$. An immediate observation is that a commitment equilibrium cannot be implementable whenever the discount factors of some agents differ. Indeed, when $\beta_1 > \beta_I$, the interest yield in the commitment equilibrium converges monotonically (from above) to the reciprocal of the largest discount factor $1/\beta_1$. The most patient agents (those with discount factor β_1) accumulate wealth in the early periods of their life to maintain a constant positive consumption level in the long–run. Other (less patient) agents borrow in the early periods of their life, and their consumption tends to zero as $t \rightarrow \infty$. This implies that the commitment allocation cannot be an equilibrium without commitment for any L since the participation constraint (3) is necessarily violated for all less patient agents at some sufficiently large t ; these agents are always better off if they do not repay their loans and revert to autarky when their market consumption becomes sufficiently low. When rates of time preference differ, less patient agents must necessarily be debt constrained in equilibrium. As a result, the commitment equilibrium is not implementable for any L .

Suppose now that all discount rates are equal, $\beta_i = \beta$, $i \in I$. In this case the commitment equilibrium involves a constant interest rate $R = 1/\beta$ and constant consumption levels for all agents, c_i , which are determined by the agents' intertemporal budget constraints. Let us suppose that $L \geq 0$ is an even number or infinity (the analysis for odd numbers is very similar). Note that low–income agents hold positive assets at the beginning of each period and do not default. If agent i with high income in period $t + 1$ defaults, he is excluded from intertemporal trade for the next L periods. When the agent returns to the market in period $t + L + 1$ with high income and a zero asset position, he can freely save and borrow again at rate $1/\beta$. Just like any agent who did not default, the defaulting agent i is unconstrained and smoothes consumption perfectly. However, since he starts from a zero asset position, his consumption c_i^d is different from the consumption of a solvent household; it is

determined from the intertemporal budget constraint as

$$c_i^d = y_i \left(1 + \alpha_i \frac{1 - \beta}{1 + \beta} \right).$$

Therefore, the utility of the defaulting agent in period $t + 1$ is

$$\bar{V}_i = \frac{1 - \beta^L}{1 + \beta} \left(u_i(y_i^H) + \beta u_i(y_i^L) \right) + \beta^L u_i(c_i^d).$$

The participation constraint for agent i requires that $u_i(c_i) \geq \bar{V}_i$. We can express the participation constraint equivalently as $c_i \geq \Phi_i(y_i, \alpha_i, \beta, L)$ where $\Phi_i(y_i, \alpha_i, \beta, L)$ is defined by

$$u_i(\Phi_i(y_i, \alpha_i, \beta, L)) = \frac{1 - \beta^L}{1 + \beta} \left(u_i(y_i(1 + \alpha_i)) + \beta u_i((1 - \alpha_i)y_i) \right) + \beta^L u_i \left(y_i \left(1 + \alpha_i \frac{1 - \beta}{1 + \beta} \right) \right). \quad (4)$$

It is easy to show that each Φ_i is increasing in y_i and decreasing in (β, L) . From the market clearing condition, $\sum_{i \in I} c_i = \Omega_d$, we obtain as a necessary condition for the commitment equilibrium to be implementable that

$$\sum_{i \in I} \Phi_i(y_i, \alpha_i, \beta, L) \leq \Omega_d. \quad (5)$$

Conversely, whenever (5) is satisfied, there exists an initial wealth distribution that leads to a consumption allocation satisfying the participation constraints of all agents, so that the corresponding commitment equilibrium is implementable.³ Hence, we have

Proposition 1:

- (a) If $\beta_i \neq \beta_j$ for some $i, j \in I$, the commitment equilibrium is not implementable for any exclusion length $L \geq 0$.
- (b) If $\beta_i = \beta$ for all $i \in I$, there exists an initial wealth distribution so that the commitment equilibrium is implementable for the exclusion length L , if and only if, (5) is satisfied.

³If (5) holds, but the participation constraint of some agents is not satisfied at the given initial wealth distribution, there exists still an equilibrium, with low wealth agents constrained in the first or second period, which leads effectively to a wealth redistribution from unconstrained to constrained agents. After the second period, all agents are forever unconstrained.

From this result and the monotonicity properties of the functions Φ_i we obtain the following extension (see Appendix):

Corollary 1: Let $\beta_i = \beta \in (0, 1)$ for all $i \in I$. Then

- (a) For any (β, L) and any initial distribution of wealth, the commitment equilibrium is not implementable whenever $d \geq 0$ and $\alpha_1, \dots, \alpha_I > 0$ are sufficiently small.
- (b) For any $\alpha_1, \dots, \alpha_I, \beta$ and L , a commitment equilibrium is implementable, if d is sufficiently large.
- (c) For any $d \geq 0$ and $\alpha_1, \dots, \alpha_I$, a commitment equilibrium is implementable whenever β and L are sufficiently large.

The interpretation of these results is straightforward. If income variability is too small and capital productivity too low, agents get a small payoff from trading in asset markets, and the participation constraint cannot be satisfied in a commitment equilibrium. Conversely, whenever capital is sufficiently productive, agents do not need to borrow much in order to smooth consumption perfectly. Hence, the commitment equilibrium can be implemented. Finally, if agents are sufficiently patient and if punishment is sufficiently long (β and L are large enough), the threat of market exclusion will deter fairly large debtors from defaulting, and the commitment allocation is again implementable.

We turn now to the second specific equilibrium allocation, an autarky equilibrium in which no agents trade in intertemporal markets. Autarky turns out to be an equilibrium in the following two cases: (i) The dividend is zero and the interest rate is so low that all agents (even those with high income) want to borrow in every period; then all agents face zero debt constraints (see also Alvarez and Jermann (2000, Proposition 4.3)). (ii) The dividend is positive and agents of type $i = 1$ are sufficiently more patient than all other agents. Each period, all impatient agents are constrained, whereas type 1 agents are unconstrained and hold the entire capital stock. Asset prices fluctuate to induce type 1 agents not to trade capital.

Proposition 2: For any $L \geq 0$, there exists an autarky equilibrium without commitment if either (i) $d = 0$ or (ii) $d > 0$ and $\beta_i < k_i \beta_1$, $i = 2, \dots, I$, for some constants $k_i \in (0, 1]$.

4 Symmetric economies

We analyze in this section the class of symmetric economies whose equilibria are stationary and easy to characterize. Formally, a *symmetric economy* has an even number of agent types and for every $i \in J \equiv \{1, 3, \dots, I - 1\}$, $u_i = u_{i+1}$, $\beta_i = \beta_{i+1}$, $\alpha_i = \alpha_{i+1}$, $y_i = y_{i+1}$, and $y_{it} = y_i^H$ whenever $y_{i+1,t} = y_i^L$. Hence, agents i and $i + 1$ are “mirror images” of each other with the same utility functions, discount factors and income profiles; their individual income fluctuations are exactly offsetting.

It turns out that symmetric economies exhibit a unique determinate steady state equilibrium at some interest rate $R \geq 1$. We concentrate our analysis to the two polar cases $L = 0$ and $L = \infty$ which have very similar features. The intermediate cases are difficult to analyze in this general framework, but we discuss possible extensions in Section 7.

To understand why symmetric economies have a steady state equilibrium, consider the consumption demand of agent i when he has high and low income as a function of the interest yield, $c_i^H(R)$ and $c_i^L(R)$. The definition of these demand functions assumes that agent i faces a sequence of constant interest rates and that he can be constrained or unconstrained at the prevailing rate of interest (a formal definition is given in the proof of Proposition 3 in the Appendix). Since the economy is symmetric, $c_i^H(R) = c_{i+1}^H(R)$ and $c_i^L(R) = c_{i+1}^L(R)$ for every $i \in J$. Therefore, the aggregate excess demand is the same in every period, $Z(R) \equiv \sum_{i \in J} (c_i^H(R) + c_i^L(R) - 2y_i)$. Boundary and monotonicity properties of Z guarantee that there exists a unique steady state equilibrium $R^* \in [1, 1/\beta_1]$ satisfying $Z(R^*) = d$. To ensure that this equilibrium is different from the autarky equilibrium of Proposition 2 (i), we assume that the lowest intertemporal rate of substitution of high-income agents is less than one:

$$\underline{R} \equiv \min_{i \in I} \underline{R}_i < 1 \quad \text{where} \quad \underline{R}_i \equiv \frac{u_i'(1 + \alpha_i)}{\beta_i u_i'(1 - \alpha_i)} \quad . \quad (\text{A1})$$

An outcome of this assumption is

Proposition 3: Let $L = \infty$ or $L = 0$ and assume (A1). Then:

- (a) Any symmetric economy has a unique non-autarkic stationary equilibrium without commitment at some $R^* \in [1, 1/\beta_1]$. If $d = 0$ and $L = 0$, then $R^* = 1$; otherwise $R^* > 1$.
- (b) Agent $i \in I$ is unconstrained if $R^* = 1/\beta_i$, agent i is periodically constrained if $R^* \in [\underline{R}_i, 1/\beta_i)$, and agent i is autarkic if $R^* < \underline{R}_i$.

The second part of Proposition 3 implies that asset markets are segmented whenever the population is sufficiently heterogenous. Suppose first that the discount factors of all agents are equal to β . Then the equilibrium is at $R^* = 1/\beta$ if either capital is sufficiently productive or agents are patient enough (Proposition 1). In this case there is no market segmentation since all agents are unconstrained and trade every period. However, when equilibrium occurs at some $R^* < 1/\beta$, all agents are rationed in low-income periods; but there is trade among them provided that $\underline{R} < 1$. Nevertheless, households with autarkic yields \underline{R}_i above the equilibrium interest rate R^* do not trade at all. This is necessarily the case whenever α_i is small enough. Intuitively, agents with small income variability value market participation too little; they are not permitted to borrow in equilibrium and the capital price is too high for them; therefore they remain autarkic. If average incomes y_i and income variability α_i are positively correlated (as is empirically plausible), agents with lower income will be less likely to trade in asset markets.

The segmentation of markets is even more pronounced when discount factors differ. If the equilibrium is then at $R^* = 1/\beta_1$ (which is the case if capital is productive enough or if the most patient agents are sufficiently patient), the most patient agents are unconstrained whereas all other agents are periodically rationed or autarkic. Agents are more likely to be excluded from asset markets if (i) their income variability is small, (ii) their discount factor is low, or (iii) their intertemporal elasticity of substitution in consumption is high.

It is worth pointing out that the equilibrium in the case of $L = 0$ and zero debt rations coincides with the equilibrium under an enforcement mechanism that excludes defaulting agents from all future borrowing but not from lending. Indeed it is easy to show that both models produce the same equilibrium allocations. This result resembles a finding of Bulow and Rogoff (1989) who consider a model of a small open economy that is cut off only from international borrowing but not from lending in the case of default; they show that this enforcement is too weak to establish an international loan market.

In the class of symmetric economies, the stationary equilibrium turns out to be determinate (locally unique). If the initial wealth distribution differs from the stationary wealth distribution, the dynamic adjustment to the stationary equilibrium takes either place within the first period (if $R^* < 1/\beta_1$) or by monotone convergence to the steady state (if $R^* = 1/\beta_1$).⁴ It is a remarkable feature of any equilibrium

⁴If the periodic endowment process has periodicity $n > 2$, monotonic convergence to the steady state is no longer guaranteed. Since the equilibrium dynamics shares similarities with

that the stationary consumption of constrained agents is independent of their initial wealth distribution, depending only on their preferences and endowment profiles. This is quite unlike the commitment equilibrium where the consumption allocation depends critically on the wealth distribution, a property that holds even in the long-run for the most patient agents. When capital is not productive ($d = 0$), there also exists an infinity of indeterminate equilibria converging to autarky, analyzed in Bewley (1980) and elsewhere. These results are collected in

Proposition 4: Assume that either $L = \infty$ or ($L = 0$ and intertemporal consumption goods are gross substitutes). Then the stationary equilibrium at R^* is determinate. Convergence to the stationary equilibrium takes place in one period if $R^* < 1/\beta_1$, and on a monotone adjustment path otherwise. When $d > 0$, the dynamic equilibrium is unique and converges to the stationary equilibrium. When $d = 0$, there also exists an infinity of dynamic equilibria converging to the autarky equilibrium $R = \underline{R}$.

5 Asset price fluctuations in asymmetric economies

We show in this section that equilibrium without commitment displays fluctuations in rates of return and individual consumption whenever the economy is not symmetric. The population of a generic economy in our class can be split into two subgroups: agents in group I_A have high income in even-numbered periods and agents in group I_B have high income in odd-numbered periods. Since the groups I_A and I_B are heterogeneous in non-symmetric economies, the characteristics of lending (unconstrained) and borrowing (constrained) agents fluctuate in a two-periodic fashion, so do asset demand and supply; this gives rise to periodic equilibrium fluctuations of interest rates and asset prices. As we show in Section 6, the crucial feature of time-varying characteristics of lenders and borrowers is not just confined to deterministic endowment processes, but applies equally well to a Markovian endowment process that preserves heterogeneity.

Suppose now that the interest yield fluctuates between R_A and R_B , the first applying in state A , i.e. when agents in group I_A have high income. Similarly to the

an overlapping-generations model of n -period living households, the results of Azariadis, Bullard, and Ohanian (2001) suggest that the adjustment dynamics might exhibit damped oscillations.

symmetric case, we can define excess demand correspondences for group I_A agents as $Z_A^H(R_A) \equiv \sum_{i \in I_A} (c_i^H(R_A) - y_i^H)$ and $Z_A^L(R_A) \equiv \sum_{i \in I_A} (c_i^L(R_A) - y_i^L)$. Excess demand correspondences for group I_B are $Z_B^H(R_B)$ and $Z_B^L(R_B)$. Markets clear in every period if

$$Z_A^H(R_A) + Z_B^L(R_B) = d \quad \text{and} \quad Z_A^L(R_A) + Z_B^H(R_B) = d . \quad (6)$$

Asset prices are $p_A = d(1 + R_B)/(R_A R_B - 1)$ and $p_B = d(1 + R_A)/(R_A R_B - 1)$. Existence, uniqueness and determinacy of such an equilibrium is shown in the following Proposition, generalizing Propositions 3 and 4 to asymmetric economies.

Proposition 5: Assume that either $L = \infty$ or ($L = 0$ and intertemporal consumption goods are gross substitutes). Then there exists a unique equilibrium without commitment in which interest yields (and asset prices) fluctuate between R_A and R_B (p_A and p_B) with $1 < R_A R_B \leq 1/\beta_1^2$. If $R_A R_B < 1/\beta_1^2$, all agents are debt constrained; otherwise some agents are unconstrained. The equilibrium is determinate, and convergence to the asymptotic state takes place either within one period or on a monotone adjustment path.

To understand how heterogeneity affects asset price fluctuations and to explore the impact of the economy's fundamentals on the size of these fluctuations, we discuss an example of an economy with symmetric endowments and an asymmetry in the rates of time preference. In Section 6 we discuss a stochastic economy in which the only asymmetry is in endowment profiles.

Example:

Consider an economy made up of two agents who are identical in everything except their rates of time preference. Formally, $\beta_1 = \beta > \beta_2 = \gamma$, $u_1 = u_2 = u$, $y_1 = y_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$, and $y_{10} = 1 + \alpha$, $y_{20} = 1 - \alpha$. We also assume that $d > 0$. The commitment equilibrium in this economy features a monotonically decreasing sequence of interest yields converging to $1/\beta$. Agent 1 saves in the earlier periods of his life to support an increasing consumption profile that converges to a positive number. Agent 2 borrows in the beginning and his consumption tends monotonically to zero as $t \rightarrow \infty$. Thus the commitment equilibrium involves no fluctuations of prices and individual consumption. In contrast, the same economy without commitment has permanently fluctuating interest rates whenever $\beta \neq \gamma$, and the consumption of each agent is positively correlated with his income.

We start by assuming an infinite exclusion length. Agent 2 (the less patient agent) is necessarily constrained in any equilibrium without commitment. The cyclical equilibrium of Proposition 5 consists of a pair of interest rates (R_A, R_B) and a consumption allocation $(c_i^H, c_i^L)_{i=1,2}$, and can be of two different types.

(UC) Agent 1 is unconstrained and agent 2 is constrained or autarkic.

(CC) Both agents 1 and 2 are rationed when their income is low.

It is easy to see that there can be no equilibrium in which agent 1 is constrained in every other period and agent 2 is autarkic. Productive capital must be held by some household in every period, and this agent cannot be constrained. Thus, at least one household must be unconstrained at every point in time.

Let us start with the equilibrium of type (CC). Since both agents are periodically constrained, their participation constraint binds when they have low income. The equilibrium can therefore be thought of as the intersection E of the two indifference curves IC_1 and IC_2 in the Edgeworth–box diagram of Figure 1. We need also to make sure that the Euler equations are satisfied. They are for agent 1 if

$$u'(c_1^H) = \beta R_A u'(c_1^L) , \quad u'(c_1^L) > \beta R_B u'(c_1^H) , \quad (7)$$

and for agent 2 if

$$u'(c_2^H) = \gamma R_B u'(c_2^L) , \quad u'(c_2^L) > \gamma R_A u'(c_2^H) . \quad (8)$$

From the binding first Euler equation in (8), the strict second equation in (7), and market clearing we obtain

$$\frac{u'(\Omega_d - c_2^H)}{u'(\Omega_d - c_2^L)} > \frac{\beta u'(c_2^H)}{\gamma u'(c_2^L)} . \quad (9)$$

This inequality is equivalent to the condition $c_2^L < \Phi(c_2^H)$, where Φ is increasing function that satisfies $\Phi(0) = 0$ and $\Phi(\Omega_d) = \Omega_d$, is convex if $\beta/\gamma > 1$, and equals c_2^H if $\beta = \gamma$. Φ is illustrated in Figure 1 as well and may be interpreted as the relevant contract curve in the Edgeworth–box diagram.⁵ When the intersection between the

⁵Note that the indifference curves in this diagram are those of high–income agents *at different points in time*. Therefore, tangency points of these indifference curves do not define the contract curve. Instead, the contract curve is defined by points of tangency between indifference curves of a high and a low income agent *at the same point in time*. One such contract curve (the curve Φ) is defined by tangency between IC_2 (the indifference curve of agent 2 with high income) and the indifference curve of agent 1 at low income. The other contract curve would be above the diagonal. With equal rates of time preference, the two contract curves fall together. The fact that they differ when $\beta \neq \gamma$ indicates that the asymptotic commitment equilibrium does not provide constant positive consumption for both agents.

two indifference curves is below the contract curve Φ (as at point E in Figure 1), then (9) is satisfied. Agent 1 is constrained at the interest rate R_B when he has low income (it can be easily checked that agent 2 is also constrained when he has low income). Therefore, the equilibrium is of type (CC).

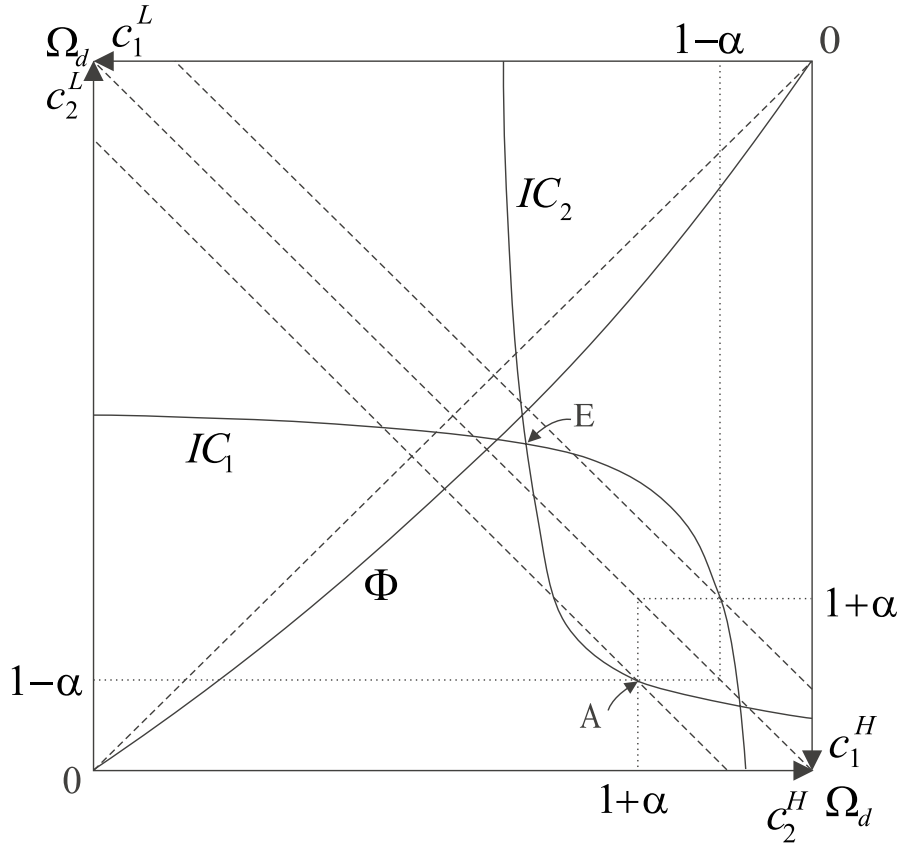


Figure 1: Equilibrium in an asymmetric two-agent economy (type (CC)).

Now let us turn to a type (UC) equilibrium in which agent 1 is unconstrained. Since both Euler equations in (7) must be satisfied with equality, (9) is satisfied with equality which means $c_2^L = \Phi(c_2^H)$. Therefore the equilibrium is at the unique intersection between the indifference curve of agent 2 and the contract curve Φ . Agent 1's participation constraint holds whenever this intersection is below agent 1's indifference curve in the Edgeworth-box diagram of Figure 1. Technically, this would occur if the graph of Φ shifts downward because of an increase in the ratio β/γ ; another factor that would work in the same direction is an increase in the dividend

d that shifts outward both indifference curves in the Edgeworth box. When β/γ becomes large, the intersection between IC_2 and Φ falls below the autarkic allocation at point A in Figure 1. In this case, the autarky equilibrium of Proposition 2 emerges in which agent 1 is unconstrained and holds the entire capital stock in every period, while agent 2 is autarkic in both periods.

To see that all equilibria involve asset price fluctuations, we start with an equilibrium of type (CC), i.e. at an intersection between the two indifference curves. Since agent 2 is more impatient than 1, his indifference curve is steeper. Therefore the intersection is at a point above the line $c_2^H + c_2^L = \Omega_d$ in Figure 1 where agent 2's marginal rate of substitution, R_B , exceeds agent 1's which is R_A . Hence, the interest rate fluctuates between R_A and R_B . Because the intersection is below Φ (and thus below the diagonal), both agents' consumption are positively correlated with their incomes, that is, $c_i^L < c_i^H$ for both agents. When the equilibrium is of type (UC), again both agents' consumption are positively correlated with income, which implies again that the interest rate fluctuates between R_A and R_B .

It is interesting to analyze how the size of fluctuations and the volume of trade in the asset depend on the fundamentals of the economy. As β/γ increases, individual consumption growth of both agents becomes more volatile, as do asset prices. At the same time, the volume of trade falls since agent 1 holds a larger fraction of the capital stock in both states. On the other hand, when d increases, agents are able to smooth consumption better, asset prices fluctuate less and the volume of trade goes up. Across different economies parameterized by β/γ and d , we observe the following two regularities: (i) the more productive the economy's asset is, the lower is the relative asset price volatility; (ii) the volume of trade in asset markets and the asset price volatility are negatively correlated. These two features seem to be in line with empirical observations. Whether these features are confined to specific examples or extend to more general environments remains a subject for further research.

What happens in this example economy if exclusion is of zero length and households cannot borrow at all? When an agent is debt constrained, he enters every period in which he has high income with zero assets and faces a zero debt constraint in every period in which he has low income. This implies that a constrained agent's savings behavior coincides with the behavior of a two-period living agent. Geometrically, this implies that the binding participation constraints (indifference curves) are to be replaced by offer curves of both agents which characterize the savings behavior of two period living agents with income profiles $(1 + \alpha, 1 - \alpha)$. Again, since agent

1 is more patient than agent 2, the two agents' offer curves intersect at some point above the line $c_2^H + c_2^L = \Omega_d$, and this is an equilibrium if this intersection is below Φ . If the intersection between the two offer curves is above Φ , the equilibrium is at the intersection between the offer curve of the constrained agent 2 and the curve Φ which describes the first-order condition for the unconstrained agent. Asset prices fluctuate in the same manner as before, but both agents are more likely to be constrained, and also their consumption is more sensitive to individual income fluctuations. Thus, c_i^H/c_i^L is larger with $L = 0$ than with $L = \infty$. This also suggests that asset price volatility is larger in an economy with a weaker enforcement mechanism.⁶

6 Uncertainty

This section constructs a simple two-state example which suggests that our main result extends to environments with uncertainty and complete asset markets. Heterogeneity in preferences and/or endowments generates asset price fluctuations that tend to be more pronounced if productivity is low or if enforcement is weak. This example also shows that the volatility of the return on the capital asset is greater than the volatility of the return on the short-term safe asset. Rate-of-return fluctuations in our example are thus in line with the so-called *equity volatility puzzle* (Campbell (2001)). With a reasonable amount of aggregate risk, the example economy with logarithmic preferences also generates a substantially bigger *equity premium* than either the corresponding economy with commitment or the corresponding symmetric economy without commitment.

Suppose there are two agents $i = 1, 2$ with identical preferences $E_0 \sum_{t=0}^{\infty} \beta^t u(c_{it})$. Endowments follow a Markov process defined on the two states $s = A, B$. In state A agent 1 has high income $1 + \alpha$ and agent 2 has low income $1 - \alpha$. State B is the opposite. Capital productivity equals $d \geq 0$ in each state, and aggregate income $\Omega_d = 2 + d$ is state-independent. The transition probability from state A to state B is π_A and the transition probability from state B to state A is π_B . The economy is symmetric whenever $\pi_A = \pi_B$, and the (symmetric) deterministic economies of

⁶Even in a symmetric economy, the similarity of the Bewley economy ($L = 0$) to an OLG model with two-period lived agents suggests an additional source of fluctuations. When substitutability between dated consumption goods is sufficiently weak, offer curves become backward-bending and the stationary equilibrium becomes indeterminate, creating the possibility of deterministic and stochastic cycles (see Grandmont (1985)).

Section 4 correspond to $\pi_A = \pi_B = 1$.

By analogy with the deterministic case, we look for a Markovian equilibrium in which prices and consumption depend only on the current state, and not on state history. Assuming complete markets, we denote by $q_{ss'}$, $s, s' \in \{A, B\}$, the price of an Arrow security in state s that promises to pay one unit of the consumption good if state s' prevails next period, and zero otherwise. Again we denote by R_s and p_s the safe gross interest rate and the capital price in state s . The absence of arbitrage opportunities requires that

$$1/R_A = q_{AA} + q_{AB} \quad , \quad 1/R_B = q_{BA} + q_{BB} \quad , \quad (10)$$

$$p_A = q_{AA}(d + p_A) + q_{AB}(d + p_B) \quad , \quad p_B = q_{BA}(d + p_B) + q_{BB}(d + p_A) \quad . \quad (11)$$

Let us first consider the equilibrium in the economy *with commitment*. Since agents smooth consumption perfectly across states and across time, the prices of Arrow securities fall out immediately from the Euler equations,

$$q_{AA} = \beta(1 - \pi_A) \quad , \quad q_{AB} = \beta\pi_A \quad , \quad q_{BA} = \beta\pi_B \quad , \quad q_{BB} = \beta(1 - \pi_B) \quad .$$

Hence asset prices and returns are constant in the commitment equilibrium:

$$R_A = R_B = \frac{1}{\beta} \quad , \quad p_A = p_B = \frac{d\beta}{1 - \beta} \quad .$$

Consider now the economy without commitment and infinite exclusion after default ($L = \infty$). Let us denote agent i 's equilibrium consumption profile by (c_{iA}, c_{iB}) , and let V_{iA} , V_{iB} denote the utility payoff to agent i in state A , B respectively. A simple recursive calculation shows that the market utility payoff for agent 1 with high income in state A is

$$V_{1A} = \left(1 - \beta(1 - \pi_A) - \frac{\beta^2 \pi_A \pi_B}{1 - \beta(1 - \pi_B)}\right)^{-1} \left(u(c_{1A}) + \frac{\beta\pi_A}{1 - \beta(1 - \pi_B)} u(c_{1B})\right) \quad .$$

Since agent 1's utility payoff from autarky (\bar{V}_{1A}) is defined similarly, we conclude that agent 1's participation constraint $V_{1A} \geq \bar{V}_{1A}$ is satisfied if

$$u(c_{1A}) + \frac{\beta\pi_A}{1 - \beta(1 - \pi_B)} u(c_{1B}) \geq u(1 + \alpha) + \frac{\beta\pi_A}{1 - \beta(1 - \pi_B)} u(1 - \alpha) \quad . \quad (12)$$

Similarly, agent 2's participation constraint in state B (when agent 2 has high income) is

$$u(c_{2B}) + \frac{\beta\pi_B}{1 - \beta(1 - \pi_A)} u(c_{2A}) \geq u(1 + \alpha) + \frac{\beta\pi_B}{1 - \beta(1 - \pi_A)} u(1 - \alpha) \quad . \quad (13)$$

A situation in which both agents are constrained in their supply of claims contingent on high-income states requires that both participation constraints (12) and (13) are satisfied with equality. Geometrically this happens to be the case at the intersection of the two indifference curves as shown previously in Figure 1, with an appropriate adjustment of notation. Whenever $\pi_A \neq \pi_B$, the “effective discount factors” in (12) and (13) differ, and the intersection of indifference curves is off the cross-diagonal that defines symmetric allocations.

An equilibrium with rationing requires that agents be constrained in their supply of claims contingent on their high-income state. For agent 1 this means $q_{BA} > \beta \frac{u'(c_{1A})}{u'(c_{1B})} \pi_B$. Since agent 2 is unconstrained in his demand for claims contingent on his low-income state A , we must have that $q_{BA} = \beta \frac{u'(c_{2A})}{u'(c_{2B})} \pi_B$. From these two conditions we see that agent 1 is constrained if, and only if,

$$\frac{u'(c_{1B})}{u'(c_{1A})} > \frac{u'(c_{2B})}{u'(c_{2A})} . \quad (14)$$

By a similar argument, agent 2 is constrained in his supply of claims contingent on his high-income state B if, and only if, (14) holds. Hence agent 1 is constrained in some state if, and only if, agent 2 is constrained in the other state. Moreover, market clearing implies that (14) holds if $c_{1A} > c_{1B}$ (or $c_{2A} < c_{2B}$), that is, if agents do not smooth consumption perfectly. Thus, the intersection of the indifference curves (12) and (13) is below the diagonal in Figure 1 (which is the contract curve in this case). Either both agents are rationed in the economy without commitment or neither is. Note that a simultaneous decrease in both transition probabilities lowers the effective discount factors of high-income households in (12) and (13) and makes both more likely to be constrained. Intuitively, if the income process is very persistent, agents need the market less and are more likely to be debt constrained (cf. also Kehoe and Levine (2001)).

It is now easy to show that asset prices fluctuate whenever the Markovian dynamics is asymmetric ($\pi_A \neq \pi_B$) and agents are constrained. Prices of Arrow securities are determined by the Euler equations of unconstrained agents, i.e.

$$q_{AA} = \beta \frac{u'(c_{1A})}{u'(c_{1A})} (1 - \pi_A) = \beta(1 - \pi_A) , \quad q_{AB} = \beta \frac{u'(c_{1B})}{u'(c_{1A})} \pi_A ,$$

$$q_{BB} = \beta \frac{u'(c_{1B})}{u'(c_{1B})} (1 - \pi_B) = \beta(1 - \pi_B) , \quad q_{BA} = \beta \frac{u'(c_{2A})}{u'(c_{2B})} \pi_B .$$

If $\pi_A > \pi_B$, the effective discount factor of agent 1 is higher than the one of agent 2; agent 1 needs the credit market more because his low-income state is more persistent. Therefore the intersection of indifference curves is, as in Figure 1, above the cross-diagonal which implies that $1 > c_{2A}/c_{2B} > c_{1B}/c_{1A}$ and therefore

$$1 < \frac{u'(c_{2A})}{u'(c_{2B})} < \frac{u'(c_{1B})}{u'(c_{1A})} . \quad (15)$$

It follows that $q_{AB} > q_{BA}$ and $q_{AA} < q_{BB}$. These inequalities mean that agent 2 pays a lower price to insure against the bad state than agent 1. From (10) and (15) we see that the safe rate is greater in state B :

$$\frac{R_B}{R_A} = \frac{q_{AA} + q_{AB}}{q_{BB} + q_{BA}} = \frac{1 + \pi_A(u'(c_{1B})/u'(c_{1A}) - 1)}{1 + \pi_B(u'(c_{2A})/u'(c_{2B}) - 1)} > 1 .$$

Moreover, the capital price is lower in state B :

$$\frac{p_B}{p_A} = \frac{q_{BB} + q_{BA} - q_{AA}q_{BB} + q_{AB}q_{BA}}{q_{AA} + q_{AB} - q_{AA}q_{BB} + q_{AB}q_{BA}} < 1 .$$

Since agent 1 has less persistently high income, he pays a higher price for capital and borrows at a higher interest rate than agent 2. We will discuss at the end of this section how much both the safe return and the return on the productive asset fluctuate.

Before we do that we take a brief look at an environment in which agents cannot be excluded from asset market trade ($L = 0$). Without any collateral, agents with low income cannot borrow against their future high income. Thus, markets for contingent claims are shut down and agents can only self-insure by trading the productive asset. As Kehoe and Levine (2001, Proposition 7) show, symmetric economies with incomplete asset markets do not have a stochastic steady state equilibrium. Similarly, asymmetric economies cannot have an asymptotic Markovian equilibrium in which current prices and allocations are time-invariant functions of the current state alone. The reason for this result is that a low-income household sells a fraction of its capital endowment in order to smooth consumption which implies that it enters next period with smaller capital holdings. If the household has low income in the next period as well, then it must consume less than in the period before. Thus, the equilibrium allocation in the incomplete markets economy cannot be just a function of the current state alone, but will depend on the whole history of state events.

We can however circumvent this problem by reverting to an alternative enforcement mechanism which appears to be the natural extension of the deterministic

Bewley model to stochastic environments with complete markets.⁷ Following the idea of Lustig (2001), we assume that the capital endowment of an agent serves as collateral. Hence, although a defaulting agent cannot be excluded from future asset market trading, his capital endowment can be seized in the case of default. With this alternative enforcement mechanism it turns out that there exists a stationary stochastic equilibrium in which low income agents do not sell their capital endowment, but they borrow against their future high income up to the debt limit implied by their capital collateral. When an agent is debt constrained, the binding debt limit implies that the agent enters any high-income period with zero net asset holdings; the value of his capital holdings exactly equals the loan that he has to repay. Therefore, the model is analogous to the deterministic Bewley economy in the sense that high-income agents enter every period with zero net asset holdings. Moreover, the stationary equilibrium in this economy is a stochastic cycle which is again determined by the intersection of offer curves of an appropriately defined two-period utility maximization problem. These utility functions are the same as the ones used in the Kehoe/Levine economy; they are defined by the left-hand sides of equations (12) and (13). Consequently, the equilibrium consumption allocation $(c_{1A}, c_{1B}, c_{2A}, c_{2B})$ solves the two offer-curve identities (a detailed derivation is contained in the Appendix)

$$0 = 1 + \alpha - c_{1A} + (1 - \alpha - c_{1B}) \frac{\beta \pi_A}{1 - \beta(1 - \pi_B)} \frac{u'(c_{1B})}{u'(c_{1A})}, \quad (16)$$

$$0 = 1 + \alpha - c_{2B} + (1 - \alpha - c_{2A}) \frac{\beta \pi_B}{1 - \beta(1 - \pi_A)} \frac{u'(c_{2A})}{u'(c_{2B})}, \quad (17)$$

plus market clearing. Whenever the two effective discount factors differ, the equilibrium allocation is asymmetric and asset prices fluctuate in the same manner as described before, though with greater volatility. We summarize our findings as follows.

Proposition 6: Let $L = \infty$ or ($L = 0$ and capital serves as collateral) and suppose that there are two identical agents with offsetting income profiles in the two states of the world A and B . Then there exists a unique Markovian equilibrium with fluctuations in interest rates and asset prices whenever the transition probabilities between the two states differ.

⁷In our interpretation of sectoral income shocks, complete asset markets are a less demanding assumption than it would be in an economy with idiosyncratic shocks and a large number of agents because completeness does not require a very large number of markets for state-contingent claims.

As the state changes over time, both the safe interest rate and the asset price fluctuate. It is interesting to analyze whether these fluctuations are in line with the empirical observations that the mean and the volatility of stock returns substantially exceed the mean and the volatility of the return of safe assets. The challenges that these phenomena imply for consumption-based asset pricing models have been called the *equity premium puzzle* and the *equity volatility puzzle* (see Campbell (2001)). Closed-form expressions for the mean and the volatility of the real interest rate R_s , $s = A, B$, and of the real stock return $R_{s s'}^e \equiv (d + p_{s'})/p_s$, $s, s' = A, B$, are difficult to obtain theoretically. Therefore we use numerical results with zero exclusion to illustrate the relation between these numbers.

Our example performs reasonably well in terms of the volatility of the two assets. Figure 2 shows the (unconditional) standard deviation of the stock return (dashed line) and the unconditional standard deviation of the safe return (solid line) as the transition probability π_B varies from zero to one. We have set $\pi_A = 0.2$ and $\beta = 0.9$, $\alpha = 0.2$, $d = 0.05$, $u(c) = \ln(c)$, and we used the economy with zero exclusion length and collateral capital.⁸ The symmetric economy ($\pi_B = 0.2$) produces no fluctuations, and neither does an asymmetric economy with sufficiently high transition probability ($\pi_B \geq 0.58$). In the latter case, agent 1 has persistently high income in state A and is rewarded with high average consumption for buying claims from agent 2 who has persistently low income and a desire to transfer consumption from state B to state A . Both agents have strong incentives to use asset markets; the outcome is a commitment equilibrium with no fluctuations in asset prices. However, if the economy is asymmetric and $\pi_B < 0.58$, agents are debt constrained, prices fluctuate and the volatility of the capital return exceeds the volatility of the safe return by roughly a factor of 2.

On the other hand, the differences between the mean stock return and the mean safe return turn out not to be substantial in this example economy without aggregate risk. Since the capital asset is intrinsically safe (the dividend is certain), it is not too surprising that its mean return is close to the mean safe return. For this reason, we also perform an experiment with aggregate risk. State A is the “good” state with high dividend and aggregate endowment $(d_A, \Omega_A) = (0.04, 2.11)$, and state B is the “bad” state with $(d_B, \Omega_B) = (0.03, 2.0)$. Instead of assuming an asymmetry in transition probabilities (which we now set to $\pi_A = \pi_B = 0.05$), we introduce another asymmetry that amplifies fluctuations: agent (or sector) 1 has large idiosyncratic

⁸In all variations of fundamentals and of the enforcement mechanism, results turn out to be similar.

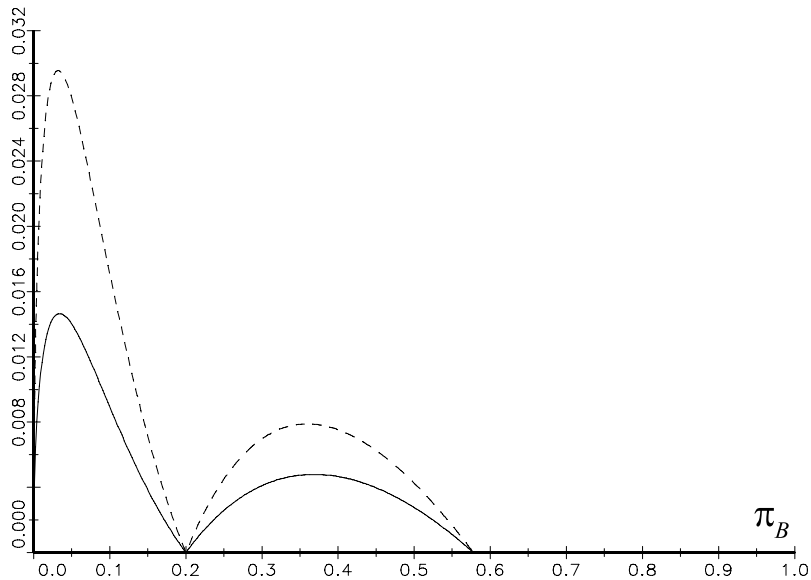


Figure 2: Return volatility of stocks (dashed) and safe asset (solid) for varying π_B ($\pi_A = 0.2$).

income risk and low mean income $(\alpha_1, y_1) = (0.95, 0.33)$; the opposite is true for agent (or sector) 2, i.e. $(\alpha_2, y_2) = (0.15, 1.72)$. For both agents we again set $\beta = 0.9$ and $u(c) = \ln(c)$. With this parameterization the volatility (standard deviation) of aggregate consumption growth is 1.28% and the volatility of dividend growth is 6.58%. Table 1 shows the mean and the volatility of the stock return R^e and of the safe return R for three economies: the first column shows the numbers for the economy *with commitment*. Here the equity premium is negligible and the mean stock return and the mean safe return are similar and close to $1/\beta$ which is too large if we interpret the length of the period to be one year. The volatility of the stock return is very low. In contrast, the *asymmetric no-commitment* economy (with $L = 0$) produces a significant equity premium and a low safe rate, and the volatility of the stock return is reasonably large. The only dimension in which this example does not perform well is the volatility of the riskless rate which turns out to be too high in relation to the data. The third column shows the results for the same economy but with *symmetric* endowment fluctuations, $\alpha_1 = \alpha_2 = 0.5$.⁹ The commitment economy

⁹Note that this economy could only be “symmetric” if there was no aggregate risk. With aggregate risk, the agents are not mirror images of each other since individual incomes are correlated with aggregate income.

is unchanged since it is independent of the size of individual income fluctuations, but the no–commitment economy with symmetry performs worse than the asymmetric economy. The equity premium is negligible and the stock return volatility falls even below the corresponding value for the commitment economy. In a richer example with more states or types of agents, similar outcomes may be obtained without extreme variations in individual endowments.

| | Commitment | No commitment (Asymmetry) | No commitment (Symmetry) |
|---------------|------------|------------------------------|-----------------------------|
| $E(R)$ | 1.11103 | 1.02090 | 1.04373 |
| $E(R^e)$ | 1.11157 | 1.05239 | 1.04376 |
| $\sigma(R)$ | 0.00319 | 0.08082 | 0.00229 |
| $\sigma(R^e)$ | 0.04362 | 0.16414 | 0.01495 |

Table 1: Mean and volatility of returns for a two–state, two–agent economy with aggregate risk and $L = 0$.

7 Conclusions and extensions

We have studied a class of pure exchange economies with stationary tastes, stationary consumption possibilities and no commitment to intertemporal trades. Default is punished by seizure of assets and market exclusion for $L \geq 0$ periods. If consumers are sufficiently heterogeneous and have correlated incomes, equilibria in these economies have unusual properties: rationing, market segmentation and, above all, non–stationarity. For $L = 0$ or $L = \infty$, most economies in this class have the following features:

1. Equilibrium is unique and cyclical without shocks to economic aggregates. The asymptotic equilibrium is a unique cycle in which yields and asset prices fluctuate while dividends and aggregate consumption are constant. In these economies, asset returns at each point in time reflect the marginal rates of substitution of agents who are not rationed at that point. As the set of unrationed agents changes over time, so do asset prices.
2. Rationing occurs every period, with some exceptions noted in Proposition 1 (b). Some, but not all, agents never engage in intertemporal trades; exceptions are noted in Propositions 1 and 2.

3. Symmetric economies have steady states in which all but the most patient agents are constrained if dividends are high, all agents are constrained if dividends are low.
4. Initial wealth is of no importance for the subsequent consumption of rationed agents.
5. Examples in Section 5 suggest that the amplitude of the limit cycle in asset prices and in individual consumption is negatively correlated with the productivity of assets and with the strength of the mechanism that enforces intertemporal trades. Fluctuations are quantitatively large if assets are unproductive and default is punished lightly.
6. The stochastic example in Section 6 shows that the factors responsible for asset price movements in deterministic economies work as well in environments with random individual incomes and complete markets, both with and without aggregate risk. The returns on risky assets and on safe assets fluctuate; but the volatility of the former appears to be considerably larger even if dividends show little volatility.

Keys to this behavior are correlated individual incomes, heterogeneity in tastes or endowments, and an extreme exclusion length $L = 0$ or $L = \infty$. Consumer heterogeneity and correlated individual incomes rule out steady states in all non-symmetric economies. And the exclusion length $L = 0$ or $L = \infty$ means that *current debt limits show very little or no sensitivity to changes in current prices*.

The remarkable similarity of equilibrium in economies with no exclusion and with infinite exclusion is notable in Kehoe and Levine (2001) who study an exchange economy like ours but with a pair of symmetric agents. In each case they find a unique steady state with either no rationing (if dividends are high) or rationing of low-income agents (if dividends are low). This similarity is unlikely to extend to environments in which the length of exclusion is some finite positive integer. Finite exclusion is relevant because credit rating agencies in actual economies have a finite memory of loan default or bankruptcy actions. Azariadis and Lambertini (2002), for example, examine the case $L = 1$ in an exchange economy of overlapping generations with identical agents and 3-period lifecycles. They find that equilibrium is no longer unique because the sensitivity of credit limits to expected yields generates income effects that overcome the intertemporal substitutability of consumption for

rationed agents. The outcome is several steady states, and another potential source of fluctuations.

Outside the scope of this paper lie important questions about economies with production which connect business cycles to growth. Ramey and Ramey (1995), for example, report a robust negative correlation between the mean growth rate of per capita income and its standard deviation in a cross section of 92 countries for the period 1960–1985. Acemoglu and Zilibotti (1997) attribute this fact to incomplete diversification in less developed countries. Our own results suggest that, even if markets are complete, volatile and slow growth may be the equilibrium outcome in environments with low productivity or with weak enforcement technologies.

Appendix

Proof of Corollary 1: (a) We first show that $\Phi_i(y_i, \alpha_i, \beta, L) > y_i$ whenever $\alpha_i > 0$ is sufficiently small. Indeed, $\Phi_i(y_i, 0, \beta, L) = y_i$ follows immediately from the definition (4). Furthermore, differentiation of (4) with respect to α_i at $\alpha_i = 0$ yields

$$u'_i(y_i)\Phi'_{i\alpha}(y_i, 0, \beta, L) = \frac{1 - \beta^L}{1 + \beta}(u'_i(y_i) - \beta u'_i(y_i)) + \beta^L u'_i(y_i) \frac{1 - \beta}{1 + \beta} > 0 ,$$

which implies $\Phi'_{i\alpha}(y_i, 0, \beta, L) > 0$. Therefore, $\Phi_i(y_i, \alpha_i, \beta, L) > y_i$ when α_i is small enough. Since $\sum_{i \in I} (y_i^H + y_i^L) = 2\Omega$, we have $\sum_{i \in I} y_i = \Omega$, and therefore

$$\sum_{i \in I} \Phi_i(y_i, \alpha_i, \beta, L) > \sum_{i \in I} y_i = \Omega .$$

Hence, when d is sufficiently small, (5) is violated, and Proposition 1 (b) implies that no commitment equilibrium is implementable.

(b) is obvious from Proposition 1 (b).

(c) When $\beta = 1$ and $L = \infty$,

$$u_i(\Phi_i(y_i, \alpha_i, 1, \infty)) = \frac{1}{2}(u_i(y_i(1 + \alpha_i)) + u_i(y_i(1 - \alpha_i))) < u(y_i) .$$

Hence, $\Phi_i(y_i, \alpha_i, \beta, L) < y_i$ when $\beta < 1$ and $L < \infty$ are large enough. Therefore,

$$\sum_{i \in I} \Phi_i(y_i, \alpha_i, \beta, L) < \sum_{i \in I} y_i \leq \Omega_d .$$

□

Proof of Proposition 2:

(i) Suppose $d = 0$. If agent i has high income in period t , he wants a loan at any yield $R_t \leq \underline{R}_i \equiv \frac{u'_i(1 + \alpha_i)}{\beta_i u'_i(1 - \alpha_i)}$. Hence, all high-income agents in period t want to borrow if $R_t = R \leq \underline{R} \equiv \min_{i \in I} \underline{R}_i$ for all $t \geq 0$. But when debt constraints are zero for all agents, no agent is allowed to borrow and can only consume his endowment in each period. Moreover, zero debt constraints are not too tight since participation constraints for all agents are satisfied with equality.

(ii) Suppose that agent 1 holds the entire capital stock. Agent 1 is then unconstrained in every period (i.e. he does not want to trade) whenever the interest rate fluctuates between the two values

$$R_A = \frac{u'_1(y_1^H + d)}{\beta_1 u'_1(y_1^L + d)} \quad \text{and} \quad R_B = \frac{u'_1(y_1^L + d)}{\beta_1 u'_1(y_1^H + d)} > R_A .$$

For every other agent $i > 1$, let

$$\underline{R}_i = \frac{u'_i(y_i^H)}{\beta_i u'_i(y_i^L)} \quad \text{and} \quad \bar{R}_i = \frac{u'_i(y_i^L)}{\beta_i u'_i(y_i^H)} > \underline{R}_i$$

denote the interest rates supporting the autarkic allocation. If agent i 's income ($i > 1$) is positively correlated with agent 1's income, he is debt constrained in both periods iff

$$\underline{R}_i > R_A \quad \text{and} \quad \bar{R}_i > R_B . \quad (18a)$$

If agent i 's income is negatively correlated with agent 1's income, agent i is debt constrained in both periods iff

$$\underline{R}_i > R_B \quad \text{and} \quad \bar{R}_i > R_A . \quad (18b)$$

(18a) is satisfied iff

$$\beta_i < \beta_1 \min \left(\frac{u'_1(y_1^L + d)u'_i(y_i^H)}{u'_1(y_1^H + d)u'_i(y_i^L)}, \frac{u'_1(y_1^H + d)u'_i(y_i^L)}{u'_1(y_1^L + d)u'_i(y_i^H)} \right) .$$

The term in the bracket defines the constant $k_i \leq 1$ for these agents. (18b) holds whenever $\underline{R}_i > R_B$ (since then $\bar{R}_i > \underline{R}_i > R_B > R_A$). But this is equivalent to

$$\beta_i < \beta_1 \frac{u'_1(y_1^H + d)u'_i(y_i^H)}{u'_1(y_1^L + d)u'_i(y_i^L)} .$$

The fraction on the right-hand side defines the constant $k_i < 1$ for these agents. \square

Proof of Proposition 3:

We assume first $L = \infty$ and look for a steady state equilibrium that consists of an interest rate R and a stationary allocation $(c_i^H, c_i^L)_{i \in J}$ (with identical consumption for the mirror agents $i \in I \setminus J$) satisfying market clearing $\sum_{i \in J} (c_i^H + c_i^L) = \Omega_d$ and the appropriate first-order conditions. For any agent i , the first-order conditions are the Euler equation and the participation constraint. In any period in which agent i has high income, these conditions are

$$u'_i(c_i^H) \geq \beta_i R u'_i(c_i^L) \quad , \quad u_i(c_i^L) + \beta_i u_i(c_i^H) \geq u_i(y_i^L) + \beta_i u_i(y_i^H) \quad , \quad (19)$$

with at least one inequality binding. Similarly, the first-order condition in a period of low income must be

$$u'_i(c_i^L) \geq \beta_i R u'_i(c_i^H) \quad , \quad u_i(c_i^H) + \beta_i u_i(c_i^L) \geq u_i(y_i^H) + \beta_i u_i(y_i^L) \quad , \quad (20)$$

again with at least one binding inequality. From (19) and (20), the agent can be in one of the following three situations: (i) The agent is always unconstrained; this requires that $R = 1/\beta_i$, and $c_i^H = c_i^L = c_i$ satisfies the participation constraint $u_i(c_i)(1 + \beta_i) \geq u_i(y_i^H) + \beta_i u_i(y_i^L)$; (ii) The agent is constrained when he has low income and unconstrained when he has high income. This implies that the Euler equation in (19) and the participation constraint in (20) are binding:

$$u'_i(c_i^H) = \beta_i R u'_i(c_i^L) \quad \text{and} \quad u_i(c_i^H) + \beta_i u_i(c_i^L) = u_i(y_i^H) + \beta_i u_i(y_i^L) \quad .$$

Moreover, the other constraints require that $R \in [\underline{R}_i, 1/\beta_i)$ and $c_i^H \leq y_i^H$, $c_i^L \geq y_i^L$; (iii) The agent is autarkic (always constrained), which requires that $R \leq \underline{R}_i$, $c_i^H = y_i^H$ and $c_i^L = y_i^L$.

Let agent i 's (stationary) demand correspondence $c_i^H(R)$ and $c_i^L(R)$ be defined by (i), (ii) and (iii) for any $R \leq 1/\beta_i$, and let $z_i(R) = c_i^H(R) + c_i^L(R) - 2y_i$ be the excess demand correspondence of agent i and his mirror agent $i + 1$. It is obvious from the definition that z_i has the following features:

1. z_i is a continuous function for all $R \in [0, 1/\beta_i)$ and vertical at $R = 1/\beta_i$.
2. $z_i(R) = 0$ if $R \leq \underline{R}_i$.
3. If $\underline{R}_i < 1$, $z_i(R) < 0$ for all $R \in (\underline{R}_i, 1]$.
4. $z'_i(R) > 0$ if $R > \max(1, \underline{R}_i)$.

From these features we immediately obtain the features of the excess demand correspondence of all agents $Z(R) \equiv \sum_{i \in J} z_i(R)$ which are stated in the following Lemma.

Lemma 1:

1. Z is a continuous function for all $R \in [0, 1/\beta_1)$ and vertical at $R = 1/\beta_1$.
2. $Z(R) = 0$ if $R \leq \underline{R} \equiv \min_{i \in J} \underline{R}_i$.
3. If $\underline{R} < 1$, $Z(R) < 0$ for all $R \in (\underline{R}, 1]$.
4. $Z(R)$ is strictly increasing for $R > \max(1, \underline{R})$.

From Lemma 1, which is illustrated in Figure 1, follows the existence of a unique stationary equilibrium $R^* > 1$ satisfying $Z(R^*) = d$:

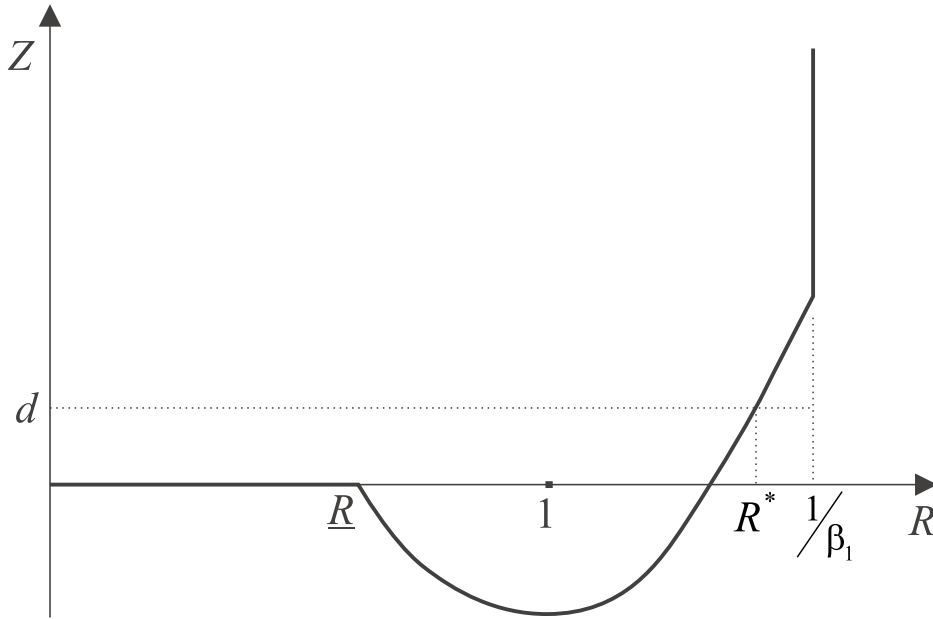


Figure 3: Excess demand in a symmetric economy with $L = \infty$.

Now consider the economy with $L = 0$. Again, every agent i can be in one of the three states (i), (ii) or (iii). The autarkic state (iii) is exactly as specified before, but the two other states differ since the participation constraint in (19) and (20) is replaced by zero debt constraints. First, an agent is unconstrained in both periods if

$$R = 1/\beta_i \quad , \quad c_i^H = c_i^L \geq y_i \left(1 + \alpha_i \frac{1 - \beta_i}{1 + \beta_i} \right) . \quad (21)$$

Second, an agent who is constrained at low income will move next to an unconstrained period of high income and zero initial debt. Hence, when $R \in [\underline{R}_i, 1/\beta_i)$, agent i faces every other period the two-period problem

$$\begin{aligned} & \max u_i(c_i^H) + \beta_i u_i(c_i^L) \\ \text{s.t. } & c_i^H + \frac{c_i^L}{R} \leq y_i(1 + \alpha_i + \frac{1 - \alpha_i}{R}). \end{aligned} \quad (22)$$

We now denote agent i 's consumption demand by the continuous correspondence $(\tilde{c}_i^H(R), \tilde{c}_i^L(R))$ which satisfies autarky if $R \leq \underline{R}_i$, solves (22) if $R \in [\underline{R}_i, 1/\beta_i)$ and satisfies (21) for $R = 1/\beta_i$. Let $\tilde{z}_i(R) = \tilde{c}_i^H(R) + \tilde{c}_i^L(R) - 2y_i$ be the excess demand correspondence of agent i and its mirror agent $i + 1$. From the observation that $(\tilde{c}^H, \tilde{c}^L)$ describes now the offer curve of the two-period problem (22) for $R \in (\underline{R}_i, 1/\beta_i)$, the features 1. and 2. for z_i hold also for \tilde{z}_i , whereas we now have

3. If $\underline{R}_i < 1$, $\tilde{z}_i(R) < 0$ for all $R \in (\underline{R}_i, 1)$ and $\tilde{z}_i(1) = 0$.
4. $\tilde{z}_i'(R) > 0$ if $R > \max(1, \underline{R}_i)$.

3. is obvious and 4. follows from the fact that dated consumption goods are normal. We denote the aggregate excess demand correspondence by $\tilde{Z}(R) \equiv \sum_{i \in J} \tilde{z}_i(R)$, $0 \leq R \leq 1/\beta_1$. \tilde{Z} has the following features:

Lemma 2:

1. \tilde{Z} is a continuous function for all $R \in [0, 1/\beta_1)$ and vertical at $R = 1/\beta_1$.
2. $\tilde{Z}(R) = 0$ if $R \leq \underline{R} \equiv \min_{i \in J} \underline{R}_i$.
3. If $\underline{R} < 1$, $\tilde{Z}(R) < 0$ for all $R \in (\underline{R}, 1)$ and $\tilde{Z}(1) = 0$.
4. $\tilde{Z}(R)$ is strictly increasing for $R > \max(1, \underline{R})$.

From these features follows the existence and uniqueness of a unique steady state $R^* \geq 1$ satisfying $\tilde{Z}(R) = d$. Moreover, when $d > 0$, $R^* > 1$. \square

Proof of Proposition 4: Assume first $L = \infty$. When the interest rate in period t is $R_t < 1/\beta_i$, the high income agent i is constrained with demands $c_{it}^H = c_i^H(R_t)$ and $c_{i,t+1}^L = c_i^L(R_t)$ defined by the Euler equation and the binding constraint

$$u_i'(c_{it}^H) = \beta_i R_t u_i'(c_{i,t+1}^L) \quad \text{and} \quad u_i(c_{it}^H) + \beta_i u_i(c_{i,t+1}^L) = u_i(y_i^H) + \beta_i u_i(y_i^L).$$

Suppose first that the stationary equilibrium is at $R^* < 1/\beta_1$ which defines a unique wealth distribution. Suppose that the initial wealth distribution is close to the

stationary wealth distribution so that in a dynamic equilibrium all agents are again constrained when they have low income or are autarkic. Market clearing in period $t \geq 2$ requires that

$$Z(R_t, R_{t-1}) \equiv Z^H(R_t) + Z^L(R_{t-1}) = d , \quad (23)$$

where $Z^H(R) \equiv \sum_{i \in J} (c_i^H(R) - y_i^H)$ and $Z^L(R) \equiv \sum_{i \in J} (c_i^L(R) - y_i^L)$. Since c_i^H is strictly decreasing and c_i^L is strictly increasing (when $R_t, R_{t+1} \in (\underline{R}_i, 1/\beta_i)$), we have

$$Z_1(R_t, R_{t+1}) > 0 , \quad Z_2(R_t, R_{t+1}) < 0 , \quad R_t, R_{t+1} \in (\underline{R}, 1/\beta_1) . \quad (24)$$

Using $\frac{d}{dR} Z(R, R)|_{R=R^*} > 0$ to obtain $Z_1(R^*, R^*) > -Z_2(R^*, R^*)$ and thus the steady state equilibrium at $R = R^*$ is determinate. Any dynamic equilibrium starting from $R_1 < R^*$ must satisfy $R_2 < R_1$ because of (24) and $Z(R_1, R_1) < d$. Consequently, $R_{t+1} < R_t$ for all $t \geq 1$. When $d > 0$, this sequence must be infeasible since it eventually converges to a negative interest rate. When $d = 0$, all these sequences converge to the autarky equilibrium at $R = \underline{R}$. On the other hand, any dynamic equilibrium starting from $R_0 > R^*$ must be infeasible as well since it involves ever increasing interest rates. Therefore, when $d > 0$, the dynamic equilibrium $(R_t)_{t \geq 0}$ satisfies $R_t = R^*$, $t \geq 1$. In periods 0 and 1, excess demands depend on the initial wealth of agents and on the initial borrowing constraints imposed on low-income agents which are $b_t = (b_{it})_{i \in J}$, $t = 0, 1$. Excess demands of the agents with high income in period zero are $Z_0^H(R_0, b_1)$ in period 0 and $Z_1^L(R_0, b_1)$ in period 1 since these agents are constrained in period 1; excess demands of the agents with low income in period 0 are $Z_0^L(b_0)$ in period 0 and $Z^H(R_1)$ in period 1. Hence, market clearing in periods 0 and 1 implies that

$$Z_0^H(R_0, b_1) + Z_0^L(b_0) = d , \quad Z_1^L(R_0, b_1) + Z^H(R_1) = d . \quad (25)$$

These equations together with the agents' budget constraints determine the initial interest rate R_0 and the debt constraints b_0 and b_1 .

Second, suppose that the stationary equilibrium is at $R^* = 1/\beta_1$, and let $I^u \subset I$ denote the group of most patient agents which has cardinality $n \geq 1$. These agents are unconstrained and their consumption satisfies the Euler equations that we write as

$$c_{i,t+1} = \Phi_i(R_t, c_{i,t}) \quad , \quad i \in I^u . \quad (26)$$

Let again $Z^H(R_t)$ and $Z^L(R_t)$ denote the excess demands of the group $I \setminus I^u$ of agent who are constrained. Then market clearing in period $t + 1 \geq 2$ requires that

$$Z^H(R_{t+1}) + Z^L(R_t) + \sum_{i \in I^u} (c_{i,t+1} - y_i) = d . \quad (27)$$

Equations (26) and (27) plus market clearing conditions analog to (25) in the two initial periods define the dynamic equilibrium. There are n initial values which are the capital shares of the unconstrained agents and one transversality condition requiring that the present value of the capital stock held by the group of unconstrained agents must be zero.¹⁰ Hence, the steady state equilibrium is determinate if the $(n + 1)$ -dimensional system (26) and (27) has n stable roots. At the steady state we have

$$\frac{dR_{t+1}}{dR_t} = -\frac{Z^{L'} + \sum_{i \in I^u} \Phi'_{i1}}{Z^{H'}} \equiv a \quad , \quad \frac{dR_{t+1}}{dc_{i,t}} = -\frac{\Phi'_{i2}}{Z^{H'}} = -\frac{1}{Z^{H'}} \equiv b \quad ,$$

$$\frac{dc_{i,t+1}}{dR_t} = \Phi'_{i1} \quad , \quad \frac{dc_{i,t+1}}{dc_{it}} = \Phi'_{i2} = 1 \quad .$$

Note that (24) implies that

$$a - b \sum_{i \in I^u} \Phi'_{i1} = -\frac{Z^{L'}}{Z^{H'}} > 1 \quad . \quad (28)$$

The characteristic polynomial is

$$(a - \lambda)(1 - \lambda)^n - \left(\sum_{i \in I^u} \Phi'_{i1} \right) b (1 - \lambda)^{n-1} = 0 \quad .$$

Therefore, $(n - 1)$ eigenvalues are equal to one, reflecting the fact that there is a $(n - 1)$ -dimensional subset of stationary consumption allocations among the n unconstrained agents.¹¹ The remaining roots are determined from the equation

$$(a - \lambda)(1 - \lambda) = \left(\sum_{i \in I^u} \Phi'_{i1} \right) b \quad .$$

Because of (28), the right-hand side is larger than the left-hand side at $\lambda = 0$ and for large λ , and it is smaller than the left hand side at $\lambda = 1$. Hence, the two roots satisfy $0 < \lambda_1 < 1 < \lambda_2$, so that the steady state is determinate and convergence is monotonic.

If $L = 0$, the constrained agent's demand is defined by the offer curve $c_{it}^H = \tilde{c}_i^H(R_t)$ and $c_{i,t+1}^L = \tilde{c}_i^L(R_t)$ so that market clearing, in a constrained equilibrium, is described

¹⁰The initial capital holdings of constrained agents are further initial conditions, but they determine uniquely the initial borrowing constraints imposed on these agents.

¹¹If period utility functions of agents are identical homothetic, the dynamical system reduces to a simpler two-dimensional system with one stable and one unstable root.

by an analogous identity $\tilde{Z}(R_t, R_{t+1}) = 0$. Since we assume gross substitutes, \tilde{c}_i^H is decreasing and \tilde{c}_i^L is increasing, and therefore \tilde{Z} has the same features as Z in (24). The rest of the proof is the same. \square

Proof of Proposition 5: Assume first $L = \infty$. We define excess demand correspondences for agents in group I_A as follows. Agent $i \in I_A$ can be in one of the following situations: (i) unconstrained if

$$u'_i(c_i^H) = \beta_i R_A u'_i(c_i^L) \quad \text{and} \quad R_A = \frac{1}{\beta_i^2 R_B},$$

together with the usual participation constraints; (ii) constrained when he has low income if

$$u'_i(c_i^H) = \beta_i R_A u'_i(c_i^L), \quad u_i(c_i^H) + \beta_i u_i(c_i^L) = u_i(y_i^H) + \beta_i u_i(y_i^L), \quad (29)$$

whenever $R_A \in [\underline{R}_i, 1/(\beta_i^2 R_B))$; (iii) autarkic when $R_A \leq \underline{R}_i$. Similar to the symmetric case, these conditions give rise to the aggregate demand correspondences $Z_A^H(R_A) \equiv \sum_{i \in I_A} (c_i^H - y_i^H)$ and $Z_A^L(R_A) \equiv \sum_{i \in I_A} (c_i^L(R_A) - y_i^L)$, both of which are continuous functions for $R_A < 1/(\beta_A^2 R_B)$ and vertical at $R_A = 1/(\beta_A^2 R_B)$ where $\beta_A \equiv \max_{i \in I_A} \beta_i$. The aggregate excess demand correspondences of group I_A have the following features.

Lemma 3:

1. Z_A^H and Z_A^L are piecewise differentiable functions for $R_A < 1/(\beta_A^2 R_B)$ and they are vertical at $R_A = 1/(\beta_A^2 R_B)$.
2. $Z_A^H(R_A) = Z_A^L(R_A) = 0$ when $R_A \leq \underline{R}_A \equiv \min_{i \in I_A} \underline{R}_i$.
3. $Z_A^{H'}(R_A) < 0$ and $Z_A^{L'} > 0$ for $R_A \in (\underline{R}_A, 1/(\beta_A^2 R_B))$.
4. The function $\phi_A(R_A) \equiv -\frac{Z_A^{H'}(R_A)}{Z_A^{L'}(R_A)}$ is downward-sloping.

Proof: 1., 2. and 3. are obvious from the definition. To show part 4, we need first another Lemma.

Lemma 4: The individual consumption demands as defined by (29) satisfy

- (a) $\frac{c_i^{H''} R}{c_i^{H'}} = \frac{c_i^{L''} R}{c_i^{L'}} - 1$,
- (b) $R c_i^{H'} + c_i^{L'} = 0$.

Proof: We use the short notation c_H instead of c_i^H and u_H instead of $u_i(c_i^H)$ (the same for c_L and u_L). Differentiate the two identities in (29) to arrive at

$$u_H'' c_H' = \beta u_L' + \beta R u_L'' c_L' , \quad (30)$$

$$u_H' c_H + \beta u_L' c_L = 0 . \quad (31)$$

(31) together with the Euler equation in (29) yields part (b). Differentiate (31) again to obtain

$$u_H'' (c_H')^2 + u_H' c_H'' + \beta u_L'' (c_L')^2 + \beta u_L' c_L'' = 0 .$$

Inserting (30) gives

$$c_H' (\beta u_L' + \beta R u_L'' c_L') + u_H' c_H'' + \beta u_L'' (c_L')^2 + \beta u_L' c_L'' = 0 .$$

Using the Euler equation again and part (b) gives the formula in part (a). \square

To complete the proof of part 4 of Lemma 3, note that ϕ_A is downward sloping iff

$$\frac{Z_A^{H''} R_A}{Z_A^{H'}} < \frac{Z_A^{L''} R_A}{Z_A^{L'}} , \quad (32)$$

Observe that this inequality holds whenever this inequality is true for the aggregate consumption demands C_A^H and C_A^L of group I_A instead of their excess demands. Using Lemma 4, we find that

$$\begin{aligned} \frac{C_A^{H''} R}{C_A^{H'}} &= \sum_{i \in I_A} \frac{c_i^{H''} R}{c_i^{H'}} \cdot \frac{c_i^{H'}}{C_A^{H'}} \\ &= \sum_{i \in I_A} \left(\frac{c_i^{L''} R}{c_i^{L'}} - 1 \right) \cdot \frac{c_i^{L'}}{C_A^{L'}} \\ &= \frac{C_A^{L''} R}{C_A^{L'}} - 1 . \end{aligned}$$

This proves (32) and thus Lemma 3. \square

To show Proposition 5 we need to find a unique solution to the two market clearing equations

$$Z_A^H(R_A) + Z_B^L(R_B) = d , \quad (33)$$

$$Z_A^L(R_A) + Z_B^H(R_B) = d . \quad (34)$$

for some $R_A \in [\underline{R}_A, 1/(\beta_A^2 R_A)]$ and $R_B \geq \underline{R}_B$ (w.l.o.g. we assume that $\beta_A \geq \beta_B$). We say that (R_A, R_B) are *interior* if $R_A R_B < 1/\beta_A^2$, and *on the boundary* otherwise.

(33) and (34) are upward-sloping curves in (R_A, R_B) -space because of Lemma 3, part 3. Because of Lemma 3, parts 2 and 3, (33) has a solution (\underline{R}_A, R_B) with $R_B > \underline{R}_B$, and (34) has a solution (R_A, \underline{R}_B) with $R_A > \underline{R}_A$.

The two curves can have at most one interior intersection. Suppose this is not the case, so that there are two interior intersections at $(R_A, R_B) \ll (\tilde{R}_A, \tilde{R}_B)$. This implies that (34) is steeper than (33) at the lower return equilibrium, but flatter at the other. (34) is steeper than (33), if and only if,

$$\left. \frac{dR_B}{dR_A} \right|_{(34)} = -\frac{Z_A^L}{Z_B^H} > \left. \frac{dR_B}{dR_A} \right|_{(33)} = -\frac{Z_A^H}{Z_B^L},$$

which holds whenever $1 > \phi_A(R_A)\phi_B(R_B)$. Therefore, $\phi_A(R_A)\phi_B(R_B) < 1 < \phi_A(\tilde{R}_A)\phi_B(\tilde{R}_B)$, a contradiction to Lemma 3, part 4, and $R_A < \tilde{R}_A$, $R_B < \tilde{R}_B$. Hence, there is at most one interior intersection.

Suppose the curve (33) cuts the line $R_A R_B = 1/\beta_A^2$ at some (\hat{R}_A, \hat{R}_B) and suppose the curve (34) cuts this line at some $(\tilde{R}_A, \tilde{R}_B)$. If there exists an interior intersection between (33) and (34), then $\hat{R}_A > \tilde{R}_A$ and $\hat{R}_B < \tilde{R}_B$. In this case there cannot be an equilibrium on the boundary; the first market clearing condition (33) is only fulfilled at the boundary for some $R_B \leq \hat{R}_B$ (lowering Z_B^L) and a higher level of Z_A^H (since some agents in group I_A are unconstrained); the second condition (34) would only be fulfilled for $R_B \geq \tilde{R}_B > \hat{R}_B$ (lowering Z_B^H) and a higher level of Z_A^L (since some agents in group I_A are unconstrained). Therefore, no equilibrium on the boundary exists, and the interior equilibrium is the unique equilibrium. If, on the contrary, no interior equilibrium exists, then $\hat{R}_A \leq \tilde{R}_A$ and $\hat{R}_B \geq \tilde{R}_B$. In this case a simple continuity argument shows that there must be a unique boundary equilibrium (R_A, R_B) with $\hat{R}_A \leq R_A \leq \tilde{R}_A$ and $\hat{R}_B \geq R_B \geq \tilde{R}_B$. In any case, this proves the existence of a unique two-cyclical equilibrium. $R_A R_B > 1$ can be easily shown by adding up the budget constraints of all individuals and using market clearing.

To show that the limit cycle is determinate, consider the dynamic equations

$$Z_A^H(R_t) + Z_B^L(R_{t-1}) = d \quad , \quad Z_A^L(R_t) + Z_B^H(R_{t+1}) = d \quad ,$$

that apply in any period $t \geq 2$ (with obvious variations in the initial periods). These two curves define monotonically increasing functions $R_t = \psi_1(R_{t-1})$ and $R_{t+1} = \psi_2(R_t)$. Hence, the second iterate $\psi_2(\psi_1(\cdot))$ is an increasing function mapping R_{t-1} into R_{t+1} . If there is an interior limit cycle (R_A, R_B) , this map has a steady state at R_B and we obtain that

$$\left. \frac{dR_{t+1}}{dR_{t-1}} \right|_{R_B} = \frac{1}{\phi(R_A)\phi(R_B)} > 1.$$

Therefore the limit cycle is determinate and there are no other dynamic equilibria except those which coincide with the limit cycle after period $t \geq 2$. If the limit cycle is on the boundary (some agents are unconstrained), a similar analysis to Proposition 5 shows that the limit cycle has a saddle-path property, so that the unique dynamic equilibrium converges monotonically to the limit cycle.

Finally, consider the case $L = 0$. The existence proof is exactly the same as before, except that we need to show that there can be at most one interior solution to the equations (33) and (34). With zero exclusion, we have

$$Z_A^H(R_A) = -s_A(R_A) \quad , \quad Z_A^L(R_A) = R_A s_A(R_A) \quad ,$$

where $s_A(R_A) = -\sum_{i \in I_A} (y_i^H - \tilde{c}_i^H)$ is the aggregate savings function of group I_A which is non-decreasing because of gross substitutes. The market clearing equations become

$$R_B s_B(R_B) - s_A(R_A) = d \quad , \quad R_A s_A(R_A) - s_B(R_B) = d \quad . \quad (35)$$

Both equations define upward-sloping curves in (R_A, R_B) -space, but when we rearrange them, we obtain

$$s_A(R_A) = d(1 + R_B)/(R_A R_B - 1) \quad , \quad s_B(R_B) = d(1 + R_A)/(R_A R_B - 1) \quad . \quad (36)$$

These both equations define downward-sloping curves. Hence, one of the downward-sloping equations in (36) together with one of the upward-sloping equations in (35) imply that any interior equilibrium must be unique. This completes the proof of Proposition 5. \square

Derivation of (16):

Let x_{iA}, x_{iB} denote the capital shares of agent $i = 1, 2$ and let $a_{iss'}$ be the claims of agent i against state s' tomorrow if the state today is s . If state B prevailed yesterday, the budget constraint of agent 1 in state A today is

$$0 = 1 + \alpha + a_{1BA} + (p_A + d)x_{1B} - c_{1A} - q_{AA}a_{1AA} - q_{AB}a_{1AB} - p_A x_{1A} \quad . \quad (37)$$

The budget constraint of agent 1 in state A , provided that state A prevailed yesterday, is

$$0 = 1 + \alpha + a_{1AA} + (p_A + d)x_{1A} - c_{1A} - q_{AA}a_{1AA} - q_{AB}a_{1AB} - p_A x_{1A} \quad .$$

From these two identities follows that

$$a_{1AA} - a_{1BA} = (p_A + d)(x_{1B} - x_{1A}) \quad . \quad (38)$$

By the same argument as in the case $L = \infty$, agent 1 is constrained in his supply of claims against the high income state if and only if agent 2 is. Let us thus assume that both agents are constrained. Agent 1's supply of claims against state A in state B is constrained by his collateral:

$$a_{1BA} = -(p_A + d)x_{1B} . \quad (39)$$

(38) implies then that

$$a_{1AA} = -(p_A + d)x_{1A} .$$

Similarly, for the constrained agent 2 we obtain

$$a_{2AB} = -(p_B + d)x_{2A} \quad , \quad a_{2BB} = -(p_B + d)x_{2B} . \quad (40)$$

(40) and asset market clearing implies that

$$a_{1AB} = (p_B + d)(1 - x_{1A}) \quad , \quad a_{1BB} = (p_B + d)(1 - x_{1B}) . \quad (41)$$

Therefore, the net asset holdings of agent 1 at the beginning of any low-income period (state B) are

$$a_{1BB} + (p_B + d)x_{1B} = a_{1AB} + (p_B + d)x_{1A} = p_B + d . \quad (42)$$

Hence, low income agents hold the value of the total capital stock and high income agents hold a zero net asset position. Using (11), (39), (41), (42), the budget constraint of agent 1 in state B is

$$\begin{aligned} 0 &= 1 - \alpha + a_{1BB} + (p_B + d)x_{1B} - c_{1B} - q_{BA}a_{1BA} - q_{BB}a_{1BB} - p_Bx_{1B} \\ &= 1 - \alpha + p_B + d - c_{1B} + q_{BA}(p_A + d)x_{1B} - q_{BB}(p_B + d)(1 - x_{1B}) - p_Bx_{1B} \\ &= 1 - \alpha - c_{1B} + (p_B + d)(1 - q_{BB}) . \end{aligned} \quad (43)$$

Similarly, (37) can be rewritten as

$$0 = 1 + \alpha - c_{1A} - q_{AB}(p_B + d) . \quad (44)$$

The two budget constraints (43) and (44) together yield

$$0 = 1 + \alpha - c_{1A} + (1 - \alpha - c_{1B}) \frac{q_{AB}}{1 - q_{BB}} .$$

From $q_{BB} = \beta(1 - \pi_B)$ and $q_{AB} = \beta\pi_A u'(c_{1B})/u'(c_{1A})$ we obtain the offer curve equation for agent 1 in the text. The one for agent 2 is similar. \square

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