

Simple Markov equilibria  
for overlapping generations

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### **Abstract**

In Markovian economies of overlapping generations, simple Markov equilibria may not exist; this is the case with intra-generational heterogeneity and multiple commodities or long life spans.

At a simple Markov equilibrium, prices at a date-event are determined by the realization of the shock and the distribution of wealth.

Markov equilibria exist if prices are allowed to vary also with the realization of the shock, prices and the consumption allocation at the predecessor date-event.

Key words: Markov equilibria, state space.

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# 1 Introduction

In a market economy over time under uncertainty, prices of commodities and assets vary across date-events.

At a competitive equilibrium, optimizing individuals must know the price process. It is thus important that the price process be simple. One way to achieve simplicity is for prices to depend on a minimal set of states that follow a Markov process — the equilibrium in Lucas (1978) is an example.

A stochastic process is a Markov process if the distribution of uncertainty at a date depends on the realization at the immediately preceding date, but not before; functions with domain the realizations of a Markov process are Markovian.

An economy is Markovian if the stochastic process that generates fundamentals is a Markov process.

The existence of Markov equilibria for competitive economies was demonstrated in Duffie, Geanakoplos, Mas-Colell and McLennan(1992); the setting is general and encompasses economies of infinitely lived individuals, as well as economies of overlapping generations, either with a complete or an incomplete asset market.

A question that arises concerns the minimal state space that supports a Markov competitive equilibrium.

Most simply, the state space would consist of the state space that generates fundamentals; nevertheless, other than in cases that are exceptional, if of interest, as in the representative individual economy in Lucas (1978), such an equilibrium may not exist. Spear (1985) showed that heterogeneity of individuals suffices to preclude, generically, equilibria supported by the state-space of fundamentals.

Endogenous variables need to enter the state space that supports competitive equilibria.

A natural conjecture is that the relevant state space is determined by the variables, endogenous as well as exogenous, sufficient for the optimization of every individual at every date-event.

In an exchange economy subject to endowment or preference shocks, it is the distribution of wealth that, the intuition goes, if added to the fundamentals, defines the state-space that can support simple competitive equilibria; in a production economy, the capital stock should complete the picture. We call such equilibria simple — or recursive, as in Ljungqvist and Sargent (2000).

Here, we examine Markov equilibria in stationary economies with overlapping generations and finite life spans.

First, we give two examples in which simple Markov equilibria do not exist.

Subsequently, we characterize a state space that supports Markov equilibria.

The literature on computable equilibria focuses on simple equilibria; Heaton and Lucas (1996), Rios-Rull (1996) and Storesletten, Telmer and Yaron (1999) are instances of this practice; more recently, Geanakoplos, Magil and Quinzii (2001), follow the practice as well, even if explicitly aware that a general existence theorem is not available.

In a recent paper, Kübler and Schmedders (2002) gave an example that demonstrates the non-existence of simple equilibria in models with incomplete asset markets and infinitely lived agents: the state space that consists of the shock and the distribution of beginning of date portfolios does not suffice to support a competitive equilibrium.

The intuition that underlies the non-existence of simple equilibria in a Markovian economy is straightforward: the realization of the shock and the distribution of wealth suffice for the optimization of every individual at any date; however, with these initial conditions for the economy starting at that date, there may be multiple equilibria; the distribution of wealth is endogenous, and it is determined by the choices of individuals at preceding date-events; these choices may lead to the current distribution of wealth only if individuals expect a particular one of the possible continuation equilibria to prevail. It follows that the equilibrium at a date may depend on the realization of a state at preceding events.

If the competitive equilibrium for the economy that starts at any date-event and with any distribution of wealth is unique, then, *à fortiori*, a simple Markov equilibrium exists

The non-existence of simple equilibria thus derives from the non-uniqueness of continuation equilibria. This intuition was first explained and illustrated in Hellwig (1983).

The non-existence of simple equilibria has been demonstrated in different contexts. Santos (2000) derived non-existence for economies with externalities; Krebs (2001) derived non-existence of recursive equilibria with a continuous price function, a compact state-space in economies with infinitely-lived individuals and idiosyncratic shocks.

Markov equilibria exist if prices are allowed to vary with the realization of the shock, the prices of assets and the consumption allocation at the predecessor date-event as well.

The examples of non-existence of simple Markov equilibria are not robust; it remains a conjecture whether simple equilibria exist generically.

Another open question is to fully characterize the minimal state-space that supports Markov competitive equilibria in a general setting.

## 2 The Economy

We consider an exchange economy of overlapping generations with stochastic shocks to endowments and with individuals that live for several dates.

### Fundamentals

Dates are  $t = 0, \dots$

Exogenous shocks,  $s_t$ , follow a Markov process with finite support  $\mathcal{S} = \{s\}$  and transition  $\Pi = \{\pi(s'|s)\}$ .

Date-events or nodes are histories of shocks  $\sigma_t = (s_1, \dots, s_t)$ . The  $n$ 'th predecessor of  $\sigma_t$  is  $\sigma_{t-n}$  and  $\sigma_{t+k} = (\sigma_t, s_{t+1}, \dots, s_{t+k})$  is a generic  $k$ 'th successor of  $\sigma_t$ .

At each date-event, finitely many, perishable commodities are available for consumption; a bundle commodities is  $x(\sigma_t)$ .

The demographic structure is that of overlapping generations; at each date-event, finitely many individuals,  $\mathcal{I} = \{i\}$ , commence their economic lives; they live for  $(N + 1)$  dates; an individual,  $(i, \sigma_t)$ , exchanges and consumes commodities at the date-event  $\sigma_t$ , and the successor date-events,  $(\sigma_t, s_{t+1}), \dots, (\sigma_t, s_{t+1}, \dots, s_{t+N})$ ; the age of an individual is  $a = 0, \dots, N$  and the shock at the date that an individual is of age  $a$  is  $s_{t+a}$ .

The economy is stationary:

1. The endowment of individual  $(i, \sigma_t)$  at the date-event  $\sigma_{t+a}$  is

$$e^{i, \sigma_t}(\sigma_{t+a}) = e^{i, a}(s_{t+a});$$

2. his intertemporal, von Neumann-Morgenstern utility function is

$$U^{i, \sigma_t} = E_{s_{t+a}} \sum_{a=0}^N u^{i, a}(x(\sigma_{t-1}, \dots, s_{t+a}), s_{t+a}).$$

Standard restrictions on preferences and endowments are needed to prove the existence of an equilibrium.

**Assumption 1.** *For every individual,*

1. *the function  $U^{i, \sigma_t}$  is strictly quasi-concave;*
2. *the function  $u^{i, a}(\cdot, s)$  is monotonically increasing and continuous;*
3. *the function is strictly increasing in commodity  $l = 1$  and unbounded:  $u^{i, a}(x, s) \rightarrow \infty$  as  $x_1 \rightarrow \infty$ ;*
4. *the endowment in commodity  $l = 1$  is strictly positive:  $e_1^{i, a}(s_t) > 0$ .*

In order to close the specification, it is necessary to suppose that at the root node,  $\sigma_0 = s_0$ , there are individuals of all types and ages  $(i, -a)$ ,  $a = 1, \dots, N$ ; their initial holdings in securities determine the initial conditions of the economy.

## Markets

There are complete spot markets for commodities, at each date-event. Prices of commodities are  $p(\sigma_t)$ ; commodity  $l = 1$  is numéraire, and  $p_1(\sigma_t) = 1$ .

In order to smooth consumption individuals can trade in assets. At each date event, there are assets  $j \in \mathcal{J}$ , a non-empty set; a portfolio of assets is  $\theta(\sigma_t)$ .

Prices of assets are  $q(\sigma_t)$ .

The payoff, at the date-event  $\sigma_{t+1}$ , of asset  $j$  traded at the date-event  $\sigma_t$  is

$$d^j(s_{t+1}) = (d_1^j(s_{t+1}), d_2^j(s_{t+1})),$$

where  $d_1^j(s_{t+1})$  are units of the numéraire commodity, while  $d_2^j(s_{t+1})$  are units of the asset  $j$ ;

Assets  $j \in \mathcal{J}_1$  are one date assets:  $d_2(s) = 0$ , for all  $s$ ; assets  $j \in \mathcal{J}_2$  are long-lived or trees:  $d_2^j(s) = 1$ , for all  $s$ .

For long-lived assets, individuals face a short-sales constraint:  $\theta(\sigma_t) \geq 0$ ; this guarantees the continuity of demand.

## Equilibrium

The budget set of individual  $(i, \sigma_t)$  is

$$\mathcal{B}^{i, \sigma_t}(p, q) = \left\{ (x, \theta) : \begin{array}{l} p(\sigma_t)(x(\sigma_t) - e^{i, \sigma_t}(\sigma_t)) \leq -q(\sigma_t)\theta(\sigma_t), \\ p(\sigma_{t+a})(x(\sigma_{t+a}) - e^{i, \sigma_t}(\sigma_{t+a})) \leq \\ \quad \sum_{j \in \mathcal{J}} (1, q(\sigma_{t+a})) d^j(s_{t+a}) \theta_j(\sigma_{t+a-1}) - \\ \quad q(\sigma_{t+a}) \theta(\sigma_{t+a}), \quad a = 1, \dots, N-1, \\ p(\sigma_{t+N})(x(\sigma_{t+N}) - e^{i, \sigma_t}(\sigma_{t+N})) \leq \\ \quad \sum_{j \in \mathcal{J}} (1, q(\sigma_{t+N})) d^j(s_{t+N}) \theta_j(\sigma_{t+N-1}), \\ x \geq 0, \quad \theta_j \geq 0, \quad j \in \mathcal{J}_2 \end{array} \right\}.$$

**Definition 1.** For initial holdings of assets  $\bar{\theta}^{i, -a}$ , a competitive equilibrium is a collection of consumption plans, portfolio holdings and prices of commodities and assets,  $(x^{i, \sigma_t}, \theta^{i, \sigma_t}, p, q)$ , such that

1. markets clear:

$$\sum_{i \in \mathcal{I}} \sum_{a=0}^N x^{i, \sigma_t - a}(\sigma_t) - e^{i, \sigma_t - a}(\sigma_t) = \sum_{j \in \mathcal{J}_2} d_1^j(s_t),$$

$$\sum_{i \in \mathcal{I}} \sum_{a=0}^N \theta_j^{i, \sigma_t - a}(\sigma_t) = 0, \quad j \in \mathcal{J}_1,$$

$$\sum_{i \in \mathcal{I}} \sum_{a=0}^N \theta_j^{i, \sigma_t - a}(\sigma_t) = 1, \quad j \in \mathcal{J}_2,$$

2. and individuals optimize:

$$(x^{i, \sigma_t}, \theta^{i, \sigma_t}) \in \arg \max \{U^{i, \sigma_t}(x) : (x, \theta) \in \mathcal{B}^{i, \sigma_t}(p, q)\}.$$

### 3 Non-existence of a simple equilibria

A Markov equilibrium is characterized by a pair of functions, a policy function that maps the current state into current date endogenous variables and a transition function that describes the evolution of the state. This description leaves open what variables can be used to describe the current state of the economy and how to define the set of admissible states.

The applied literature assumes that the shock and beginning-of-date portfolio-holdings constitute a sufficient statistics for the future evolution of the economy.

**Definition 2.** *A simple Markov or recursive equilibrium is described by a set of holdings of assets,  $\Theta$ , that contains an open set, and a collection of functions,  $\{f_x, f_\theta, f_p, f_q\}$  with domain  $\Theta \times \mathcal{S}$ , such that for every  $\bar{\theta} \in \Theta$ , there exist a competitive equilibrium with*

$$\begin{aligned} x^{i,\sigma_t}(\sigma_{t+a}) &= f_x^{i,a}(\theta(\sigma_{t+a-1}, s_t), \\ \theta^{i,\sigma_t}(\sigma_t) &= f_\theta(\theta(\sigma_{t-1}), s_t), \\ p(\sigma_t) &= f_p(\theta(\sigma_{t-1}), s_t), \\ q(\sigma_t) &= f_q(\theta(\sigma_{t-1}), s_t). \end{aligned}$$

We present two simple examples to illustrate that even in standard economies recursive equilibria may fail to exist; and it seems impossible to derive general conditions that ensure the existence of a recursive equilibrium.

Subsequently, we introduce alternative concepts and we prove the existence of equilibria.

There are two individuals per generation,  $i = 1, 2$  two commodities,  $l = 1, 2$ , and two dates in the life-spans of individuals,  $a = 0, 1$ .

Shocks,  $s_t \in \mathcal{S}$  are equiprobable, and they are distributed independently over time.

At each date-event there is a complete set of elementary securities of one date maturity; there are no long-lived assets.

#### Example 1

Utilities are independent of the shock, which only affects the endowments of individuals.

The intertemporal utility function of individual  $(1, \sigma_t)$  is

$$U^{1,\sigma_t} = -\frac{1}{x_1(\sigma_t)} + \mathbb{E}_{s_t} \left[ -\frac{1}{x_1(\sigma_t, s_{t+1})} + x_2(\sigma_t, s_{t+1}) \right],$$

and for individual  $(2, \sigma_t)$ ,

$$U^{2,\sigma_t} = -\frac{1}{x_1(\sigma_t)} + \mathbb{E}_{s_t} \left[ x_1(\sigma_t, s_{t+1}) - \frac{1}{x_2(\sigma_t, s_{t+1})} \right].$$

The endowments of individuals in the first date of their lives are stochastic, while in the second they are not.

For individual  $(1, \sigma_t)$ ,

$$e^{1, \sigma_t}(\sigma_t) = (e_1^1(s_t), 0), \quad e^{1, \sigma_t}(\sigma_{t+1}) = (0, \kappa^1),$$

and for individual  $(2, \sigma_t)$ ,

$$e^{2, \sigma_t}(\sigma_t) = (e_1^2(s_t), 0), \quad e^{2, \sigma_t}(\sigma_{t+1}) = (\kappa^2, 0),$$

with

$$\kappa^1 > \frac{e_1^2(s_t)}{e_1^1(s_t)} > \kappa^2.$$

Since individuals are endowed with and consume only commodity  $l = 1$  in the first date of their lives, there can be no intergenerational trade.

It follows that intertemporal competitive equilibria decompose into equilibria of sub-economies,  $\sigma_t$ , each with two individuals, and two dates.

Economies associated with different realizations of the shock,  $s_t$ , are different, since the shock affects the endowments of individuals in the first date.

Randomness translates into heterogeneity across sub-economies, which, however, are not subject to uncertainty.

A perfect capital market is operative in each sub-economy; the only asset is a risk-free bond.

The competitive equilibrium at each sub-economy is unique.

The construction is as follows:

1. There is no trade in commodity  $(2, \sigma_t)$ .
2. The value, at  $\sigma_t$ , of the aggregate demand for the numéraire commodity is

$$q(\sigma_t)x^{1, \sigma_t}(1, \sigma_{t+1}) + q(\sigma_t)x^{2, \sigma_t}(1, \sigma_{t+1}) = e_1^1(s_t) + q(\sigma_t)\kappa^1 - \sqrt{q(\sigma_t)} - q(\sigma_t)\sqrt{p(\sigma_{t+1})} + q(\sigma_t)\sqrt{p(\sigma_{t+1})},$$

where, by a slight abuse of notation,  $p(\sigma_{t+1})$  is the price of the non-numéraire commodity, for which, the value of the aggregate demand is

$$q(\sigma_t)p(\sigma_{t+1})x^{1, \sigma_t}(2, \sigma_{t+1}) + q(\sigma_t)p(\sigma_{t+1})x^{2, \sigma_t}(2, \sigma_{t+1}) = q(\sigma_t)\sqrt{p(\sigma_{t+1})} + e_1^2(s_t) + q(\sigma_t)p(\sigma_{t+1})\kappa^2 - \sqrt{q(\sigma_t)p(\sigma_{t+1})} - q(\sigma_t)\sqrt{p(\sigma_{t+1})}.$$

3. Since the value of the supply of the numéraire commodity is  $q(\sigma_t)\kappa^1$ , and of the non-numéraire commodity  $q(\sigma_t)p(\sigma_{t+1})\kappa^2$ , the unique competitive equilibrium prices are

$$q(\sigma_t) = [e_1^1(s_t)]^2, \quad p(\sigma_{t+1}) = \left[ \frac{e_1^2(s_t)}{e_1^1(s_t)} \right]^2.$$

The price of the non-numéraire commodity is determined by the realization of the shock at the predecessor of a date-event.

A reason that recursive equilibria fail to exist is that, in the sub-economy, there is a multiplicity of equilibria conditional on the distribution of wealth at the start of the second date.

## Example 2

Utilities are independent of the shock, which only affects the endowments of individuals.

The intertemporal utility function of individual  $(1, \sigma_t)$  is

$$U^{1, \sigma_t} = -\frac{1024}{x_1^4(\sigma_t)} + E_{s_{t+1}|s_t} \left[ -\frac{1024}{x_1^4(\sigma_{t+1})} - \frac{1}{x_2^4(\sigma_{t+1})} \right],$$

and for individual  $(2, \sigma_t)$ ,

$$U^{2, \sigma_t} = -\frac{1}{x_1^4(\sigma_t)} + E_{s_{t+1}|s_t} \left[ -\frac{1024}{x_2^4(\sigma_{t+1})} - \frac{1}{x_1^4(\sigma_{t+1})} \right].$$

The endowments of individuals in the first date of their lives are stochastic, while in the second they are not.

For individual  $(1, \sigma_t)$ ,

$$e^{1, \sigma_t}(\sigma_t) = \begin{cases} (10.4, 0), & \text{if } s_t = s_1, \\ (8.6313, 0), & \text{if } s_t = s_2, \end{cases} \quad e^{1, \sigma_t}(\sigma_{t+1}) = (12, 1),$$

and for individual  $(2, \sigma_t)$ ,

$$e^{1, \sigma_t}(\sigma_t) = \begin{cases} (2.6, 0), & \text{if } s_t = s_1, \\ (4.3687, 0), & \text{if } s_t = s_2, \end{cases} \quad e^{1, \sigma_t}(\sigma_{t+1}) = (1, 12).$$

As above, there is no intergenerational trade, and equilibria decompose into equilibria for sub-economies.

A perfect capital market is operative in each sub-economy; the only asset is a risk-free bond.

The competitive equilibrium at each sub-economy is unique.

The construction is as follows:

1. There is no trade in commodity  $(2, \sigma_t)$ .
2. Since there the aggregate supply of the numéraire commodity is the same at the two dates and the cardinal utility index for the commodity at the two dates is the same, the price of the risk-free bond is  $q(\sigma_t) = 1$ .

3. The aggregate excess demand for the non-numéraire commodity at date 2 is

$$\frac{e^{1,\sigma_t}(s_t) + 12 + p(\sigma_{t+1})}{p(\sigma_{t+1})^{1/5}(8 + p(\sigma_{t+1})^{4/5})} + \frac{4(e^{2,\sigma_t}(s_t) + 1 + 12p(\sigma_{t+1}))}{p(\sigma_{t+1})^{1/5}(2 + 4p(\sigma_{t+1})^{4/5})} - 13 = 0.$$

4. One can verify that for  $s = 1, 2$  competitive equilibria are unique.
5. Whenever  $s_t = 1$ , prices of both elementary securities are equal to  $1/2$ , and the spot price of the non-numéraire commodity in the following date is  $p(\sigma_{t+1}) = 1$ ; the consumption of individuals in the second dates of their lives is  $(10.4, 2.6)$ , for individual  $i = 1$ , and  $(2.6, 10.4)$ , for individual  $i = 2$ . Whenever  $s_t = 2$ , the prices of both elementary securities are equal to  $1/2$ , and the spot price of the non-numéraire commodity in the following date is  $p(\sigma_{t+1}) = 7.8574$ ; the consumption of individuals in the second date of their lives is  $(8.6313, 1.4288)$  for individuals  $i = 1$ , and  $(4.3687, 11.5812)$ , for individuals  $i = 2$ .

As in example 1, the failure of existence of a recursive equilibrium relies on the fact that, in the sub-economy, at date 2, there are multiple spot-market equilibria — the example is from Kehoe (1992). This multiplicity is robust, which is not the case in example 1. Nevertheless, the failure of existence of recursive equilibria is not robust, since a perturbation in date 2 endowments leads to different portfolio-holdings for different realizations of the shock.

## 4 Self-justified sets and Markov equilibria

In this section we prove that, under our assumptions on preferences, assets and endowments, Markov competitive equilibria exist.

The method of proof follows Duffie et al. (1994).

We first state a proposition from Duffie et al. (1994) that allows us to establish existence by verifying that equilibria exist for truncated economies and that equilibrium variables remain uniformly bounded for all truncations.

### 4.1 Markov equilibria for expectation correspondences

The state space consist of all exogenous and endogenous variables that describe the state of the economy at some date-event:

$$\mathcal{Y} = \mathcal{S} \times \mathcal{Z},$$

where  $\mathcal{S}$  is the finite set of exogenous shocks and  $\mathcal{Z}$  is the set of all possible endogenous variables.

An expectations correspondence,

$$G : \mathcal{Y} \Rightarrow \bigotimes_{s \in \mathcal{S}} \mathcal{Z},$$

embodies all short run equilibrium conditions: (first order) conditions for individual optimization and market clearing conditions; it assigns possible next period endogenous variables to each state today.

A (time-homogeneous) Markov equilibrium for an expectations correspondence,  $G$ , consists of a subset  $\mathcal{Z}^* \subset \mathcal{Z}$  and a function  $h : \mathcal{S} \times \mathcal{Z}^* \rightarrow \bigotimes_{s \in \mathcal{S}} \mathcal{Z}^*$ , such that

$$y \in \mathcal{S} \times \mathcal{Z}^* \Rightarrow h(y) \in G(y);$$

the set  $\mathcal{Z}^*$  supports the equilibrium.

A  $T$ -horizon equilibrium for the correspondence  $G$  consists of a subset  $\tilde{\mathcal{Z}} \subset \mathcal{Z}$  and  $y(\sigma_t) : t = 1, \dots, T$ , with  $y(\sigma_t) = (s_t, z(\sigma_t))$ , and  $z(\sigma_t) \in G_{s_t}(y(\sigma_{t-1}))$ , that satisfies  $y(\sigma_t) \in \tilde{\mathcal{Z}}$ ; the set  $\tilde{\mathcal{Z}}$  supports the equilibrium.

**Proposition 1.** *If  $G$  is an expectations correspondence, such that*

1. *there exists a compact subset  $\mathcal{K} \subset \mathcal{Y}$  that supports a  $T$ -horizon equilibrium for  $T = 1, \dots$ ,*
2. *the graph of  $G$  is closed,*

*then, a Markov equilibrium for  $G$  exists.*

**Proof** Define sets  $\mathcal{T}_0 = \mathcal{K}$ , and, recursively,

$$\mathcal{T}_n = \{y \in \mathcal{Y} \cap \mathcal{K} : \text{there exists } (z_1, \dots, z_S) \in G(y), \text{ with } (z_s, s) \in \mathcal{T}_{n-1}\}.$$

Since, by assumption, the expectations correspondence has a closed graph and since  $\mathcal{K}$  is closed each  $\mathcal{T}_n$  is closed. Since  $\mathcal{T}_n \subset \mathcal{T}_{n-1}$ , and since, by assumption, all  $\mathcal{T}_n$  are non-empty, the set

$$\mathcal{Y}^* = \bigcap_{n=0}^{\infty} \mathcal{T}_n$$

is non-empty. The set  $\mathcal{Y}^* = \mathcal{S} \times \mathcal{Z}^*$  supports a Markov equilibrium.  $\square$

## Existence of Markov equilibria

For the stochastic economy of overlapping generations, endogenous state variables are portfolios at the preceding date and current excess demand for commodities, portfolios and prices for commodities and assets:

$$\mathcal{Z} = \left\{ z = (\theta_-, \xi, \theta, p, q) : \begin{cases} \sum_{i \in \mathcal{I}} \sum_{a=0}^{N-1} \theta_j^{i,a} = 0, & j \in \mathcal{J}_1, \\ \sum_{i \in \mathcal{I}} \sum_{a=0}^{N-1} \theta_j^{i,a} = 1, & j \in \mathcal{J}_2, \\ \sum_{i \in \mathcal{I}} \sum_{a=0}^N \xi^{i,a} = 0. \end{cases} \right\}.$$



At a competitive equilibrium, individuals optimize, and markets for commodities and assets clear.

**Lemma 1.** *For every  $T$ , competitive equilibria exist.*

**Proof** The economy ends at date  $T$ ; budget sets and utility functions of individuals born after  $T - N$  are adjusted accordingly.

The domain of prices is

$$\mathcal{D} = \{(p, q) : \|(p, q)\| = 1, p \geq 0, q_j \geq 0, j \in \mathcal{J}_2\};$$

here normalization involves the norm of prices of commodities and assets across date-events, and it does not single out commodity  $l = 1$  as numéraire.

Given bounds  $\bar{x}, \bar{\theta}, \underline{\theta}$ , and any  $\epsilon > 0$ , there exist  $\theta_\epsilon^{i, \sigma_t}$  and  $x_\epsilon^{i, \sigma_t}$ , for every individual, as well as  $p_\epsilon$  and  $q_\epsilon$ , such that

$$(x_\epsilon^{i, \sigma_t}, \theta_\epsilon^{i, \sigma_t}) \in \arg \max \{u^{i, \sigma_t}(x) : (x, \theta) \in \mathcal{B}^{i, \sigma_t}(p_\epsilon, q_\epsilon), \bar{x} \geq x(\sigma_t), \bar{\theta} \geq \theta(\sigma_t) \geq \underline{\theta}\},$$

and

$$(p_\epsilon, q_\epsilon) \in \arg \max \left\{ \sum_{\sigma_t \in \Sigma_T} p(\sigma_t) \xi(\sigma_t) + q(\sigma_t) (\phi(\sigma_t)) : (p, q) \in \mathcal{D}, p_l(\sigma_t) \geq \epsilon \right\},$$

where

$$\xi(\sigma_t) = \sum_{i \in \mathcal{I}} \sum_{a=0}^N (x_\epsilon^{i, \sigma_t-a}(\sigma_t) - e^{i, \sigma_t-a}),$$

$$\phi_j(\sigma_t) = \sum_{i \in \mathcal{I}} \sum_{a=0}^N \theta_\epsilon^{i, \sigma_t-a}(\sigma_t) \text{ for } j \in \mathcal{J}_1,$$

$$\phi_j(\sigma_t) = \sum_{i \in \mathcal{I}} \sum_{a=0}^N \theta_\epsilon^{i, \sigma_t-a}(\sigma_t) - 1 \text{ for } j \in \mathcal{J}_2.$$

This follows from a standard fixed point argument, since the budget correspondences is continuous.

Since by the budget constraints,

$$\sum_{\sigma_t \in \Sigma_T} p_\epsilon(\sigma_t) \xi(\sigma_t) + q_\epsilon(\sigma_t) \phi(\sigma_t) \leq 0$$

for sufficiently small  $\epsilon > 0$  it must be true that there is also a fixed point with

$$\xi = 0 \text{ and } \phi = 0.$$

This implies by market clearing that whenever  $\bar{x}$  exceeds aggregate endowments  $\bar{x} > x_l^h(\sigma)$  the additional constraint on consumption is not binding. Again by market clearing, whenever  $\bar{\theta}_j > 1$ ,

$$\bar{\theta}_j > \theta_{j\epsilon}^h(\sigma_t), \quad j \in \mathcal{J}_2.$$

Since consumption is non-negative and the payoffs of short-lived assets are independent, there exists sufficiently small  $\underline{\theta}_j$  which ensure that

$$\theta_{j\epsilon}^h > \underline{\theta}_j, \quad j \in \mathcal{J}_2.$$

Again by market clearing the upper bounds will not bind either.

To complete the argument it has to be shown that as  $\epsilon \rightarrow 0$   $(p_\epsilon, q_\epsilon, \theta_\epsilon, x_\epsilon) \rightarrow (p^*, q^*, \theta^*, x^*)$  that solve the above maximization problems.

For this it suffices to show that  $p_{\epsilon 1}(\sigma_t)$  remains bounded away from zero for all  $\sigma_t$ . But if for some  $\sigma_T$   $p_{\epsilon 1}(\sigma_T) \rightarrow 0$ , demand for the numéraire must explode because utility is strictly increasing in that good. If, for some non-terminal  $\sigma_t$ ,  $p_{\epsilon 1}(\sigma_T) \rightarrow 0$ , demand would only stay bounded if the prices of all other commodities in that date event tend to zero. But then by our normalization there must exist an asset whose price remains non-zero. If this asset is long-lived, some agents must hold positive amounts and can sell (if price is positive) or buy (if negative) some of the asset to buy commodity 1. If the asset is short-lived, the current young can sell it short and buy commodity 1.  $\square$

It now suffices to establish that there exist uniform bounds such that for every  $T$  all equilibrium asset holdings and asset prices lie within these bounds. Since agents have finite lives and there are short-sale constraints for the long-lived assets bounds on portfolios are immediate. By the assumption on utility, if the price of some commodity or some asset would be sufficiently large, agents who in the equilibrium consume the commodity or are able to sell some of the asset can do so and strictly increase their utility by buying some of commodity 1.

Therefore,

**Lemma 2.** *Across truncated economies, equilibrium asset prices, consumptions and portfolio-holdings are uniformly bounded.*

The expectations correspondence  $G$  has a closed graph.

From Proposition 1 and the fact that first order conditions are necessary and sufficient for optimization, it follows that

**Theorem 1.** *For the stochastic economy of overlapping generations, a Markov equilibrium exists.*

The construction of the Markov equilibrium implies that there also exists a Markov equilibrium where portfolios, prices and consumptions today just depend on the wealth distribution, the exogenous shocks as well as the shock, asset prices and consumptions at the predecessor node.

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