The Stock Market in the Overlapping Generations Model with Production

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1. Introduction

This paper addresses the following question: does the stock market influence the process of capital accumulation? If exchanging ownership of firms on a stock market is equivalent to exchanging the ownership of their capital on a capital goods market, then introducing a stock market will not affect the predictions of the real models of capital accumulation—the Ramsey model (1928) if agents are infinitely lived, or Diamond's overlapping generations (OLG) model (1965) if agents are finitely lived. The assumption that ownership of firms is transferred through the stock market rather than the capital goods market can lead to a different outcome only if there is a friction which makes the stock market into a financial entity distinct from the real capital goods market.

The friction that we study in this paper is the *firm specificity of capital* which makes it costly, if not impossible, to detach part of the tangible or intangible capital of a firm to sell it on a (second-hand) market for capital goods. To quote Tobin (1998, p. 147) "The various physical assets of a business enterprise are often designed, installed and used in complex combinations specific to the technology. It is costly or impossible to detach and move individual assets or to apply them to alternative purposes." We take this observation to the theoretical limit by assuming that capital, once installed, is a sunk cost: it cannot be transformed back into a consumption good or used by other firms.

Under this assumption, when capital is durable, firms must be long lived and if transferred, must be kept intact in their entirety. If, as we shall assume, economic agents are short lived, then there is a need for a market which makes such transfers possible, and this is one of the important roles of the stock market: each firm becomes a separate legal entity which issues equity shares to its future income stream, and ownership of firms can be transferred in perfectly divisible amounts across an indefinite succession of finite-lived shareholders, while retaining in perpetuity the full physical and organizational entity of the firm.[1]

We are thus led to study the role of the stock market as an instrument for transferring firms in the setting of the standard Diamond model to which we add the friction that capital once installed in a firm cannot subsequently be sold (i.e. has a zero price) on the market for current output. We do not assume any frictions on "new" investment: thus the financial value of a firm cannot exceed its replacement cost, for the young agents could always recreate the capital of the firm out of current output if it were less expensive

[1]Blackstone (1765) in his *Commentaries on the Laws of England*, (Book I, Chapter XVIII), referred to "perpetual succession" as the "very end of incorporation: for there can not be a succession forever without an incorporation". He explained "it has been found necessary when it is for the advantage of the public to have any particular rights kept on foot and continued, to constitute artificial persons, who may maintain a perpetual succession, and enjoy a kind of legal immortality. These artificial persons are called... *corporations*."

to do so, and the firm would not sell at its current equity price. However, and this is the important point, the assumption that previously installed capital is a sunk cost, permits the equity price to be less than the replacement cost without creating arbitrage opportunities.

We are thus led to study a stock market equilibrium in which the price of equity is given by a *two-part pricing formula*: the equity price of a firm is equal to its replacement cost less a lump sum discount. Since young agents can invest in both equity and bonds, absence of arbitrage requires that the rate of return on these two securities be the same: two-part pricing of equity is consistent with no arbitrage if the discount grows at the rate of interest, and is consistent with positive investment if the discount does not become too large—in a sense made precise in Section 2.

The two-part pricing formula for equity leads to an interesting new mechanism by which the stock market influences investment, especially for the class of economies regarded by many economists (see e.g. Abel et al (1989)) as empirically the most relevant, namely those characterized by underaccumulation. For in such economies the savings of the young are scarce and, in the standard Diamond model, do not suffice to lead the economy to the Golden Rule: as the term of 'underaccumulation' suggests, the Diamond equilibrium, although dynamically efficient is not long-run efficient. However when there is a discount on the equity prices of firms, this discount — no matter how small — frees some of the scarce savings of the young and enables them to be used to purchase new investment rather than paying for previously installed capital. Although the investment behavior of firms in our model is the same as in Diamond's model, it is "as if" there were more savings in the economy (thanks to the discount), so that the equality "savings = investment" occurs at a lower interest rate than in the Diamond equilibrium. As a result there is more investment, and hence more output, wages and savings in the next period, and this virtuous cycle fuels a sufficient increase in investment to lead the economy to the Golden Rule rather than to the Diamond steady state. Since the Golden Rule is the efficient steady state, the dynamic analysis reveals a new benefit derived from the stock market, as an instrument for the transfer of ownership of firms between generations, which is separate from its liquidity role mentioned above, and its risk-sharing role which typically takes preeminence in models of financial economics such as CAPM, but which is inevitably absent in the simple deterministic framework of this model.

The idea that frictions, or adjustment costs, may importantly influence the process of capital accumulation has a long tradition in economics (Lucas (1967), Gould (1968), Uzawa (1969), Kydland-Prescott (1982)) and some authors have derived from such adjustment costs the existence of a Tobin's q different from 1 (Hayashi (1982), Basu (1987), Abel (1999)). These papers study the effects of adjustment costs in installing new capital, and typically assume that these costs are convex. Our approach is different in that it focuses on the adjustment costs that would need to be incurred if previously installed capital were to be put to an alternative use, rather than on the cost of installing new capital. As a consequence in our model, Tobin's q is always less than or equal to one.

Since the model has explicit equity and bond markets, in Section 4 we analyze the properties of the equilibria from the financial perspective. Models of financial economies over an open-ended future have been developed mainly in the framework of models with infinite-lived agents in which equilibria have the following properties: the price of a security in positive supply like equity is equal to the discounted sum of its future stream of dividends (its *fundamental value*), and the present value of its debt at a future date T tends to zero as $T \to \infty$ i.e. the present value of debt 'at infinity' is zero. Although these properties are considered 'normal', it is known since the work of Tirole (1985) that they do not always hold in an OLG model. Tirole exhibited a variant of Diamond's model in which a security with a zero dividend has a positive price, so that its price exceeds its fundamental value (its price is said to have a *bubble component*). However in Tirole's model such a bubble component can only arise in economies with overacccumulation. In our model, for almost all initial conditions, the equilibrium price of equity has a bubble component and/or the present value of debt at infinity is positive (there is a bubble on debt). Only for an economy with underaccumulation for which the initial discount on equity is zero—i.e. for the Diamond equilibrium—is the price of equity equal to its fundamental value and the present value of debt at infinity is zero. Since in the dynamics of the model this trajectory is (locally) saddle-point stable, this may be considered an exceptional trajectory.

The presence of a bubble component on either equity or debt (or both) is not necessarily a sign of inefficiency, since equilibria of an economy with underaccumulation which converge to the Golden Rule are both dynamically and long-run efficient. If, in keeping with the literature, we refer to the property that the present value of the equity price and debt at date T tend to zero as $T \to \infty$ as a "transversality condition", then in economies with underaccumulation, stock market equilibria with two-part pricing satisfy the Cass criterion (1972) for dynamic efficiency, but do not satisfy the transversality condition. For, while a transversality condition typically implies that the Cass criterion is satisfied (see for example Scheinkman (1983) and Dechert-Yamamoto (1992)), the converse is not true. Thus for example when the stock market equilibrium converges to the Golden Rule , the interest rate converges to the rate of growth n and the present values of equity and/or debt at infinity are non-zero, as we shall show in Section 4.

The paper is organized as follows. Section 2 describes the model and introduces the concept of a stock market equilibrium. The asymptotic properties of such an equilibrium are studied in Section 3. The comparison between firms' market values and the discounted sums of their dividends is the subject of Section 4.

2. The Stock Market Model

Consider a standard OLG model with production: at each date t, N_t young agents are born who live for two periods, t and $t+1$, and each of these agents is endowed with 1 unit of labor when young, having no initial resources when old. Agents of all generations are identical, with the same endowment (1 unit of labor when young) and the same preferences, represented by a utility function $u(c_0^t, c_1^t)$ over consumption streams $c^t = (c_0^t, c_1^t)$, where $c_s^t,\ s = 0, 1$, represents the consumption at date $t+s$ of an agent born at date t. The population is assumed to grow at the exogenous rate $n, (n \geq 0)$, i.e. $N_{t+1} = (1+n)N_t$.

On the production side, there is a collection of J firms ($j = 1, \ldots J$), each firm producing at each date t an all-purpose good — which we will call the output—from capital and labor, with the time-invariant technology $Y_t^j = F(K_t^j, L_t^j)$ where the function F is the same for all firms and is smooth, concave, strictly increasing and homogeneous of degree 1. The output of firms can be used either directly for consumption or to create new capital, where it takes one unit of the good to produce one unit of new capital for any firm. Capital in each firm is durable and depreciates at the rate β $(0 < \beta < 1)$, and needs to be installed one period before it is used: thus the capital K_t^j used by firm j is the capital that it has carried over from date $t-1$.

The model that we introduce differs from that of Diamond (1965) by the assumption that capital once installed in a firm cannot be "unbolted" and transformed back into the homogeneous current output or transferred to another firm, without incurring significant adjustment costs — which for simplicity we take to be infinite. Thus once capital has been installed in a firm, it cannot be used for consumption, nor can it be used for new investment (i.e. additional capital) by any other firm: in short, it is sunk in the firm. As indicated in the introduction this assumption is designed to capture the fact that many resources invested in a firm have to be adapted in a way which is firm specific to make the whole production process function smoothly and efficiently. Since the precise way in which the resources have been adapted typically makes them inappropriate for use by other firms, such installed capital has limited value on a resale market. For example, software written specifically for a firm and incorporating its specific needs may be very expensive — it consumes a great deal of labor not used for producing the consumption good — but has essentially no resale value. Even those capital goods which have a resale value, for example plant and machinery, usually have a low value on the used capital market relative to their replacement cost, since significant "adjustment costs" have to be incurred to adapt them for use by other firms. To capture this phenomenon in a simple way, we study the theoretical limit in which the installed capital of a firm is completely firm specific, so that no part of it has a positive resale value on the second-hand market.

In such an economy capital accumulation will only take place if the market structure permits firms to be infinitely lived. Invested capital has no value if the firm is liquidated, and has value only if the firm retains its identity as an income generating unit in the economy. The natural market structure which permits short-lived agents to transfer ownership of long-lived firms from one generation to the next is an equity market for ownership shares of firms. Thus to have a market structure consistent with the firm specificity of capital, we assume that each firm is a corporation with an infinite life whose ownership shares are transmitted from one generation to the next through the stock market. Let Q_t^j denote the equity price of firm j at date t.

At each date t, in addition to the stock market, there are three other markets: a market for current output, a labor market, and a bond market. Since this is a real (as opposed to a monetary) model, the price of a unit of current output is normalized to be 1. Let w_t denote the wage rate at date t on the labor market on which the (homogeneous) services of labor supplied by the young generation are sold to the firms. The bond market provides firms with a source of external funds for financing investment which is an alternative to issuing new equity shares, and gives young agents a way of borrowing and lending. Let r_{t+1} denote the interest rate on a loan from date t to date $t+1$ and let $(1, (Q_t^j)_{j=1}^J, w_t, r_{t+1})$ denote the vector of prices on these four markets at date t $(t = 0, 1, \ldots)$.

Two-part Pricing. The assumption that firms are transferred through the equity market can lead to an equilibrium that is different from the standard Diamond equilibrium only if the equity price of a firm differs from the value of its underlying capital. Let ξ^j denote the installed capital of firm j at the time it is to be sold on the equity market. ξ^j is the result of past capital accumulation and is equal to the accumulated sum of past investments, once depreciation has been taken into account. Since we assume that one unit of good can be transformed into one unit of capital, ξ^j is also the *replacement cost* of firm j: by this we mean that agents could recreate a firm equivalent to firm j by purchasing ξ^j units of good on the current output market.[2] If firms can be recreated in this way, then the equity price of firm j cannot exceed its replacement

[2]Thus we assume that if a firm consists of both tangible and intangible capital, then both types of capital can be reproduced. If there is some capital which cannot be replaced at any (or only at a very large) cost due to some special knowledge or some first-mover advantage, then ξ^j, taken as the accumulated sum of past investments, can be less than the replacement cost of the firm, and the equity price could be more than ξ^j. In this paper we do not

cost. On the other hand the equity price can be below ξ^j without creating an arbitrage opportunity since, after buying the firm, agents cannot turn around and recover ξ^j since there is no market for installed capital. The object of this section is to show that a *two-part pricing formula* of the form

$$Q_t^j(\xi^j) = \xi^j - V_t^j, \quad V_t^j \geq 0 \tag{1}$$

leads to an equilibrium with positive investment provided the fixed parts $(V_t^j)_{t\geq 0}$ satisfy appropriate restrictions introduced below: we refer to the fixed part V_t^j as the *discount* on the equity of firm j arising from the non-salability of installed capital.

The assumption that capital, once installed, can not be transformed back into the consumption good, nor transferred to other firms, has two consequences: the first is that investment must be non-negative—the *irreversibility constraint*; the second is that the equity price of a firm can be less than its replacement cost—the *sunk cost effect.* The irreversibility constraint can be incorporated into Diamond's model and, when binding, has consequences for equity prices, which have been studied by Huffman (1986) in a stochastic Diamond model. In this paper we consider only equilibria where investment is positive at all dates: we thus by-pass the effect of the irreversibility constraint. This amounts to restricting attention to economies for which the initial level of capital is low. Our objective is to focus attention on the second effect—the sunk-cost effect—the associated two-part pricing formula (1), and its consequences for the process of capital accumulation.

Corporation's decision problem. Firms are owned by the equity holders and are managed so as to maximize the payoff to the current owners. Suppose the stock market opens at date t, after production has taken place and capital has depreciated: young agents buy the shares of firm j, endowed with a capital $(1-\beta)K_t^j$, from the old for the price Q_t^j and decide on the investment I_t^j to be made. The date t investment is chosen so as the maximize the net present value of investment

$$-I_t^j + \frac{1}{1+r_{t+1}}\left[F\Big((1-\beta)K_t^j + I_t^j, L_{t+1}^j\Big) - w_{t+1}L_{t+1}^j + Q_{t+1}^j\Big((1-\beta)^2K_t^j + (1-\beta)I_t^j\Big)\right] \tag{2}$$

anticipating the next-period labor decision L_{t+1}^j and the effect of the investment I_t^j on the resale price Q_{t+1}^j of equity next period. In order for the problem (2) to be well defined, the formula (1) needs to be made more precise. For if we assume that there is "free disposal of securities", then old agents cannot be forced to sell their equity contracts for a negative price: thus to be defined for all positive values of ξ^j, formula (1)

attempt to model such non-reproducible capital.

must be written as

$$Q_t^j(\xi^j) = \max(\xi^j - V_t^j, 0), \quad V_t^j \geq 0 \tag{3}$$

Since the investment at date t is added to the existing capital $(1-\beta)K_t^j$ which becomes $(1-\beta)^2 K_t^j$ at date $t+1$, the relevant values of ξ^j to consider for the investment problem (2) are $\xi^j \geq (1-\beta)^2 K_t^j$. If $V_{t+1}^j > (1-\beta)^2 K_t^j$, then the first units of investment do not increase the resale value of the firm which stay equal to zero, up to the point where $(1-\beta)^2 K_t^j + (1-\beta) I_t^j = V_{t+1}^j$. By studying the maximum problem (2) —which in view of (3) is a non-convex problem—it can be shown that in this case the optimal investment is $I_t^j = 0$.

Proposition 1: *If the anticipated price for equity at date $t+1$ is given by (3), and if $V_{t+1}^j > (1-\beta)^2 K_t^j$, then the optimal solution to the investment problem (2) at date t is $I_t^j = 0$.*

Proof (see Appendix).

Since we focus on equilibrium paths on which (gross) investment is positive at all dates, the discounts $(V_t^j)_{t\geq 1})$ must satisfy

$$V_{t+1}^j \leq (1-\beta)^2 K_t^j, \quad t \geq 0 \tag{4}$$

Under condition (4) formula (1) and (3) for the anticipated price at date $t+1$ coincide and the capital value term in the maximum problem (2) is equal to $(1-\beta)^2 K_t^j + I_t^j - V_{t+1}^j$. The term $(1-\beta)^2 K_t^j - V_{t+1}^j$ is a constant term which does not affect the solution to the maximum problem. It follows that the first-order conditions characterizing the optimal decisions of investment at date t and labor at date $t+1$ are given by

$$\begin{aligned} F'_K\big((1-\beta)K_t^j + I_t^j, L_{t+1}^j\big) &= r_{t+1} + \beta, \quad t \geq 0 \\ F'_L\big((1-\beta)K_t^j + I_t^j, L_{t+1}^j\big) &= w_{t+1}, \quad t \geq 0 \end{aligned} \tag{5}$$

Since the production function is homogeneous of degree one, conditions (5) only determine an optimal capital-labor ratio for each firm at each date. Since each firm has the same production function, this ratio is the same for all firms. If we let $k = K/L$ denote the capital-labor ratio and introduce the production function per unit of labor

$$f(k) = \left(\frac{1}{L}\right) F(K, L) \;=\; F(k, 1)$$

then $F'_L(K,L) = f\left(\frac{K}{L}\right) - \left(\frac{K}{L}\right) f'\left(\frac{K}{L}\right)$ and $F'_K(K,L) = f'\left(\frac{K}{L}\right)$. If $(w_t, r_{t+1})_{t\geq 0}$ is a price sequence for the economy, then given K_0^j, the sequence $(K_{t+1}^j, L_t^j)_{t\geq 0}$ is profit maximizing for firm j with positive investment if $K_{t+1}^j = (1-\beta)K_t^j + I_t^j$ with $I_t^j > 0$, and the sequence $(k_t)_{t\geq 0}$ of capital-labor ratios, with $k_t = \frac{K_t^j}{L_t^j}$, satisfies[3]

$$
\begin{aligned}
f'(k_{t+1}) &= r_{t+1} + \beta, \quad t \geq 0 \\
f(k_t) - k_t f'(k_t) &= w_t, \quad t \geq 0
\end{aligned}
\tag{6}
$$

Note that to be compatible with (6), the price sequence $(w_t, r_{t+1})_{t\geq 0}$ must satisfy

$$
w_{t+1} = f(k_{t+1}) - k_{t+1}(r_{t+1} + \beta), \quad t \geq 0 \tag{7}
$$

The criterion (2) for the choice of investment at date t suggests that the shareholders directly finance the investment, receiving the output of the firm (net of labor costs) plus the resale value of their equity in the next period. Such a method of financing is not especially realistic. However if the firm finances its investment by one-period borrowing, then the decision criterion is unchanged since (2) can be written as

$$
\frac{1}{1 + r_{t+1}} \left[F\Big((1-\beta)K_t^j + I_t^j, L_{t+1}^j\Big) - w_{t+1}L_{t+1}^j - (1 + r_{t+1})I_t^j + Q_{t+1}^j\Big((1-\beta)K_{t+1}^j\Big) \right]
$$

This corresponds to the sum of the dividend D_{t+1}^j and the capital value Q_{t+1}^j, where

$$
D_{t+1}^j = F(K_{t+1}^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j - (1 + r_{t+1})I_t^j \tag{8}
$$

is the dividend received by the shareholders of firm j. This more realistic method of financing thus leads to the same investment decision. More generally it can be shown that the Modigliani-Miller theorem holds for this economy: the real outcome (firms' production and agents' consumption) is independent of the mode of financing, provided the borrowing of firms does not lead the firms to bankruptcy. To simplify the exposition, we will assume up to Section 4 that firms finance their investment using one-period loans which are reimbursed the following period.

Agent's maximum problem. The representative young agent born at date t purchases a portfolio of securities

$$
(z_t, \theta_t^1, \ldots, \theta_t^J)
$$

consisting of an amount z_t of bonds and a share θ_t^j of firm j (for $j = 1, \ldots, J$), so as to maximize lifetime utility $u(c^t)$ where $c^t = (c_0^t, c_1^t)$ subject to the budget constraints

$$
\left.
\begin{aligned}
c_0^t &= w_t - z_t - \sum_{j=1}^{J} \theta_t^j Q_t^j \\
c_1^t &= z_t\,(1 + r_{t+1}) + \sum_{j=1}^{J} \theta_t^j (D_{t+1}^j + Q_{t+1}^j)
\end{aligned}
\right\} \quad \forall\, t \geq 0 \tag{9}
$$

[3] The differences in the time indices for the two FOCs in (6) comes from the fact that there is a labor decision at date 0, while the capital K_0^j is given exogenously.

where D_t^j denotes the dividend paid by firm j at date t. The agent takes the prices $\big(1, (Q_t^j)_{j=1}^J, w_t, r_{t+1}\big)$ as given, and correctly anticipates the next-period dividends and prices of the firms $(D_{t+1}^j, Q_{t+1}^j)_{j=1}^J$. The maximum problem of the agent has a solution if and only if the no-arbitrage condition between the stock and the bond market

$$Q_t^j = \frac{1}{1+r_{t+1}}(D_{t+1}^j + Q_{t+1}^j), \quad j = 1, \dots, J, \quad \forall\, t \geq 0 \tag{10}$$

holds for the equity price of each firm. In view of (8) and Euler's theorem, D_{t+1}^j can be written as

$$D_{t+1}^j = K_{t+1}^j F_K'(K_{t+1}^j, L_{t+1}^j) + L_{t+1}^j F_L'(K_{t+1}^j, L_{t+1}^j) - w_{t+1} L_{t+1}^j - (1 - r_{t+1}) I_t^j$$

which, on a trajectory for which the FOCs (5) are satisfied reduces to

$$D_{t+1}^j = K_{t+1}^j (r_{t+1} + \beta) - (1 - r_{t+1}) I_t^j$$

If, in addition, the pricing of equity is given by (1) then

$$\frac{D_{t+1}^j + Q_{t+1}^j}{Q_t^j} = \frac{K_{t+1}^j (r_{t+1} + \beta) - (1 - r_{t+1}) I_t^j + (1 - \beta) K_{t+1}^j - V_{t+1}^j}{(1 - \beta) K_t^j - V_t^j}$$

and (10) holds if and only if

$$V_{t+1}^j = (1 + r_{t+1}) V_t^j, \quad t \geq 0 \tag{11}$$

When (11) is satisfied the rate of return on the bond and each of the equity contracts is the same, and the agent is indifferent between investing in any firm or investing in the bond market: all that matters is the total sum invested in the capital markets, namely the agent's total savings s_t. When (10) holds the budget equations (9) can be written as

$$\begin{aligned} c_0^t &= w_t - s_t \\ c_1^t &= s_t (1 + r_{t+1}) \end{aligned} \tag{12}$$

where

$$s_t = z_t + \sum_{j=1}^{J} \theta_t^i Q_t^j \tag{13}$$

The maximizing behavior of the agent is summarized by the savings function $s : \mathbb{R}_t^2 \longrightarrow \mathbb{R}$ defined by

$$s(r_{t+1}, w_t) \;=\; w_t - c_0(r_{t+1}, w_t)$$

where $(c_0(r, w), c_1(r, w))$ is the solution of the problem of maximizing $u(c_0, c_1)$ subject to the budget equations (12), or equivalently the solution of the problem

$$\max_{(c_0, c_1) \in \mathbb{R}_+^2} \left\{ u(c_0, c_1) \,\middle|\, c_0 + \frac{c_1}{1+r} = w \right\}$$

Assumption $\mathcal{C}$. The utility function $u(c_0, c_1)$ is smooth, increasing, strictly quasi-concave and such that the induced savings function $s(r, w)$ satisfies

(a) $s'_w(r, w) > 0, \quad \forall\,(r, w) \gg 0$

(b) $s'_r(r, w) \geq 0, \quad \forall\,(r, w) \gg 0$

(a) is the assumption that consumption in the second period is a normal good, while (b) implies that when the interest rate increases, the substitution effect dominates the income effect, so that savings increase.

Equilibrium. A *stock market equilibrium* is defined as a sequence of prices $\big(w_t, r_{t+1}, (Q_t^j, V_t^j)_{j=1}^J\big)_{t\geq 0}$, production-investment decisions $(L_t^j, Y_t^j; I_t^j, K_{t+1}^j)_{t\geq 0}$ for each of the J corporations, and consumption-savings-portfolio decisions $\big(c^t, s_t, z_t, (\theta_t^j)_{j=1}^J\big)_{t\geq 0}$ for the sequence of representative consumers born at each date $t \geq 0$ such that

(i) the equity of each firm is priced according to the two-part pricing formula $Q_t^j = (1-\beta)K_t^j - V_t^j$ where V_t^j satisfies (4) and (11),

(ii) each firm maximizes its market value (condition (5)) with price anticipation given by (i),

(iii) each consumer maximizes lifetime utility subject to the budget constraints (9),

(iv) the output, labor and financial (bond, equity) markets clear at every date $t \geq 0$.

By Walras Law the output market clears once the labor market ($\sum_{j=1}^J L_t^j = N_t$) and financial markets clear. Given the indeterminacy of the bond-equity portfolio choice of consumers when (11) holds, market clearing on the bond and equity markets

$$N_t z_t = \sum_{j=1}^J I_t^j, \qquad N_t \theta_t^j = 1,\ j = 1, \ldots, J$$

only requires that the financial markets clear in aggregate:

$$N_t s_t = \sum_{j=1}^J I_t^j + \sum_{j=1}^J Q_t^j \tag{14}$$

The young agents must buy the equity of the firms from the old and finance new investment. Given the pricing of equity (i) and the evolution of capital $K_{t+1}^j = (1-\beta)K_t^j + I_t^j$, (14) holds if and only if

$$N_t s_t = \sum_{j=1}^J (K_{t+1}^j - V_t^j)$$

The economy starts date 0 with firms having initial capital stocks (K_0^j) and discounts V_0^j reflecting the operation of equity markets before date 0. To reduce the analysis to the study of the aggregate economy we study only balanced growth equilibria in which firms have at all times the same relative sizes and stock market values. Consider therefore initial conditions $(K_0^j, V_0^j) = \mu_j(K_0, V_0)$ with $\mu_j > 0$ and $\sum_{j=1}^J \mu_j = 1$. If the sequence of prices $(w_t, r_{t+1})_{t\geq 0}$, aggregate discounts $(V_t) \geq 0$ and labor-investment decisions $(L_t, I_t)_{t\geq 0}$ satisfy (4), (5), (11), then $(V_t^j, L_t^j, I_t^j) = \mu_j(V_t, L_t, I_t)$ also satisfy(4), (5), (11) so that for each firm j, (L_t^j, I_t^j) is optimal, its market value is positive and the return on its equity is r_{t+1}. Thus the maximizing behavior of individual firms can be summarized by the optimal choice of aggregate capital and labor. Equilibrium on the labor market, which can be expressed by $L_t = N_t$, is satisfied if we require the capital-labor ratio to be equal to the per-capita capital stock $k_t = K_t/N_t$. Using lower-case letters (k_t, i_t, v_t) to denote per-capita capital, investment and discount, a balanced-growth equilibrium can be summarized in the following per-capita aggregate form:

Definition 1. A path of savings, capital accumulation, wages and security prices $((s_t, i_t, k_{t+1}), (w_t, r_{t+1}, v_t,))_{t\geq 0}$, with initial conditions (k_0, v_0), is a *stock market equilibrium* if the following conditions are satisfied for all $t \geq 0$

$$
\begin{array}{llll}
\text{(i)} & q_t = (1-\beta)k_t - v_t & \text{(v)} & f'(k_{t+1}) = \beta + r_{t+1} \\
\text{(ii)} & (1+n)v_{t+1} = (1+r_{t+1})v_t & \text{(vi)} & s(r_{t+1}, w_t) = (1+n)k_{t+1} - v_t \\
\text{(iii)} & 0 \leq (1+n)v_{t+1} \leq (1-\beta)^2 k_t & \text{(vii)} & (1+n)k_{t+1} = (1-\beta)k_t + i_t \\
\text{(iv)} & f(k_t) - k_t f'(k_t) = w_t & \text{(viii)} & i_t > 0
\end{array}
\tag{E}
$$

Condition (i) is the two-part pricing of equity, while condition (ii) ensures that the rate of return on equity is the same as on the bond, and (iii) ensures that each firm has incentives to undertake positive investment. When these conditions hold, (iv) and (v) characterize the maximizing behavior of firms, while (vi) summarizes the maximizing behavior of consumers and equilibrium on the financial markets at each date. The consumption of the agents, while not explicitly given in Definition 1 is given by (12) for all agents born at date 0 or thereafter, and is given by the initial condition

$$
c_1^{-1} = (1+n)\big(f(k_0) - w_0 + (1-\beta)k_0 - v_0\big) = (1+n)(k_0 f'(k_0) + (1-\beta)k_0 - v_0)
$$

for the old agents at date 0.

Since equation (viii) requires investment to be positive, if the initial stock of capital is large, an equilibrium in the sense of Definition 1 may not exist. Since our objective is to study the process of 'capital accumulation' (rather than de-cumulation), and how the discount on equity affects this process, we will restrict attention to economies with a low initial level of capital.

3. Dynamics of Stock Market Equilibrium

In this section we study the long-run dynamics of a stock market equilibrium: as we shall see, the most interesting properties arise when the stock market value of each firm differs from its replacement cost. In the model outlined above, this difference between the two values of a firm was traced to an attribute of capital — namely that once installed in a firm, it ceases to be a perfect substitute for current output or current investment. The model however contains as a special case the classic Diamond model in which capital is perfectly malleable and firms can be liquidated at any time, their capital being sold on the market for current output: in this case the financial value of a firm must coincide with its replacement cost. More precisely, if $v_0 = 0$, equation (ii) of Definition 1 implies that $v_t = 0$ for all t, and the equations (iv)-(vii) are the equations defining a Diamond equilibrium. In particular equation (vi)

$$(1+n)k_{t+1} \;=\; s(r_{t+1}, w_t) \tag{15}$$

is the basic "investment = savings" equation of Diamond's model. Since the properties of a stock market equilibrium in which there is a discount on the equity prices of firms (equilibrium with $v_0 > 0$) depend in an essential way on the properties of the underlying Diamond equilibrium ($v_0 = 0$), we recall briefly the requisite properties of such an equilibrium.

Diamond Equilibrium. (iv) and (v) in Definition 1 define the wage and interest rate (w_t, r_{t+1}) as functions of the capital-labor ratios (k_t, k_{t+1}): substituting these functions into (15) gives the first-order difference equation

$$\Phi(k_{t+1}, k_t) \equiv (1+n)k_{t+1} - s\big(r(k_{t+1}), w(k_t)\big) = 0, \qquad \forall\, t \geq 0 \tag{E_D}$$

with initial condition $k_0 > 0$, which defines an equilibrium path of capital accumulation of Diamond's model. A *Diamond steady state* k_D is a solution of the equation

$$(1+n)k_D - s\big(r(k_D), w(k_D)\big) = 0 \tag{16}$$

For general preferences and technology (u, F, n) there can be several non-trivial steady states and the dynamics (E_D) can exhibit complex behavior. We restrict attention to economies $\mathcal{E}(u, F, n)$ for which there is a unique positive steady state k_D and every solution of (E_D) converges to k_D: as noted by Galor-Ryder (1989), the standard Assumption $\mathcal{C}$ on preferences combined with Inada (and the usual concavity and homogeneity) conditions on F do not suffice to give this property. Assumption $\mathcal{C}$(b) and the concavity of F imply that there exists a unique solution

$$k_{t+1} = \phi(k_t) \tag{E'_D}$$

to the equation (E_D). By the implicit function theorem, ϕ is differentiable. An additional assumption is needed to ensure that the graph of ϕ cuts the diagonal with a positive slope at a unique $k_D > 0$. The following condition — which is less restrictive and simpler to verify than the one given by Ryder-Galor (1989) — is sufficient[4].

Assumption $\mathcal{S}$. Define $S(k) = s(r(k), w(k))$. The function $S(k)/k$ is decreasing for all $k > 0$, $\lim_{k\to 0^+} S(k)/k > 1+n$, and $\lim_{k\to+\infty} S(k)/k < 1$.

The property $S(k)/k$ decreasing is equivalent to log $(S(k)/k)$ decreasing and this is equivalent to the elasticity of S being less than one ($\eta_S = \dfrac{dS/S}{dk/k} < 1$): a given percentage increase in the capital stock k gives rise to a smaller percentage increase in savings S. Although this assumption is a joint assumption on preferences and technology, it can be decomposed into separate assumptions on the consumption and the production sides. For example, it holds if

- u is homothetic and satisfies Assumption $\mathcal{C}$
- f is such that $w(k)/k$ is a decreasing function with $\lim_{k\to 0^+} \dfrac{w(k)}{k} = \infty$ and $\lim_{k\to\infty} \dfrac{w(k)}{k} = 0$

These conditions are satisfied if both u and F are CES with elasticity of substitution greater than or equal to 1—which includes Cobb-Douglas utility and production functions.

Proposition 2: *Under Assumptions ($\mathcal{C}$, $\mathcal{S}$), the Diamond steady-state capital k_D is globally stable for the dynamics (E'_D): for any initial condition $k_0 > 0$, the per-capita capital stock on an equilibrium trajectory of the Diamond economy converges to $k_D > 0$.*

Proof: See Appendix.

There are a number of different criteria which can be used to evaluate the efficiency of OLG economies. One is the usual Pareto criterion which, in OLG economies, is often called dynamic efficiency: we recall briefly the definition for the case where agents in all generations are identical and treated equally, so that the allocations can be defined in per-capita terms.

Definition 2. An allocation $\left(c_1^{-1}, (c^t, i_t, k_t)_{t\geq 0}\right)$, where i_t and k_t denote per-capita investment and capital,

[4] A similar assumption was used by Weil (1987). For sake of completeness we prove in the Appendix that Assumption $\mathcal{S}$ implies uniqueness and global stability of the Diamond steady state.

is *feasible* for the economy $\mathcal{E}(u, F, n)$ if for all $t \geq 0$

$$c_0^t + \frac{1}{1+n} c_1^{t-1} + i_t = f(k_t), \qquad (1+n)k_{t+1} = (1-\beta)k_t + i_t$$

A feasible allocation $\left(c_1^{-1}, (c^t, i_t, k_t)_{t \geq 0}\right)$ is *Pareto optimal* or *dynamically efficient* if there does not exist another feasible allocation $\left(\tilde{c}_1^{-1}, (\tilde{c}^t, \tilde{i}_t, \tilde{k}_t)_{t \geq 0}\right)$ with the same initial capital ($\tilde{k}_0 = k_0$) such that $\tilde{c}_1^{-1} \geq c_1^{-1},\ u(\tilde{c}^t) \geq u(c^t)$ for $t \geq 0$, with at least one strict inequality.

For an allocation which converges to a steady state one can also study the long-run efficiency of the allocation, which will hold if the limiting steady state is optimal in the set of feasible steady states.

Definition 3. A steady state allocation (c_0, c_1, i, k) is *feasible* if

$$\left.\begin{array}{c} c_0 + \frac{1}{1+n} c_1 + i = f(k) \\ (1+n)k = (1-\beta)k + i \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{r} c_0 + \frac{1}{1+n} c_1 = f(k) - (n+\beta)k \\ i = (n+\beta)k \end{array}\right.$$

A feasible steady state allocation (c_0, c_1, i, k) is *steady-state optimal* if there does not exist another feasible steady state allocation $(\tilde{c}_0, \tilde{c}_1, \tilde{i}, \tilde{k})$ such that $u(\tilde{c}_0, \tilde{c}_1) > u(c_0, c_1)$. An allocation $\left(c_1^{-1}, (c^t, i_t, k_t)_{t \geq 0}\right)$ which converges to a steady state $(\bar{c}_0, \bar{c}_1, \bar{i}, \bar{k})$ is *long-run efficient*, if the steady state $(\bar{c}_0, \bar{c}_1, \bar{i}, \bar{k})$ is steady-state optimal.

A steady state allocation (c_0, c_1, i, k) can be steady-state inefficient even though the allocation defined by $c_1^{-1} = c_1,\ c^t = (c_0, c_1),\ i_t = i,\ k_t = k,\ t \geq 0$, is dynamically efficient. This occurs when the feasible steady state allocations $(\tilde{c}_0, \tilde{c}_1, \tilde{i}, \tilde{k})$ which dominate (c_0, c_1, i, k) in the sense of Definition 3 are such that $\tilde{c}_0 > c_0,\ \tilde{c}_1 < c_1$. Viewed as allocations on $[0, \infty)$ such steady state allocations do not lead to a Pareto improvement since the old agents who loose consumption at date 0 cannot be compensated in their youth at date -1, since the economy is assumed to "start" at date 0. More generally when an allocation is dynamically efficient but long-run inefficient it is typically possible to change the path of consumption and investment so as to converge to a limiting steady state in which the utility of the representative agent is higher: however, since the allocation is dynamically efficient such changes must decrease the utility of some agents in the early generations in order to improve the utility of an infinite number of future generations. Allocations which are dynamically efficient but long-run inefficient involve an inherent conflict of interest between the welfare of current and future generations: market structures which leads to equilibrium allocations which are both dynamically and long-run efficient, and avoid this conflict between long-run growth and current welfare, are thus of special interest.

A steady state (c_0^*, c_1^*, i^*, k^*) is steady state optimal if it is feasible and satisfies the first-order conditions

$$f'(k^*) - \beta = n, \qquad \frac{u_0'(c^*)}{u_1'(c^*)} - 1 = n \tag{17}$$

The capital labor k^* satisfying the first condition in (17) maximizes aggregate per-capita permanent consumption and is called the *Golden Rule* capital-labor ratio. The second condition requires that the distribution of consumption between young and old agents corresponds to the choice of the representative agent when faced with the interest rate $r(k^*) = n$ which supports the Golden rule capital k^*.

The Golden Rule k^* is determined purely by technological (f, β) and demographic factors (n): the Diamond steady state k_D defined by (16) depends in addition on agents' preferences (savings behavior). Thus for typical economies $k_D \neq k^*$, so that for most economies the Diamond steady state is steady-state inefficient. When $k_D < k^*$, the interest rate $r_D = f'(k_D) - \beta$ at the Diamond steady state exceeds the Golden Rule interest rate $r(k^*) = f'(k^*) - \beta = n$: the Diamond economy converges to a steady state of *underaccumulation* characterized by a low level of capital, low output and a high interest rate. Under Assumption $\mathcal{S}$, $k_D < k^*$ is equivalent to $s\big(r(k^*), w(k^*)\big) < (1+n)k^*$, so that an alternative definition of *underaccumulation* is that the savings of the consumers at the prices $(r(k^*), w(k^*))$ at the Golden Rule are not sufficient to sustain the Golden Rule capital stock. To attain a steady state with a higher level of capital and consumption, more investment than that undertaken in the Diamond equilibrium would be needed at some date. This is not feasible without a decrease in the welfare of some generation: in the case of underaccumulation, a Diamond equilibrium, though long-run inefficient, is dynamically efficient.

When $k_D > k^*$, $r_D < n$ and $s\big(r(k^*), w(k^*) > (1+n)k^*$ so that the savings of consumers at k^* can "buy" more capital than k^* : the Diamond economy converges to a steady state of *overaccumulation* characterized by a high level of capital, a low interest rate r_D and high output level y_D, much of which is absorbed by the need to maintain the capital stock rather than being used for consumption. In this case the Diamond steady state, taken as an allocation on $[0, \infty)$ is dynamically inefficient since an improvement can be obtained by discarding some of the capital at the initial date. For any $k_0 > 0$, a Diamond equilibrium which converges to k_D is both dynamically and long-run inefficient.

Stock Market Equilibrium. In the general case where $v_0 > 0$, a stock market equilibrium can be summarized by the path of the capital and the discount on equity (k_t, v_t) with initial conditions (k_0, v_0). As before equations (iv) and (v) of Definition 1 define the wage and interest rate as a function of the capital-labor ratio, while (i) defines the equity price, and (vii) gives the investment. The path $(k_t, v_t)_{t \geq 0}$ must satisfy for all $t \geq 0$

$$\begin{aligned}(1+n)k_{t+1} &= s(r(k_{t+1}), w(k_t)) + v_t \\ (1+n)v_{t+1} &= (1+r(k_{t+1}))v_t\end{aligned} \qquad (\mathrm{E}_S)$$

$i_t > 0$, and the inequality

$$0 \leq (1+n)v_{t+1} \leq (1-\beta)^2 k_t \qquad (18)$$

to ensure that $q_t \geq 0$.

The first equation in (E_S) is the "savings=investment" equation (vi) of a stock market equilibrium. When there is a discount $v_t > 0$, the (per-capita) capital stock that young agents are able to acquire for use in the subsequent period $\big((1+n)k_{t+1}\big)$, exceeds their savings because firms are sold on the equity market at a discount relative to their replacement cost. The discount v_t in essence acts like an additional "source of funds" that enables them to finance a higher level of capital accumulation than would be warranted by their savings in a Diamond equilibrium, where firms are sold for their replacement cost.

A steady state solution (k, v) of (E_S) must satisfy

$$(1+n)k = s(r(k), w(k)) + v_t \qquad (19)$$

$$(1+n)v = (1+r(k))v \qquad (20)$$

and the inequality

$$0 \leq (1+n)v \leq (1-\beta)^2 k \qquad (21)$$

(20) is equivalent to $(r(k) - n)v = 0$ and thus has two solutions, $v = 0$ and $r(k) = n$. When $v = 0$, (19) gives the Diamond steady state k_D, defined by (16), and (21) is satisfied. Thus the Diamond steady state is a steady-state stock market equilibrium. $r(k) = n$ is equivalent to $k = k^*$, the Golden Rule capital-labor ratio, and (19) defines the associated discount

$$v^* = (1+n)k^* - s(r(k^*), w(k^*)) \qquad (22)$$

As we have seen above, in the case of an economy with underaccumulation, $(1+n)k^* - s(r(k^*), w(k^*)) < 0$ so that (21) is not satisfied: for such economies the Golden Rule is not a steady-state stock market equilibrium. For an economy with underaccumulation, v^* defined by (22) is positive, but in order that the righthand inequality in (21) be satisfied, the characteristics (u, F, n) of the economy must be such that

$$s\big(r(k^*), w(k^*)\big) \geq (1+n)k^* - \frac{(1-\beta)^2}{1+n}k^* \qquad (23)$$

Although the savings at the Golden Rule are not sufficient to cover the replacement cost of capital, they must not be too deficient in a sense made precise by (23). There is thus a limit to the extent to which the discount on equity can make up for the deficiency in savings needed to sustain the Golden Rule. This condition can also be written as

$$s\big(r(k^*), w(k^*)\big) \geq \; i^* \; + \; i^* \frac{(1-\beta)}{1+n}$$

where $i^* = (n+\beta)k^*$ is the (per-capita) investment needed to sustain the Golden Rule: thus, while the savings of the young may not be sufficient to cover the combined costs of new investment and installed capital at replacement value, they must be sufficient to cover current new investment and the depreciated investment of the previous period. For economies with underaccumulation, we focus on those that satisfy the inequality (23)

Assumption $\mathcal{GR}$. For economies with underaccumulation the characteristics (u, F, n) are such that (23) is satisfied with strict inequality.

The system of equations (E_S) is related in an interesting way to the equations studied by Tirole (1985). He considered a Diamond economy in which the young, in addition to financing the capital used in the next period, could also use their savings to purchase an asset paying a zero dividend, which he called a "bubble". This leads to the system of equations

$$\begin{aligned} s_t &= (1+n)k_{t+1} + b_t \\ (1+n)b_{t+1} &= (1+r_{t+1})b_t \end{aligned}$$

where b_t is the (per-capita) price of the bubble asset, and $b_t \geq 0$, since by free disposal the asset cannot have a negative price. This system of equations is the same as (E_S) with $v_t = -b_t$, but with $v_t \leq 0$. If formally when $v_t > 0$, the discount on equity plays the role of a "negative bubble" for Tirole's equations, in our market structure v_t is *not* a negative bubble. A security price is said to have a bubble component if the price differs from the present value of its future stream of dividends (the fundamental value): the difference between the price and the fundamental value is called the bubble (component). With free disposal of securities, all security prices must be non-negative, and as Tirole has shown, this implies that the bubble component can only be non-negative. When inequality (21) is satisfied, the equity price is non-negative, so that if there is a bubble in our model it can only be non-negative. As we shall see in Section 4, when $v_t > 0$ the fundamental value of equity is not equal to $(1-\beta)k_{t+1}$ so that $-v_t$ is not the difference between the price of equity and its fundamental value. In short, in our model there is free disposal of securities and there are no negative bubbles.

Tirole's equations also hold for a Diamond economy in which a government incurs a debt b_0 with the young at date 0, and rolls it over indefinitely, borrowing from the young agents of generation t to reimburse the contemporaneous old agents. The case $b_t > 0$ (or $v_t < 0$) corresponds to government debt, so that formally the case obtained for $b_t < 0$ or ($v_t > 0$) corresponds to a government surplus. Most authors in macroeconomics have found it implausible to study a setting where the government runs a perpetual surplus used to finance investment in the private sector (see for example Azariadis (, p.)): as a result the dynamics of the system (E_S) with $v_t > 0$ has not been studied in the macro literature. Since as we shall see shortly, for economies with underaccumulation, the trajectories of the system (E_S) with $v_t > 0$ have good normative properties, it seems important to establish that this dynamics can be generated by a realistic market structure under plausible assumptions, without invoking government intervention and without violating any rationality assumption.

Under Assumption $\mathcal{C}$, (E_S) can be written as

$$\begin{aligned} k_{t+1} &= \psi(k_t, v_t) \\ v_{t+1} &= \frac{1-\beta+f'\big(\psi(k_t,v_t)\big)}{1+n} v_t \end{aligned} \tag{E'_S}$$

where $\psi : \mathbb{R}^2_t \longrightarrow \mathbb{R}$ is an increasing, differentiable function. The phase diagram is determined by the curves $\mathcal{V}$ and $\mathcal{K}$ defined by

$$\begin{aligned} \mathcal{V} &= \big\{(k_t, v_t) \in \mathbb{R}^2_+ \,|\, v_{t+1} = v_t\big\} = \big\{(k_t, v_t) \in \mathbb{R}^2_+ \,|\, v_t = 0 \text{ or } \psi(k_t, v_t) = k^*\big\} \\ &= \big\{(k_t, v_t) \in \mathbb{R}^2_+ \,|\, v_t = 0 \text{ or } v_t = (1+n)k^* - s(r^*, w(k_t))\big\} \end{aligned}$$

and

$$\mathcal{K} = \big\{(k_t, v_t) \in \mathbb{R}^2_+ \,|\, k_{t+1} = k_t\big\} \;=\; \big\{(k_t, v_t) \in \mathbb{R}^2 \,|\, v_t = (1+n)k_t - s\big(r(k_t), w(k_t)\big)\big\}$$

Under Assumptions $(\mathcal{C}, \mathcal{S})$, $\mathcal{V}$ is the union of the axis $v = 0$ and the graph of a decreasing function, while $\mathcal{K}$ is a U-shaped curve passing through the origin. The resulting phase diagrams which suggest—but are by themselves insufficient to prove—the stability properties of the steady states, are shown in Figure 1(a) for an economy with overaccumulation, and in Figure 1(b) for an economy with underaccumulation.

As Figure 1(a) suggests, the Diamond steady state k_D is globally stable for an economy with overaccumulation.

Proposition 3: *Under Assumptions* $(\mathcal{C}, \mathcal{S})$*, if* $k^* < k_D$ *then any solution* $(k_t, v_t)_{t\geq 0}$ *of* (E_S) *with* $k_0 > 0, v_0 \geq 0$ *converges to* $(k_D, 0)$.

Proof: If $v_0 = 0$, then by Proposition 2, the trajectory converges to the Diamond steady state. Suppose $v_0 > 0$. Consider the three regions A, B, C shown in Figure 1(a). A is defined by $k \leq \hat{k}$ where $\hat{k}$ is such that $(1+n)k^* - s(n, w(\hat{k})) = 0$: since $(1+n)k^* - s(n, w(\hat{k}^*)) < 0$, $0 < \hat{k} < k^*$. The definitions of B and C

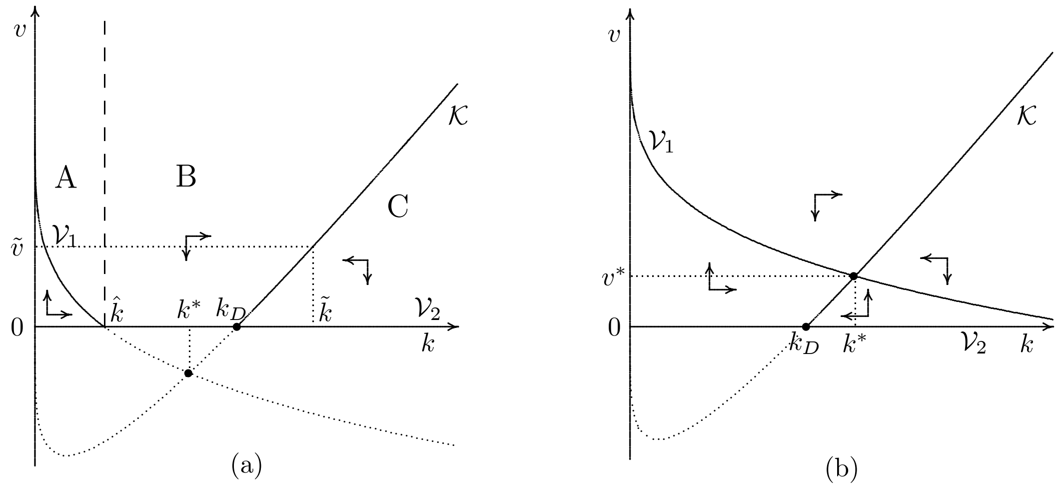

Figure 1: Phase diagram for the stock market equilibrium equations (E'_S) in the cases of (a) overaccumulation and (b) underaccumulation.

are clear from Figure 1(a). Let us show that if $(k_0, v_0) \in A$ then the trajectory must leave the region A in a finite number of periods and enter $B \cup C$. $v_t > 0$, $t \geq 0$ implies by induction that for $t \geq 1$, $k_t > k_t^D$ where $(k_t^D)_{t \geq 0}$ is the Diamond trajectory beginning at $(k_0, 0)$: $k_{t+1} = \psi(k_t, v_t) > \psi(k_t, 0) > \psi(k_t^D, 0) = k_{t+1}^D$. Since k_t^D converges to $k^D > \hat{k}$ the property follows. Let us show that if $(k_0, v_0) \in B \cup C$, the trajectory stays in $B \cup C$. If $(k_t, v_t) \in B$ then $k_{t+1} > k_t$, so that $(k_{t+1}, v_{t+1}) \in B \cup C$. If $(k_t, v_t) \in C$, then $k_t > k_D$ so that $k_{t+1} = \psi(k_t, v_t) > \psi(k_D, 0) = k_D$. Thus $(k_t, v_t) \in B \cup C$. When $(k_t, v_t) \in B \cup C$ the sequence $(v_t)_{t \geq 0}$ is a decreasing sequence which is bounded below since $v_t > 0$, $\forall\, t \geq 0$: thus $v_t \longrightarrow \bar{v}$. Either $\bar{v} > 0$ or $\bar{v} = 0$. Suppose $\bar{v} > 0$ then $k_t \longrightarrow \bar{k}$ defined by $\bar{v} = \frac{1}{1+n}\left(1 - \beta + f'\big(\psi(\bar{k}, \bar{v})\big)\right)\bar{v} \iff \psi(\bar{k}, \bar{v}) = k^*$ so that $(\bar{k}, \bar{v})$ lies on the curve $\mathcal{K}$. Since there is no intersection of the $\mathcal{V}_1$ curve and the $\mathcal{K}$ curve in the non-negative orthant, it follows that $\bar{v} \in \mathcal{V}_2$. Thus $\bar{v} = 0$ and $\bar{k} = k_D$. □

A solution of (E_S) is an equilibrium trajectory for our economy if the inequalities $i_t > 0$, and $(1+n)v_{t+1} < (1-\beta)^2 k_t$ are satisfied at all dates. Let $\tilde{k}$, be the first value of the capital-labor ratio for which investment is zero under the Diamond dynamics, i.e. $\phi(\tilde{k}) = (1-\beta)\tilde{k}$. Then $\tilde{k} > k_D$, and $k < \tilde{k}$ implies $\phi(k) > (1-\beta)k$. Let $\tilde{v}$ be the discount such that $(\tilde{v}, \tilde{k}) \in \mathcal{K}$ (see Figure 1(a)). Consider initial conditions (v_0, k_0) such that $k_0 < k_D, v_0 < \tilde{v}$, and $(v_0, k_0) \in$ B and $(1+n)v_0 \leq (1-\beta)^2 k_0$. Since, by the reasoning above $v_t \leq v_0$ and $k_t \geq k_0$, it follows that $(1+n)v_{t+1} \leq (1+n)v_0 \leq (1-\beta)^2 k_0 \leq (1-\beta)^2 k_t, \forall\, t \geq 0$, so that the inequality (18) holds. Since $(v_0, k_0) << (\tilde{v}, \tilde{k}), \psi(v_0, k_0) < \psi(\tilde{v}, \tilde{k}) = \tilde{k}$. Thus $k_1 \leq \tilde{k}$, and since $v_1 \leq v_0, v_1 \leq \tilde{v}$. Thus

$(v_1, k_1) < (\tilde{v}, \tilde{k})$, and by induction for all $t, (v_t, k_t) << (\tilde{v}, \tilde{k})$. But $\psi(v_t, k_t) \geq \psi(k_t, 0) > (1-\beta)k_t$ since $k_t < \tilde{k}$, so that investment is positive on the whole trajectory.

Thus in the case of overaccumulation it is easy to prove the existence of a stock market equilibrium trajectory and its convergence to the Diamond equilibrium. In this case the existence of a discount on equity does not improve the long-run efficiency of the equilibrium. This was to be expected since in the case of overaccumulation the propensity to save of the young agents is too high when compared to the productivity of capital. The discount on equity which is akin to an increase in savings can only make things worse. In the long run however the effect vanishes, since the discount on equity increases at a slower rate than the population and tends to disappear in per-capita terms, so that the equilibrium converges to the stable Diamond steady state. A variety of methods have been proposed for absorbing the excess savings to restore convergence to the Golden Rule: social security, land as a third factor of production (McCallum (1987), Rhee (1991)) or unbacked debt (Pingle-Tesfatsion (1998)): each of these methods is applicable to our model.

For an economy with underaccumulation, the savings of the young are "scarce" and the discount on the equity prices acts like an additional source of funds, permitting increased investment. The phase diagram (Figure 1(b)) suggests that the equilibrium trajectories converge to the Golden Rule steady state. Global properties are more difficult to establish for economies with underaccumulation than for those with overaccumulation. Indeed even to prove the local stability of the Golden Rule (k^*, v^*), a stronger assumption is needed than that which assures the stability of the Diamond steady state under the Diamond dynamics, namely Assumption $(\mathcal{C}, \mathcal{S})$.

Assumption $\mathcal{P}$. The production function f is such that $kf'(k)$ is an increasing function of k.

$\mathcal{P}$ is satisfied only if capital and labor are sufficiently substitutable: it requires that the marginal product of capital $f'(k)$ does not decrease too fast as the capital-labor ratio k increases, so that the amount of output (per capita) going as payment to capital $(kf'(k))$ decreases. This property is satisfied for CES production functions $F(K, L)$ with elasticity of substitution greater than or equal to 1 (and hence for Cobb-Douglas production functions): thus for the class of CES functions $\mathcal{P}$ requires no additional restrictions over those needed to satisfy Assumption $\mathcal{S}$.

Proposition 4: *Under Assumptions $(C, \mathcal{S}, \mathcal{P})$, if $k^* > k_D$ then under the stock market equilibrium dynamics (E_S), the Golden Rule (k^*, v^*) is locally stable and the Diamond steady state $(k_D, 0)$ is locally saddlepoint stable.*

Proof: See Appendix

Proposition 4 implies that for all economies satisfying Assumption $\mathcal{GR}$, a stock market equilibrium which converges to the Golden Rule exists for all initial conditions (k_0, v_0) in a neighborhood of the Golden Rule (k^*, v^*). In both the case of over and underaccumulation, since there is a stable steady state (k_D in overaccumulation, k^* in underaccumulation), stock market equilibria exist for a continuum of initial conditions (k_0, v_0) or equivalently (k_0, q_0) since $q_0 = (1-\beta)k_0 - v_0$. In our model the equity price (q_t) is not rigidly tied to the amount of capital $(1-\beta)k_t$ embodied in the firm since there can be a discount (v_t). The discount v_0 must justify the price q_{-1} paid by the old agents of date 0 in their youth at date -1: however since this price is absent from the model when it is "started" at date 0, q_0 must be given exogenously. Some authors find it problematic if the model does not deliver a unique "right" price q_0 which can justify the expectations of agents at dates not included in the model, and consider the equilibria indeterminate. OLG models with infinite-lived financial (monetary) assets frequently exhibit this type of indeterminacy. We adopt the alternative, more pragmatic view, that if the economy is cut at any moment in time t and agents are given the currently available information from the markets (k_t, q_t), then the future course of prices (and capital accumulation) is determinate and can be correctly anticipated by the agents.

Stock market equilibria of an economy with underaccumulation which converge to the Golden Rule are, by definition, long-run efficient: applying the Cass criterion shows that they are also dynamically efficient.

Proposition 5: *If f is strictly concave and u differentiably strictly quasi-concave*[5]*, a stock market equilibrium of an economy with underaccumulation which converges to the Golden Rule is both dynamically and long-run efficient.*

Proof: To prove dynamic efficiency note that, since the per-capita capital satisfies the first-order condition $f'(k_t) = r_t + \beta$ and agents maximize utility under the budget constraint $c_0^t + c_1^t/(1+r_{t+1}) = w_t$, a stock market equilibrium can be viewed as an Arrow-Debreu equilibrium, with prices $p_0 = 1$, $p_t = \prod_{\tau=1}^{t} 1/(1+r_\tau)$, for an economy with the following characteristics: there is one good at each date, and each agent born at date t is the owner of a firm which which lasts for two periods, $t-1$ and t, investing k_t units of the good at date $t-1$ to obtain $f(k_t)+(1-\beta)k_t$ units at date t. Each firm maximizes its profit $p_t(f(k_t)+(1-\beta)k_t) - p_{t-1}k_t$, or equivalently maximizes $p_t(f(k_t)+(1-\beta)k_t - (1+r_t)k_t)$, by choosing k_t such that $f'(k_t) = \beta + r_t$. The profit of each firm $p_t(f(k_t) - f'(k_t)k_t)$ goes to its owner, a young agent at date t. In this market structure, which is

[5] See Mas-Collel (1985), p.). The condition insures that the indifference curves have positive Gaussian curvature.

purely formal, i.e. does not claim to be realistic, young agents own firms which operate one period before they are born, and the labor income that agents receive in the standard model becomes a profit. An agent born at date t maximizes lifetime utility $u(c_0^t, c_1^t)$ subject to the constraint $p_t c_0^t + p_{t+1} c_1^t = p_t(f(k_t) - k_t f'(k_t))$. Market clearing at date $t \geq 0$ is given by

$$N_t c_0^t + N_{t-1} c_1^{t-1} + N_{t+1} k_{t+1} = N_t(f(k_t) + (1-\beta)k_t)$$

which is the same as the equation $c_0^t + c_1^t/(1+n) + i_t = f(k_t)$ in the stock market model. Since an equilibrium trajectory $(c_t, k_t)_{t\geq 0}$ with $k_0 > 0,\ 0 < c_{-1}^1 < (1+n)(f(k_0) + (1-\beta)k_0$ which converges to the Golden Rule , lies in a compact subset of the positive orthant the curvature condition needed to apply the efficiency condition developed by Cass (1972), Benveniste-Gale (1975), and Balasko-Shell (1981) is satisfied. Thus the Arrow-Debreu allocation is Pareto optimal if

$$C = \sum_{t=1}^{\infty} \frac{1}{N_t p_t} = \sum_{t=1}^{\infty} \frac{\prod_{\tau=1}^{t}(1+r_\tau)}{N_0(1+n)^t} = \infty$$

If the equilibrium is also a stock market equilibrium converging to the Golden Rule, then there exists a sequence $v_t > 0,\ t \geq 0$ which converges to $v^* > 0$ such that $(1+n)v_{t+1} = (1+r_{t+1})v_t, t \geq 0$ so that $\prod_{\tau=1}^{t}(1+r_\tau)v_0 = (1+n)^t v_t$. Thus $C = \sum_{\tau=1}^{\infty} v_t$ and since $v_t \to v^* > 0,\ C = \infty$. □

Note that the Arrow-Debreu equilibria constitute a 2-dimensional family[6] of paths parametrized by (k_0, c_1^{-1}). The Diamond equilibria select the Arrow-Debreu equilibria with $c_1^{-1} = (1+n)(k_0 f'(k_0) + (1-\beta)k_0$, while the stock market equilibria select the class of Arrow-Debreu equilibria with $c_1^{-1} = (1+n)(k_0 f'(k_0) + (1-\beta)k_0 - v_0$ for appropriate restrictions on (k_0, v_0).

Assumption $\mathcal{GR}^*$ in essence imposes a restriction on how far the Diamond steady state k_D is from the Golden Rule k^*: if k^* is too much greater than k_D then the funds in excess of the savings of the young needed to finance investment become too large to permit them to be covered by the discount on the equity prices. To get a feel for how the equilibrium behaves and to what extent these conditions are restrictive, let us consider a family of Cobb-Douglas economies.

[6]To see this, note that, if we denote demand and supply functions by tilde, the equilibrium equations can be written as

$$\tilde{c}_0^0(p_0, p_1) + \tfrac{1}{1+n} c_1^{-1} + (1+n)\tilde{k}_1(p_0, p_1) = f(k_0) + (1-\beta)k_0$$
$$\tilde{c}_0^t(p_t, p_{t+1}) + \tfrac{1}{1+n} \tilde{c}_1^{t-1}(p_{t-1}, p_t) + (1+n)\tilde{k}_1(p_t, p_{t+1}) = f(\tilde{k}_t(p_{t-1}, p_t)) + (1-\beta)k_t(\tilde{k}_t(p_{t-1}, p_t),\ \forall\, t \geq 1$$

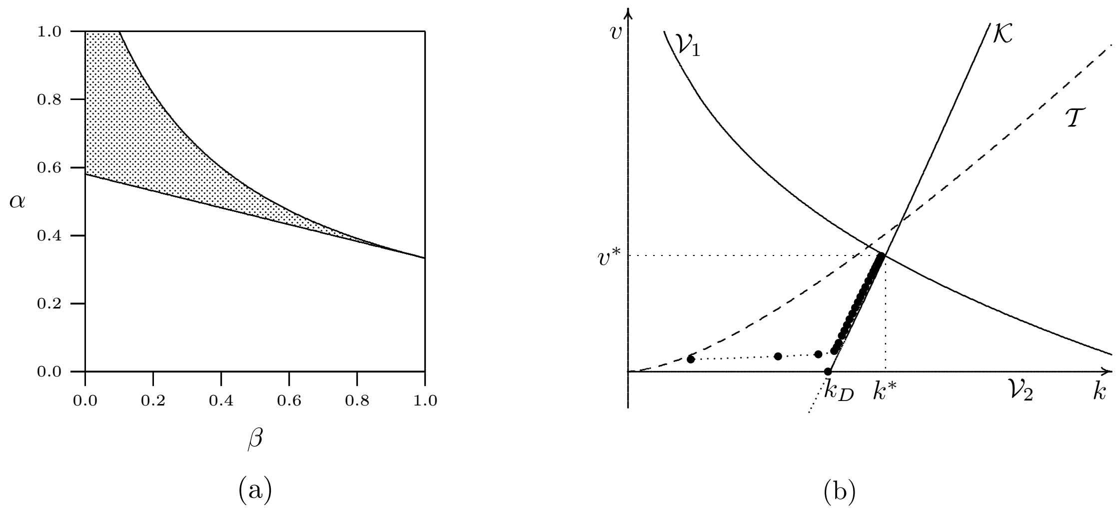

Figure 2: (a) parameters for which the Cobb-Douglas economy is characterized by underaccumulation and satisfies $\mathcal{GR}$; (b) convergence of a trajectory to the Golden Rule.

Example: Let $\mathcal{E}(u, F, n)$ be a Cobb-Douglas economy:

$$u(c_0, c_1) = c_0^{1-\alpha} c_1^{\alpha}, \quad 0 < \alpha < 1, \quad F(K, L) = AK^{\gamma} L^{1-\gamma}, \quad 0 < \gamma < 1$$

There are four parameters $(\alpha, \gamma, \beta, n)$ which characterize an economy: the parameter A is just a scale factor which does not matter for the analysis (for the graph in Figure 2 we chose $A = 50$). α gives the propensity to save of the young $\big(s(r, w) = \alpha w\big)$, γ determines the share of capital in output, $0 < \beta < 1$ is the depreciation rate of capital and n the population growth rate. Let us fix $\gamma = 0.25$ and $n = 0.35$ (which corresponds to an annual increase of population of about 1% for 30 years). The Golden Rule capital-labor ratio is $k^* = \left(\dfrac{A\gamma}{\beta + n}\right)^{\frac{1}{1-\gamma}}$ and there is underaccumulation if

$$(1+n)k^* \geq A\alpha(1-\gamma)(k^*)^{\gamma} \iff (1+n) \geq \frac{\alpha(1-\gamma)}{\gamma}(\beta + n) \iff \alpha \leq \frac{\gamma(1+n)}{(1-\gamma)(\beta+n)} \tag{24}$$

The Golden Rule k^* satisfies condition (23) if

$$A\alpha(1-\gamma)(k^*)^{\gamma} \geq (\beta+n)k^* + \frac{1-\beta}{1+n}(\beta+n)k^* \iff \alpha \geq \frac{\gamma}{1-\gamma} + \frac{(1-\beta)^2}{1+n}\frac{\gamma}{1-\gamma} \tag{25}$$

For the chosen parameters $(\gamma, n) = (0.25, 0.35)$, (24) and (25) give the admissible values of the parameters $(\alpha, \beta) \in (0,1) \times (0,1)$ for which the Golden Rule is a stock market equilibrium. Let $\ell(\beta) = \frac{\gamma}{1-\gamma}\left(\frac{1+n}{\beta+n}\right)$ denote the function in (24) defining economies with "low" savings (underaccumulation) and let $p(\beta) = \frac{\gamma}{1-\gamma}\left(1 + \frac{(1-\beta)}{1+n}\right)$ denote the function in (25) defining economies in which there is "positive" investment, then the admissible parameters (α, β) are given by the shaded region in Figure 2(a). Thus, for example, if $\beta = 0.4$ (which corresponds to an annual depreciation rate of 1.7% for 30 years) then the interval of admissible α values is [0.48, 0.6]. Figure 2(b) shows an equilibrium trajectories when $(\alpha, \beta) = (0.5, 0.4)$ beginning with a low level of (k_0, v_0). The region lying below the dashed curve $\mathcal{T}$ defines the set of (k_t, v_t) pairs satisfying condition (18) which ensures that the young have the incentive to make positive investment in their firms. All initial conditions in the region below $\mathcal{T}$, above $\mathcal{K}$, and with $v_0 < v^*$ lead to stock market equilibria converging monotonically to the Golden Rule. While for the Cobb-Douglas economy the convergence is monotone, this may not be true generally since Proposition 5 does not preclude complex or negative eigenvalues at the Golden Rule.

4. Financial Valuation

The previous section analyzed the real equilibrium outcome of an economy in which capital once installed is a sunk cost, making it possible for the equity price of a firm to be less than its replacement cost. In this section we examine the equilibrium financial valuation of firms— in particular, the relation between the equity price of a firm and its fundamental value (the discounted sum of its future dividends).

In our model, as in all models in which the sequence of equilibrium interest rates is $(r_t)_{t \geq 0}$, the equity price[7] must satisfy the rate of return condition

$$Q_t = \frac{1}{1 + r_{t+1}}(D_{t+1} + Q_{t+1}) \tag{26}$$

which by successive substitution gives

$$Q_t = \sum_{\tau=1}^{T} \frac{D_{t+\tau}}{(1 + r_{t+1}) \cdots (1 + r_{t+\tau})} + \frac{Q_{t+T}}{(1 + r_{t+1}) \cdots (1 + r_{t+\tau})}$$

Assuming that the limits exist (perhaps in the generalized sense of taking the values $+\infty$ or $-\infty$) the *fundamental value* of equity at date t is defined by

$$Q_t^f = \lim_{T \to \infty} \sum_{\tau=1}^{T} \frac{D_{t+\tau}}{(1 + r_{t+1}) \cdots (1 + r_{t+\tau})}$$

and the *bubble component* as

$$Q_t^b = \lim_{T \to \infty} \frac{Q_{t+T}}{(1 + r_{t+1}) \cdots (1 + r_{t+T})}$$

[7] In what follows we assume that all firms follow, up to scale, the same investment and financial policy and omit the firm index.

Free disposal of securities implies that the equity price is always non-negative, $Q_{t+T} \geq 0, \forall\, T \geq 0$ so that $Q_t^b \geq 0$. If $Q_t^b = 0$, the equity price is said to satisfy a *transversality condition*, and the price of equity is equal to its fundamental value $Q_t = Q_t^f$. If $Q_t^b > 0$, the equity price has a bubble component and its price exceeds its fundamental value, $Q_t > Q_t^f$.

The transversality condition holds in all economies with equilibria in which the asymptotic interest rate exceeds the rate of growth of the economy. This includes most models with infinite-lived agents who discount the future, the OLG model of Scheinkman (1983) and Dechert-Yamamoto (1992), and the Diamond equilibria of economies with underaccumulation. For, if in such economies Q_t^b were positive, the price of equity Q_t would have to grow asymptotically faster than the rate of growth of the economy and would at some date have to exceed the resources of the economy, which is not compatible with equilibrium.[8]

The transversality condition is often associated with efficiency. The reason is that, when the asymptotic interest rate exceeds the rate of growth, the Cass criterion for efficiency is satisfied. However it is known that the Cass criterion is weaker than the transversalilty condition. The analysis that follows will show that stock market equilibria of a economy with underaccumulation are an example of equilibria which do not satisfy a transversality condition, but do satisfy the Cass criterion, as we saw in Proposition 5. The effect of the discount on equity is to lower interest rates relative to the Diamond equilibrium, so that in an economy with underaccumulation the interest rate converges to the rate of growth of the population n, rather than to the Diamond steady state interest rate $r_D > n$. As shown in Proposition 6 below this downward shift in the equilibrium interest rates is sufficient to create a bubble component in the market value (equity plus debt) of firms.

So far we have restricted attention to the financial policy for firms which consists in financing investment at each date by a one-period debt which is fully reimbursed in the following period ($B_t = I_t, \forall\, t \geq 0$). Confining the analysis of equity prices to this case has two drawbacks. First, in economies with overaccumulation this financial policy leads to negative dividends, which is not realistic: for in such economies the low earnings of firms coupled with their high investment make it likely that they will either roll over their debt or issue new equity. Second, the proposition below which shows that the equity price has a bubble component may be criticized as being a result which depends on the particular financial policy chosen.To avoid these drawbacks, we note that for our model it can be shown that the Modigliani-Miller theorem holds[9]: this theorem asserts that *for financial policies which do not lead to bankruptcy,* (i) *the optimal investment policy of a firm, and*

[8]This is in essence the argument which eliminates the bubble component in the models of Scheinkman (1983), Magill-Quinzii (1996), and Santos-Woodford (1997).

[9]The proof is given in the earlier version of the paper (Magill-Quinzii (2000)).

(ii) *the market value of the firm (the sum of its equity and debt), are independent of its financial policy.*

In our economy the *market value* to the firms at date t is given by

$$M_t = Q_t + B_t = (1-\beta)K_t - V_t + I_t = K_{t+1} - V_t$$

This can be evaluated by choosing the particular financial policy $B_t = I_t$ that we have considered above, and the timing in which firms are sold without debt and before their investment is made. For simplicity we restrict firms external funding to general debt policies $(B_t)_{t\geq 0}$, with $B_t \geq 0, \forall t \geq 0$ and assume that firms do not issue new equity. The following identity for the sources and uses of funds at date $t+1$ must then hold

$$F(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - (1+r_{t+1})B_t + B_{t+1} = I_{t+1} + D_{t+1} \tag{27}$$

The left side describes the sources of funds for the firm, internal (earnings) and external (borrowing), while the right side gives the two uses to which funds can be put: to pay for investment or to pay dividends to shareholders. In equilibrium, regardless of the firms' debt policy $(B_t)_{t\geq 0}$, the equity price $(Q_t)_{t\geq 0}$ must satisfy (26): adding B_t to both sides gives

$$Q_t + B_t = \frac{1}{1+r_{t+1}}\Big(D_{t+1} + (1+r_{t+1})B_t + Q_{t+1}\Big)$$

which can be written as

$$M_t = \frac{1}{1+r_{t+1}}\Big(R_{t+1} + M_{t+1}\Big) \tag{28}$$

where, in view of (27),

$$R_{t+1} = D_{t+1} + \Delta_{t+1} = F(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - I_{t+1} \tag{29}$$

D_{t+1} being the payment by the firms to to the shareholders and

$$\Delta_{t+1} = (1+r_{t+1})B_t - B_{t+1} \tag{30}$$

being the net payment to the debtholders. The right side of (29) is the real output which is available after compensating labor and deducting the part of output going to investment: this is the "real dividend" which is left to pay the "capital markets", i.e. the two claimants to the firms' income stream, the equity and debt holders. Integrating (28) gives

$$M_t = \sum_{\tau=1}^{T} \frac{R_{t+\tau}}{(1+r_{t+1})\dots(1+r_{t+\tau})} + \frac{M_{t+T}}{(1+r_{t+1})\dots(1+r_{t+T})}$$

and, by analogy with the definitions for equity, we define

$$M_t^f = \lim_{T\to\infty} \sum_{\tau=1}^{T} \frac{R_{t+\tau}}{(1+r_{t+1})\dots(1+r_{t+\tau})}, \quad M_t^b = \lim_{T\to\infty} \frac{M_{t+T}}{(1+r_{t+1})\dots(1+r_{t+T})}$$

as the fundamental value and the bubble component of the firms' market value. Since $M_{t+T} = Q_{t+T} + B_{t+T}$, if $M_t^b = 0$ then both the present value of the equity price and of the debt tend to zero: there is no bubble on equity and firms do not accumulate debt at infinity. If $M_t^b > 0$, then either $Q_t^b > 0$ and their is a bubble on equity, or $\lim_{T\to\infty} \frac{B_{t+T}}{(1+r_{t+1})\dots(1+r_{t+T})} > 0$ and firms accumulate debt at infinity, or both. If the present value of debt at infinity is positive, we will say that there is a bubble on debt. To justify this terminology, note that (30) can be written as

$$B_t = \frac{1}{1+r_{t+1}}\Big(\Delta_{t+1} + B_{t+1}\Big) \tag{31}$$

which is the rate of return equation on the firms' debt, Δ_{t+1}, the net payment by firms to debt holders,[10] playing the role of a generalized dividend. Integrating (31) leads to the definitions

$$B_t^f = \lim_{T\to\infty} \sum_{\tau=1}^{T} \frac{\Delta_{t+\tau}}{(1+r_{t+1})\dots(1+r_{t+\tau})}, \quad B_t^b = \lim_{T\to\infty} \frac{B_{t+T}}{(1+r_{t+1})\dots(1+r_{t+T})}$$

as the fundamental value and bubble component of the debt. The two components of market value can then be decomposed into their equity and debt components

$$M_t = Q_t + B_t, \quad M_t^f = Q_t^f + B_t^f, \quad M_t^b = Q_t^b + B_t^b$$

and if there is a bubble on market value, there is either a bubble on equity, or on debt, or both.

Proposition 6: *The market value of firms $(M_t)_{t\geq 0}$ in a stock market equilibrium has the following properties:*

(i) *If $(K_t, V_t)_{t\geq 0}$ is an equilibrium trajectory of an economy with underaccumulation, then*

(α) *if $V_0 = 0$, then $M_t = M_t^f$ and $M_t^b = 0$;*

(β) *if $V_0 > 0$, and the trajectory converges to the Golden Rule, then $M_t > M_t^f$ and $M_t^b > 0$.*

(ii) *If $(K_t, V_t)_{t\geq 0}$ is an equilibrium trajectory of an economy with overaccumulation, then $M_b^f = -\infty$ and $M_t^b = +\infty$*

Proof: (i)(α): Since $V_t = 0, \forall t \geq 0$,

$$\frac{M_{t+T}}{(1+r_{t+1})\dots(1+r_{t+T})} = \frac{N_t(1+n)^{T+1}k_{t+T+1}}{(1+r_{t+1})\dots(1+r_{t+T})}$$

[10] In the OLG model, $(1+r_{t+1})B_t$ goes to the old, while B_{t+1} is taken from the young.

and since the equilibrium is a Diamond equilibrium, $k_{t+T+1} \to k_D$, $r_{t+T} \to r_D$ with $r_D > n$, and $\lim_{T\to\infty} \dfrac{(1+n)^{T+1}}{(1+r_{t+1})\dots(1+r_{t+T})} = 0$ so that $M_t = M_t^f$ and $M_t^b = 0$.

(i)(β): Since $V_t > 0, \forall\, t \geq 0$,

$$\frac{M_{t+T}}{(1+r_{t+1})\dots(1+r_{t+T})} = \frac{N_t(1+n)^T((1+n)k_{t+T+1} - v_{t+T})}{(1+r_{t+1})\dots(1+r_{t+T})} = N_t\frac{v_t}{v_{t+T}}((1+n)k_{t+T+1} - v_{t+T})$$

and since the equilibrium is a stock market equilibrium, $k_{t+t+1} \to k^*$, $v_{t+T} \to v^*$ with $(1+n)k^* - v^* = s(n, w(k^*)) > 0$ and $M_t^b = N_t v_t \dfrac{s(n,w(k^*))}{v^*} > 0$.

(ii) If $V_0 = 0$, then the expression for M_{t+T} is the same as in (i)(α). Since $k_{t+T+1} \to k_D$, $r_{t+T} \to r_D$ with $r_D < n$, it follows that $\lim_{T\to\infty} \dfrac{(1+n)^T}{(1+r_{t+1})\dots(1+r_{t+T})} = +\infty$. When $V_0 > 0$, the expression for M_{t+T} is the same as in (i)(β) and since $k_{t+T+1} \to k_D, v_{t+T+1} \to 0$, it follows that $M_t^b = +\infty$. Since M_t is finite, it follows that $M_t^f = -\infty$, which can also be checked directly by evaluating M_t^f. □

The only case in which the market value of the firms concides with the fundamental value of their real dividends is in the Diamond equilibrium of an economy with underaccumulation. Since in this case there is no discount on equity, the market value also coincides with the replacement value of the capital embodied in the firms at the end of the period

$$M_t^f = M_t = K_{t+1}$$

In this case there is no bubble on equity and the equity price fits with the conventional measures: comparing the financial and the real side, the market value of the firms (equity + debt) will corrrespond with the "book value" or replacement cost, i.e. the accumulated value of investments once depreciation has been taken into account. Or, if analysts were to correctly forecast future interest rates and future dividends and evaluate the fundamental value of equity, their valuation would coincide with the observed value of equity.

In all other cases the conventional measures—replacement cost and fundamental value—will not provide exact estimates, they will only provide bounds since

$$M_t^f < M_t \leq K_{t+1}$$

with strict inequality on the right side when $V_t > 0$. To better understand the implications of Proposition 6 for the equity prices, let us study two examples with specific financial policies for economies with under and overaccumulation.

Example: *Economy with Underaccumulation*

Consider a stock market equilibrium of an economy with underaccumulation converging to the Golden Rule, in which investment is financed by one-period loans as in Section 2. Then $B_t = I_t, \forall\, t \geq 0$ and the dividend on equity is

$$\begin{aligned} D_{t+1} &= F(K_{t+1}, L_{t+1}) - w_{t+1}L_{t+1} - (1+r_{t+1})I_t \\ &= N_t\Big((1+n)k_{t+1}f'(k_{t+1}) - (1+r_{t+1})i_t\Big) \end{aligned}$$

which, after using the relation $f'(k_{t+1}) = \beta + r_{t+1}$ and re-arranging terms, can be written as

$$D_{t+1} = N_t(1-\beta)\Big((1+r_{t+1})k_t - (1+n)k_{t+1}\Big) \tag{32}$$

Thus

$$D_{t+1} \geq 0 \Longleftrightarrow \frac{k_t}{k_{t+1}} \geq \frac{1+n}{1+r_{t+1}} = \frac{v_t}{v_{t+1}} \Longleftrightarrow \frac{v_{t+1}}{k_{t+1}} \geq \frac{v_t}{k_t}$$

As can be seen from Figure 2, for the equilibrium trajectory of the Cobb-Douglas example of Section 3, the ratio v_t/k_t is increasing when (k_t, v_t) is not too far from the Golden Rule (k^*, v^*): when the capital stock is not too far from the Golden Rule, the earnings of the firms are sufficient to pay off the debt at each period and distribute positive dividends. To avoid calling on the shareholders to finance the substantial investments on the initial segment of the trajectory where the earnings are relatively low, the firms would need to roll over their debt for several periods.

Assuming that the capital stock is sufficiently close to k^*, so that the firms no longer roll over debt and the one-period debt policy is feasible, summing the present value of dividends given by (32) gives

$$\sum_{\tau=1}^{T} \frac{D_{t+\tau}}{(1+r_{t+1})\dots(1+r_{t+\tau})} = N_t(1-\beta)\left(k_t - \frac{(1+n)^T k_{t+T}}{(1+r_{t+1})\dots(1+r_{t+T})}\right) \tag{33}$$

The recursive equations (E_S) imply $\dfrac{(1+n)^T}{(1+r_{t+1})\dots(1+r_{t+T})} = \dfrac{v_t}{v_{t+T}}$, so that the fundamental value of equity is given by

$$Q_t^f = N_t(1-\beta)\lim_{T\to\infty}(k_t - \frac{v_t}{v_{t+T}}k_{t+T}) = N_t(1-\beta)(k_t - \frac{k^*}{v^*}v_t)$$

which can be written as

$$Q_t^f = (1-\beta)K_t - (1-\beta)\frac{k^*}{v^*}V_t \tag{34}$$

Since, by Assumption $\mathcal{GR}$, inequality (21) is satisfied at the Golden Rule, it follows that $v^* < \frac{(1-\beta)^2}{1+n}k^* < (1-\beta)k^*$. By (34), the fundamental value of dividends is less than the equity price $Q_t = (1-\beta)K_t - V_t$ and

$$Q_t^f < Q_t < (1-\beta)K_t$$

The equity price is bounded above by the replacement cost and below by the fundamental value. As mentioned in Section 3, $(1-\beta)K_t$ is not the fundamental value of equity so that the discount $-V_t$ cannot be considered as a negative bubble attached to the fundamental value. The bubble component of equity is $Q_t^b = ((1-\beta)\frac{k^*}{v^*} - 1)V_t > 0$.

It follows from (32) and (34) that at the Golden Rule steady state, the dividends and fundamental value of equity are zero: firms use all their earnings to pay for the growth of capital—with the financial policy consisting of one-period loans, all their earnings are used to reimburse the debt incurred to finance last-period's investment.[11] Shareholders obtain their return solely by capital gains i.e. by the increase in the price of equity that agents of the next generation will pay. Since the agents' expectations in our model are given by (1), the justification for expecting a price increase is that the firms' real stock of capital, and hence their earning power, will increase: such expectations are therefore more robust than in situations where agents invest in 'intrinsically worthless pieces of paper'. For in the case of such 'paper bubbles', the expectation of a price increase is the expectation that the *convention* of assigning positive value to this asset will be followed by the next generation, and is not 'backed' by any commensurate increase in earnings.

Example: *Economy with Overaccumulation*

In an economy with overaccumulaton, if at each date firms use a debt policy consisting of repaying the debt used to finance investment in the previous period ($B_t = I_t, \forall\, t \geq 0$), then (33) holds. Since $k_{t+T} \to k_D$ and $r_{t+T} \to r_D < n$ as $T \to \infty$, the limit Q_t^f is $-\infty$. Thus the dividends would be consistently negative, which is not realistic. To avoid calling on shareholders to finance investment, firms must roll over their debt. Let us study the consequence of a financial policy by which firms borrow sufficiently to finance current investment and reimburse previous debt, while distributing zero dividends ($D_t = 0, \forall t \geq 0$). From (27) we obtain

$$\Delta_t = (1+r_t)B_{t-1} - B_t = (\beta + r_t)K_t - I_t = (1+r_t)K_t - K_{t+1}$$

Thus

$$K_{T+1} - B_t = (1+r_t)(K_t - B_{t-1}) = (1+r_1)\dots(1+r_t)(K_1 - B_0) \tag{35}$$

To simplify, suppose the timing is such that the equity of firms is sold on the stock market after investment is made. Then the equity price is

$$Q_t = K_{t+1} - B_t - V_t = (1+r_1)\dots(1+r_t)(K_1 - B_0 - V_0)$$

[11] The absence of dividends at the Golden Rule does not depend on the particular financial policy used by the firms: it can be checked from (29) that at the GR the real dividend is zero, $R_t = 0$. Since debt cannot grow faster that the rate of growth of the population n without being unsustainable, $\Delta_t \geq 0$: if the dividends D_t are required to be non-negative, then $R_t = D_t + \Delta_t = 0$ implies $\Delta_t = D_t = 0$.

which is positive when the initial conditions are such that $K_1 - B_0 - V_0 > 0$. Thus there is a bubble component on equity. Since $Q_t = Q_t^b$ is finite, but $M_t^b = \infty$, the present value at date t of the debt at date $t+T$ must go to infinity as $T \to \infty$. From (35)

$$B_{t+T} = K_{t+T+1} - (1+r_1)\dots(1+r_{t+T})(K_1 - B_0)$$

so that

$$\frac{B_{t+T}}{(1+r_1)\dots(1+r_{t+T})} = \frac{(1+n)^{t+T+1} N_0 K_{t+T+1}}{(1+r_1)\dots(1+r_{t+T})} - (1+r_1)\dots(1+r_t)(K_1 - B_0)$$

which tends to ∞, since $r_{t+T} \to r_D < n$. $B_t^b = \infty$ is possible because the rate of growth of the population is greater than the asymptotic interest rate: the debt can grow faster than the interest rate while staying bounded per capita and is thus sustainable.

Appendix

Proof of Proposition 1: If $V_{t+1}^j < (1-\beta)^2 K_t^j$ then, when $I_t^j \in [0\ \bar{I}]$ with $\bar{I} = V_{t+1}^j - (1-\beta)^2 K_t^j$, $\tilde{Q}_{t+1}^j((1-\beta)K_{t+1}^j) = 0$. On the other hand if $I_t^j \geq \bar{I}$ then $\tilde{Q}_{t+1}^j((1-\beta)K_{t+1}^j) = (1-\beta)K_{t+1}^j - V_{t+1}^j$. Thus the objective function (2), to be maximized with respect to (I_t^j, L_{t+1}^j), that we will denote $\pi(I_t^j, L_{t+1}^j)$, is equal to

$$\pi(I_t^j, L_{t+1}^j) = \begin{cases} -I_t^j + \frac{1}{1+r_{t+1}}\left[F((1-\beta)K_t^j + I_t^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j\right] & \text{if} \quad 0 \leq I_t^j \leq \bar{I} \\ -I_t^j + \frac{1}{1+r_{t+1}}\left[\begin{array}{c} F((1-\beta)K_t^j + I_t^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j \\ +(1-\beta)^2 K_t^j + (1-\beta)I_t^j \end{array}\right] & \text{if} \quad I_t^j \geq \bar{I} \end{cases} \tag{36}$$

When $I_t^j \geq \bar{I}$, substituting $I_t^j = K_{t+1}^j - (1-\beta)K_t^j$, this function can also be written as a function of capital and labor as

$$\pi(I_t^j, L_{t+1}^j) = \frac{1}{1+r_{t+1}}\left[\begin{array}{c} F(K_{t+1}^j, L_{t+1}^j) - w_{t+1}L_{t+1}^j - (\beta + r_{t+1})K_{t+1}^j \\ +(1-\beta)(1+r_{t+1})K_t^j - V_{t+1}^j \end{array}\right] \tag{37}$$

Suppose that there exists a solution (I_t^{j*}, L_{t+1}^{j*}) to maximizing (36) with $I_t^{j*} > 0$. Suppose first that $I_t^{j*} \in (0\ \bar{I}]$. Then (I_t^{j*}, L_{t+1}^{j*}) must satisfy the first-order conditions of maximizing (36) in $[0\ \bar{I}] \times \mathbb{R}_+$, where the constraint $I_t^j \geq 0$ is not binding. This implies that

$$F_K'(K_{t+1}^{j*}, L_{t+1}^{j*}) \geq 1 + r_{t+1}, \qquad F_L'(K_{t+1}^{j*}, L_{t+1}^{j*}) = w_{t+1} \tag{38}$$

with $K_{t+1}^{j*} = (1-\beta)K_t^j + I_t^{j*}$. Consider the associated capital-labor ratio k_{t+1}^{j*} and large values of investment and labor such that $K_{t+1}^j > (1-\beta)K_t^j + \bar{I}$, $K_{t+1}^j / L_{t+1}^j = k_{t+1}^{j*}$. Then (37) evaluated at such pairs gives

$$\pi(I_t^j, L_{t+1}^j) = \frac{L_{t+1}^j}{1+r_{t+1}}\left(f(k_{t+1}^{j*}) - w_{t+1} - (\beta + r_{t+1})k_{t+1}^{j*}\right) + \frac{1}{1+r_{t+1}}\left((1-\beta)(1+r_{t+1})K_t^j - V_{t+1}^j\right)$$

which, when (38) holds, can be made arbitrarily large (since $\beta < 1$). Thus $I_t^{j*} \leq \bar{I}$ cannot be a solution.

Suppose $I_t^{j*} \in (\bar{I}, \infty)$. Then $(K_{t+1}^{j*}, L_{t+1}^{j*})$ must be an interior solution to maximizing (37), so that the first-order conditions

$$F'_K(K_{t+1}^{j*}, L_{t+1}^{j*}) = \beta + r_{t+1}, \qquad F'_L(K_{t+1}^{j*}, L_{t+1}^{j*}) = w_{t+1} \tag{39}$$

must hold and the value of the objective function is $\pi(I_t^{j*}, L_{t+1}^{j*}) = \frac{1}{1+r_{t+1}}((1-\beta)(1+r_{t+1})K_t^j - V_{t+1}^j)$. Since $V_{t+1}^j > (1-\beta)^2 K_t^j$, $\pi(I_t^{j*}, L_{t+1}^{j*}) < \frac{1}{1+r_{t+1}}((1-\beta)(\beta + r_{t+1})K_t^j)$. Let us show that the shareholders would be made better off by not investing. If they do not invest the objective function will be larger or equal to $\pi(0, \tilde{L}_{t+1}^j)$ where $\tilde{L}_{t+1}^j$ is chosen so that $((1-\beta)K_t^j)/\tilde{L}_{t+1}^j = k_{t+1}^{j*}$. When the FOC (39) hold, $\pi(0, \tilde{L}_{t+1}^j) = \frac{1}{1+r_{t+1}}(1-\beta)(\beta + r_{t+1})K_t^j > \pi(I_t^{j*}, L_{t+1}^{j*})$. Thus the problem of maximizing (36) cannot have a solution such that $I_t^{j*} > 0$. □

Proof of Proposition 2: A Diamond steady state is a solution of the equation $S(k)/k = 1 + n$ and it is clear that assumption $\mathcal{S}$ implies that the equation has a unique positive solution k_D. To prove global stability we show (i) ϕ is increasing (ii) $\phi(k) > k$ if $0 < k < k_D$ and (iii) $\phi(k) < k$ if $k > k_D$.

(i) $s(r(k_{t+1}), w(k_t)) = (1+n)k_{t+1} \iff s(r(\phi(k_t)), w(k_t)) = (1+n)\phi(k_t) \implies s'_r r'(k_{t+1})\phi'(k_t) + s'_w w'(k_t) = (1+n)\phi'(k_t) \implies [(1+n) - s'_r f''(k_{t+1})]\,\phi'(k_t) = -s'_w k_t f''(k_t) \implies$

$$\phi'(k_t) = \frac{-s'_w k_t f''(k_t)}{(1+n) - s'_r f''(k_{t+1})} > 0 \tag{40}$$

(ii) Suppose not, $\phi(k) \leq k$; then $r(\phi(k)) \geq r(k)$ and $s(r(\phi(k)), w(k)) \geq s(r(k), w(k)) > (1+n)k \geq (1+n)\phi(k)$ where the first inequality follows from $s'_r \geq 0$ and the second from $S(k)/k > \dfrac{S(k_D)}{k_D} = (1+n)$ since $k < k_D$: but this contradicts (E_D), namely $s(r(\phi(k)), w(k)) = (1+n)\phi(k)$.

(iii) Suppose not, $\phi(k) \geq k$; then $r(\phi(k)) \geq r(k)$ and $s(r(\phi(k)), w(k)) \leq s(r(k), w(k)) < (1+n)k \leq (1+n)\phi(k)$, contradicting (E_D).

To complete the proof, suppose $0 < k_0 > k_D$ (resp. $k_0 < k_D$) then (i)-(iii) imply that k_t is an increasing (decreasing) sequence which is bounded above (below) by k_D: thus $k_t \to k_D$ as $t \to \infty$. □

Proof of Proposition 4: The difference equation system (E'_S) can be written as

$$k_{t+1} = \psi(k_t, v_t)$$

$$v_{t+1} = h(k_t, v_t)$$

where ψ is defined implicitly by the equation

$$(1+n)\psi(k_t,v_t) - s(f'(\psi(k_t,v_t)) - \beta, w(k_t)) - v_t = 0$$

and $h(k_t,v_t) = g\big(\psi(k_t,v_t)\big)v_t$ with $g(x) = \frac{1-\beta+f'(x)}{1+n}$. Thus the linearized system associated with (E''_S) around a steady state $(\bar{k},\bar{v})$, expressed in terms of the deviation variables $(\kappa_t,\nu_t) = (k_t-\bar{k}, v_t-\bar{v})$ is given by

$$\begin{bmatrix}\kappa_{t+1}\\ \nu_{t+1}\end{bmatrix} = \begin{bmatrix}\psi'_k(\bar{k},\bar{v}) & \psi'_v(\bar{k},\bar{v})\\ h'_k(\bar{k},\bar{v}) & h'_v(\bar{k},\bar{v})\end{bmatrix}\begin{bmatrix}\kappa_t\\ \nu_t\end{bmatrix} \tag{L_S}$$

where

$$\psi'_k(\bar{k},\bar{v}) = \frac{-s'_w\bar{k}f''(\bar{k})}{1+n-s'_rf''(\bar{k})}, \qquad \psi'_v(\bar{k},\bar{v}) = \frac{1}{1+n-s'_rf''(\bar{k})}$$

$$h'_k(\bar{k},\bar{v}) = \frac{f''(\bar{k})}{1+n}\left(\frac{-s'_w\bar{k}f''(\bar{k})}{1+n-s'_rf''(\bar{k})}\right)\bar{v}, \qquad h'_v(\bar{k},\bar{v}) = g(\bar{k}) + \frac{f''(\bar{k})\bar{v}}{(1+n)(1+n-s'_rf''(\bar{k}))}$$

Let $\bar{M}$ denote the matrix of coefficients in (L_S) evaluated at $(\bar{k},\bar{v})$, and let $p(\lambda) = \lambda^2 - tr(\bar{M})\lambda + \det\bar{M} = 0$ denote the associated characteristic polynomial. To show that the Golden Rule steady state $(\bar{k},\bar{v}) = (k^*,v^*)$ is locally stable we show that both roots of the characteristic polynomial lie inside the unit circle ($|\lambda_i| < 1, i = 1,2$). Note that $\det M^* = \psi'_k(k^*,v^*) = \dfrac{-s'_wk^*f''(k^*)}{1+n-s'_rf''(k^*)} > 0$ by assumption $\mathcal{C}$. Since there is underaccumulation, $k^* > k_D$ and by assumption $\mathcal{S}, S'(k^*) < \dfrac{S(k_D)}{k_D} = 1+n$. Since $S'(k^*) = s'_rf''(k^*) - s'_wk^*f''(k^*)$, this implies $0 < \det M^* < 1$. Since $\det M^* = \lambda_1\lambda_2$, if both roots are complex, they lie inside the unit circle. The condition $0 < \det M^* < 1$ implies that if both roots are real they lie in the unit interval $(-1,1)$ if and only if $p(1) > 0$ and $p(-1) > 0$. Now

$$p(1) = 1 - trM^* + \det M^* = \frac{-f''(k^*)v^*}{(1+n)(1+n-s'_rf''(k^*))} > 0$$

since $v^* > 0, s'_r \geq 0, f''(k^*) < 0$ and

$$p(-1) = 1 + trM^* + \det M^* = 2 - \frac{2s'_wk^*f''(k^*)}{1+n-s'_rf''(k^*)} + \frac{f''(k^*)v^*}{(1+n)(1+n-s'_rf''(k^*))}$$

The first two terms are positive: to show $p(-1) > 0$ it suffices to show that the third term is bounded below by -1. Since $S(k^*) > 0, v^* < (1+n)k^*$ and since assumption $\mathcal{P}$ implies $k^*f''(k^*) \geq -f'(k^*) = -(\beta+n)$, it follows that

$$\frac{f''(k^*)v^*}{(1+n)(1+n-s'_rf''(k^*))} > \frac{f''(k^*)k^*}{1+n-s'_rf''(k^*)} \geq \frac{-f'(k^*)}{1+n-s'_rf''(k^*)} > -\frac{(\beta+n)}{1+n} > -1$$

Thus both roots lie inside the unit circle and (k^*,v^*) is locally stable.

At the Diamond steady state $(\bar{k}, \bar{v}) = (k_D, 0)$, $h_k'(k_D, 0) = 0$ so that M_D is triangular and $P(\lambda) = (\psi_k'(k_D, 0) - \lambda)\big(h_v'(k_D, 0) - \lambda\big)$. Since $0 < \psi_k'(k_D, 0) = -\dfrac{-s_w' k_D f''(k_D)}{1 + n - s' f''(k_D)} < 1$ where the latter inequality follows from $S'(k_D) < \dfrac{S(k_D)}{k_D} = 1 + n$, and $h_v'(k_D, 0) = g(k_D) = \dfrac{1 + r(k_D)}{1 + n} > 1$, it follows that the Diamond steady state is locally saddlepoint stable. □

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