Equations on the Derivatives of an Initial Allocation-Competitive Equilibrium Mapping for an Exchange Economy

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Abstract

Consider an exchange economy with n traders and k goods. Suppose that there exists C^2 mappings $p(\cdot)$ and $x(\cdot)$ that specify for each initial allocation w a price vector $p(w)$ and a vector of net trades $x(w)$ that define a competitive equilibrium. Three systems of determinant equations are derived that the derivatives of $p(\cdot)$ and $x(\cdot)$ necessarily satisfy regardless of the underlying preferences that determine these mappings. Each of the three systems of equations fundamentally reflects the interaction among traders in the competitive mechanism.

1 Introduction

Consider an exchange economy with n traders and k goods. Let w_i denote the initial allocation of trader i and let $w = (w_i)_{1 \leq i \leq n}$ denote an initial allocation for the economy. Assume that w lies in some open subset W of $\mathbb{R}^{kn}_{++}$. For fixed preferences of the traders, suppose that there exists C^2 mappings $p(\cdot)$ and $x(\cdot)$ that specify for each w in W a price vector $p(w)$ and a vector of net trades $x(w)$ for the n traders that together define a competitive equilibrium. I derive in this paper three systems of determinant equations that the derivatives of $p(\cdot)$ and $x(\cdot)$ necessarily satisfy regardless of the underlying preferences that determine these mappings. The first of the three systems formalizes the invariance of $p(\cdot)$

*I thank Wayne Shafer for bringing to my attention the relationship between some of my work on decentralized mechanisms and a research program of Don Brown and Rosa Matzkin, which led to this paper.

and the particular way in which $x(\cdot)$ changes as w is perturbed in a manner consistent with each trader's budget equation and balance of net trades. The other two systems follow directly from the fact that the pair (p, x) can be realized in a decentralized manner using the competitive mechanism. Each of the three systems fundamentally reflects interaction among traders in the sense that none of the systems can be derived solely from considerations of a single trader's behavior.

Equations are thus derived here concerning competitive market aggregates (prices and net trades) as functions of physical quantities (initial allocations). The paper contributes to the long-standing research program whose objective is to derive testable implications of general equilibrium theory. Most of the research in this area has been negative vis-à-vis this objective in the sense that market aggregates and individual trader behavior have typically been shown to satisfy only the most rudimentary of restrictions once the preferences of the traders are varied. Examples include the Debreu-Mantel-Sonnenschein Theorem,[1] which shows that the excess demand function in an exchange economy is restricted only near the boundary of the price simplex, and the related work of Mas-Colell (1977), who showed that any compact and nonempty subset of the interior of the price simplex is the equilibrium price set of some economy. This work takes the number of traders, their preferences, and their initial endowments as variables that are selected to support a given mapping or set as a market aggregate. The present paper takes the preferences of n traders as fixed in their determination of the mappings $p(\cdot)$ and $x(\cdot)$. Explicit determinant equations on the first and second derivatives of $p(\cdot)$ and $x(\cdot)$ are then derived that are independent of the traders' preferences, i.e., the equations hold for all $p(\cdot)$ and $x(\cdot)$ obtained from the traders' preferences, regardless of what those preferences may be. The preferences of the traders are effectively eliminated in deriving the equations, with the end result being a rather striking set of restrictions on $p(\cdot)$ and $x(\cdot)$.

The research program cited above has recently been revived by Brown and Matzkin (1996), who proved the existence of a set of polynomial equalities and inequalities that m points selected from the competitive price - initial allocation set necessarily satisfy.[2] Besides being the source of my interest in this research program, a connection between this paper and Brown and Matzkin (1996) is that the present paper also concerns the competitive price - initial allocation set for economies in which this set is locally the graph of a C^2 mapping between initial allocations and the competitive price vector. The equations derived on the first and second derivatives of this mapping can be interpreted as restricting the structure of the competitive price - initial allocation set, which is the objective of the Brown-Matzkin paper. The two papers are distinguished in (i) my use of the differential approach, (ii) their investigation of the econometrics of testing their result, and (iii) my inclusion of the equilibrium net trade mapping $x(\cdot)$ as part of the analysis. With regards to point (iii), the equilibrium net trade mapping

[1] The relevant references are Debreu (1974), Mantel (1974), and Sonnenschein (1972, 1973).

[2] Recent work that contributes to this research program includes Chiappori et al. (1999) and Nachbar (2001a, b).

plays a critical role in this paper in identifying the properties of a competitive price mapping $p(\cdot)$ that distinguish it from arbitrary mappings with the same domain and range as $p(\cdot)$.

As suggested above, two of the three systems are derived by considering the realization of the price and net trade mappings through the competitive mechanism, as modeled initially in the work of Stanley Reiter and Leonid Hurwicz. The mathematics of the differential approach to mechanisms that is employed in this paper originated in Hurwicz, Reiter and Saari (1978) with subsequent development in Saari (1984). Most analysis of mechanisms focuses upon the privacy of preference information for a trader and its affect upon what can be accomplished by the group. Here, however, the choice of a trader's initial endowment constitutes his information while his preferences remain fixed. The privacy of his information and the consequent problem of incentives is not addressed in this paper. Instead, the second two systems stem from the observation that each trader can verify that a proposed trade is optimal for him given the price vector and knowing only his own initial endowment and not those of the other traders. The reflects the decentralization of decision-making in the competitive mechanism.

The paper is organized as follows. The model is addressed in the next section. The three systems of equations are then developed in sections 3-5. Section 6 illustrates these systems in an example with $k = 2$ goods and $n = 2$ traders, each of whom has a Cobb-Douglas utility function. This form of utility allows $p(\cdot)$ and $x(\cdot)$ to be calculated explicitly. The Conclusion discusses the independence of these three systems from one another and directions for future work.

2 The Model

Let $w_{i,t}$ denote trader i's initial endowment of good t. Trader i's initial endowment $w_i \equiv (w_{i,t})_{1\leq t\leq k}$ is an element of an open subset $W_i \subset \mathbb{R}^k_{++}$. Let $W \equiv \Pi_{i=1}^n W_i$ denote the set of possible initial endowments for the economy. The price vector p is normalized to lie in the $(k-1)$-dimensional price simplex $\Delta \subset \mathbb{R}^k$.

The preferences of the traders are fixed. They are assumed to be restricted sufficiently so that the following conditions hold.

1. There exists an open set $O \subset Int\ \Delta$ and C^2 mappings

$$p : W \to O \text{ and } x : W \to \mathbb{R}^{kn}$$

   such that the price vector $p(w)$ together with the net trade vector $x(w)$ determine the unique competitive equilibrium given the initial allocation w whose price vector is in O.

2. Each trader i's excess demand $\chi_i : O \times W_i \to \mathbb{R}^k$ is a C^2 mapping that satisfies the regularity condition

$$rank\ D_{w_i}\chi_i(p, w_i) = k - 1 \tag{1}$$

on $O \times W_i$.

It is routine to place regularity conditions on utility functions of the traders so that assumptions 1. and 2. hold near a given $w^* \in \mathbb{R}^{nk}$ by virtue of the Implicit Function Theorem. I therefore do not pursue this topic in this paper.

Assumptions 1. and 2. hold throughout the paper. Three points should be noted concerning these assumptions. First, $p(w) \in Int\ \Delta$ insures that the equilibrium price of each good is positive in this paper. Second, the freedom to restrict prices by properly choosing $O \subset Int\ \Delta$ means that uniqueness of the competitive equilibrium for each choice of the initial allocation is *not* assumed in assumption 1.. Rather, it asserts local uniqueness of the competitive price vector for $w \in W$, i.e., the equilibrium price vector $p(w)$ is isolated. Third, trader i's budget equation implies that $p \cdot D_{w_i}\chi_i(p, w_i) = 0$. Condition (1) is thus a regularity condition in the sense that it requires $D_{w_i}\chi_i(p, w_i)$ to have maximum possible rank on $O \times W_i$.

Let $\mathcal{E}(w) = (p(w), x(w))$ denote the competitive equilibrium mapping. Represent trader i's net trade in $x(w)$ as $x_i(w) \in \mathbb{R}^k$ and let $x_{i,t}(w)$ denote his net trade in good t. Because $p(w)$ and $x(w)$ define a competitive equilibrium, they necessarily satisfy the *balance* equation

$$\sum_{i=1}^{n} x_i(w) = 0, \tag{2}$$

and the *budget* equation

$$p(w) \cdot x_i(w) = 0 \tag{3}$$

for each trader i. These familiar equations are of interest partly for the purpose of determining the number of equations that are derived in this paper that are not simple consequences of these equations that hold by virtue of the definition of a competitive equilibrium.

Let $p = (p_t)_{1 \le t \le k}$ so that p_t represents the price of good t. Define $p_{-k}(w)$ as the mapping

$$p_{-k}(w) = (p_1(w), \ldots, p_{k-1}(w))$$

consisting of all but the kth price. Because $p(w) \in \Delta$, the rows of the $k \times k$ matrix $D_{w_i}p$ sum to zero and hence this matrix has rank at most $k-1$. On occasion, it will be assumed that $p(w)$ satisfies the regularity condition

$$rank\ D_{w_i}p_{-k}(w) = rank\ D_{w_i}p(w) = k-1 \tag{4}$$

for each trader i.

3 The Change in $\mathcal{E}$(w) From a Change in w in the Direction of a Balanced and Feasible Net Trade

Let $\mathcal{E}(w^*) = (p^*, x^*)$. Let $N(p^*)$ denote the set of individually feasible and balanced net trade vectors given the price vector p^*,

$$N(p^*) = \left\{ m = (m_i)_{1 \leq i \leq n} \in \mathbb{R}^{kn} \middle| p^* \cdot x_i = 0 \text{ for each } i, \ \sum_{i=1}^{n} x_i = 0 \right\}.$$

Consider $m \in N(p^*)$ sufficiently small that $w^* + m \in W$. For each trader i, the budget line in $\mathbb{R}^{k+}$ determined by p^* and the initial allocation $w_i^* + m_i$ is the same as the budget line determined by p^* and the initial allocation w_i^*. The bundle $w_i^* + x_i^*$ is therefore optimal for trader i regardless of whether w_i^* is his initial allocation or $w_i^* + m_i$ is his initial allocation. Because m is balanced, the price vector p^* and the net trade vector $x^* - m$ is therefore a competitive equilibrium for the initial allocation $w^* + m$.

Assumption 1. in section 2 asserts that $\mathcal{E}(w^* + m)$ is the unique competitive equilibrium for $w^* + m$ whose price vector is in the set O. Because $p^* \in O$, it follows that

$$x\left(w^* + m\right) = x\left(w^*\right) - m \text{ and} \tag{5}$$

$$p(w^* + m) = p(w^*) = p^* \tag{6}$$

whenever $m \in N(p^*)$ and $w^* + m \in W$.

The system of equations derived in this section is obtained by expressing (5) and (6) in terms of the derivatives of $p(w)$ and $x(w)$, which allows the variable m to be eliminated. Let I^{kn} denote the $kn \times kn$ identity matrix. Because w^* is arbitrary, equations (5) and (6) are equivalent to

$$\begin{aligned} D_w x(w^*) \cdot m = -m \Leftrightarrow \\ \left(D_w x(w^*) + I^{kn}\right) \cdot m = 0 \end{aligned} \tag{7}$$

and

$$D_w p(w^*) \cdot m = 0 \tag{8}$$

holding at each $w^* \in W$ for $m \in N(p^*) = N(p(w^*))$. Let $\varepsilon_{i,t}$ denote the standard basis vector in $\mathbb{R}^{kn}$ corresponding to trader i's initial endowment $w_{i,t}$ of good t. Adding $\varepsilon_{i,t}$ to a vector thus increases the value of $w_{i,t}$ by 1. On the level of individual trader's net trades in particular goods, (7) is equivalent to

$$\left(\nabla x_{q,t}(w^*) + \varepsilon_{q,t}\right) \cdot m = 0 \tag{9}$$

holding for each trader q and each good t, while (8) expressed in terms of individual prices is equivalent to

$$\nabla p_t(w^*) \cdot m = 0 \tag{10}$$

holding for each good t.

Equations (9) and (10) are equivalent to each $\nabla x_{q,t}(w^*)$ and $\nabla p_l(w^*)$ being linearly dependent with row vectors representing the linear system that defines $N(p^*)$. The first step to eliminating the variable m is to represent the linear equations

$$p^* \cdot m_i = \mathit{0}\ \mathit{for\ each}\ i\ \mathit{and}\ \sum_{i=1}^{n} m_i = 0 \tag{11}$$

that define $N(p^*)$ in matrix form. Because $p^* \cdot m_n = 0$ is implied by the other equations in the system (11), the reduced system of $n+k-1$ independent equations

$$\sum_{i=1}^{n} m_i = 0 \tag{12}$$

and

$$p^* \cdot m_i = \mathit{0}\ \mathit{for}\ i < n \tag{13}$$

that defines $N(p^*)$ is considered below.

Let $Z^{t,s}$ be the $t \times s$ zero matrix. For traders $i < n$ let $P_i(w)$ denote the $(n+k-1) \times k$-matrix

$$P_i(w) \equiv \begin{pmatrix} I^k \\ Z^{i-1,k} \\ p(w) \\ Z^{n-i-1,k} \end{pmatrix},$$

and let

$$P_n(w) \equiv \begin{pmatrix} I^k \\ Z^{n-1,k} \end{pmatrix}.$$

The system (12)-(13) is represented in matrix form by the $(n+k-1) \times nk$-matrix

$$P(w^*) \equiv \begin{pmatrix} P_1(w^*) & \cdots & P_n(w^*) \end{pmatrix}.$$

The tth of first k rows represents the equation defined by $\Sigma_{i=1}^{n} m_i = 0$ on the tth good and the $(k+i)$th row represents the budget equation $p^* \cdot m_i = 0$ of trader $i < n$. Because $p^* \neq 0$, it is clear that $P(w^*)$ has rank equal to $n+k-1$. The dependence of the equations (9) and (10) with this system implies the following theorem.

Theorem 1 *The rank conditions*

$$rank\begin{pmatrix} P(w) \\ \nabla p_l\left(w\right) \end{pmatrix} = n+k-1 \text{ and} \tag{14}$$

$$rank\begin{pmatrix} P(w) \\ \nabla x_{q,t}\left(w\right)+\varepsilon_{q,t} \end{pmatrix} = n+k-1 \tag{15}$$

hold for each trader q, any goods t and l and at each $w \in W$.

Each of the rank conditions of (14) and (15) is equivalent to a system of $(n-1)(k-1)$ independent determinant equalities on the submatrices of the matrix in the statement of the condition.[3]

Independent Rank Conditions. The balance equation (2), the normalization of prices to the price simplex Δ, and the n budget equations (3) all imply redundancy in the rank conditions (14)-(15). The following discussion will show that that there is a total of $n(k-1)$ rank conditions determined by (14) and (15) that are mutually independent and independent of these familiar equations on competitive equilibrium. Addressing (14) is straightforward: the normalization of prices to Δ implies that (14) holds for good k if it holds for goods $l < k$, which means that (14) determines $k-1$ independent rank conditions.

Turning to (15), the following argument shows that this rank condition holds in the case of trader $q = n$ if it holds for traders $q < n$. The first line below follows from the balance equation (2):

$$rank\begin{pmatrix} P(w^*) \\ \nabla x_{n,t}\left(w^*\right)+\varepsilon_{n,t} \end{pmatrix} = rank\begin{pmatrix} P(w^*) \\ -\sum_{q<n}\nabla x_{q,t}\left(w^*\right)+\varepsilon_{n,t} \end{pmatrix} \tag{16}$$

$$= rank\begin{pmatrix} P(w^*) \\ \sum_{q=1}^{n}\varepsilon_{q,t} \end{pmatrix}. \tag{17}$$

If (15) holds for traders $q < n$, then row operations on the second matrix in (16) implies that each $-\nabla x_{q,t}\left(w^*\right)$ can be replaced with $\varepsilon_{q,t}$, which implies the second equality above. The last row of the matrix (17) is the same as

[3] Each rank condition given by (14) and (15) requires that every $(n+k)\times(n+k)$ submatrix of the given $(n+k)\times nk$ matrix A has determinant equal to zero. This can be reduced to a system of independent equations by interchange the ordering of columns in A so that its first $n+k-1$ columns are linearly independent. Successively place each of the $nk-(n+k-1)=(n-1)(k-1)$ remaining columns next to this block of $n+k-1$ independent columns and then take the determinant. Each of these $(n-1)(k-1)$ determinants equaling zero is necessary and sufficient for the matrix A to have rank equal to $n+k-1$.

its tth row, which establishes the desired result. Trader q's budget equation $p(w) \cdot x_q(w) = 0$ implies

$$\sum_{t=1}^{k} p_t \nabla x_{q,t} + \sum_{t=1}^{k} x_{q,t} \nabla p_t = 0. \qquad (18)$$

A similar argument using (18) shows that (15) holds for good $t = k$ if (i) it holds for goods $t < k$ and (ii) the rank condition (14) holds for each price function $p_l(w)$. By varying q and t, there are thus a total of $(n-1)(k-1)$ rank conditions defined by (15) that are mutually independent and also independent of the $k-1$ rank conditions given by (14).

This implies the total of $n(k-1)$ independent rank conditions from (14) and (15) that is stated above. As noted immediately after Theorem 1, each rank condition is equivalent to $(n-1)(k-1)$ independent equations on $p(w)$ and $x(w)$. A total of $n(n-1)(k-1)^2$ independent equations are thus imposed on these mappings by Theorem 1.

4 Dependence of Net Trades and Prices

The competitive mechanism operates by having each trader select his own net trade given a price vector p according to his excess demand function. The mechanism is decentralized in the sense that each trader makes his decision knowing his own preferences and initial endowment but without knowledge of the preferences and endowments of others. Given that preferences are fixed in this paper, decentralization is reflected in the independence of trader i's excess demand $\chi_i(p, w_i)$ of the initial endowments of the other traders. The second and third systems of equations are derived by considering the level sets of these excess demand functions and their relationship to the level sets of the competitive equilibrium mapping $\mathcal{E}(w)$. The decentralized nature of the competitive mechanism is reflected in the particular geometric structure of these level sets, from which the second and third systems originate.

Again let $\mathcal{E}(w^*) = (p^*, x^*)$. Define $S_i(w^*)$ as the level set of trader i's excess demand $\chi_i(\cdot)$ through w_i^* taking p^* as fixed,

$$S_i(w^*) = \{\, w_i | \, \chi_i(p^*, w_i) = x_i^* \} \,. \qquad (19)$$

The regularity condition (1) implies that $S_i(w^*)$ is a 1-dimensional C^2 submanifold of W_i. The dependence of $S_i(w^*)$ upon w_{-i}^* is through the price vector $p^* = p(w^*)$. Notice that

$$\begin{aligned} \mathcal{E}^{-1}(p^*, x^*) &= \{w \,| \chi_i(p^*, w_i) = x_i^*, \ 1 \le i \le n\} \\ &= \{w \,| w_i \in S_i(w^*), \ 1 \le i \le n\} \\ &= \prod_{i=1}^{n} S_i(w^*). \end{aligned} \qquad (20)$$

The remaining two systems originate in this equation. It is clear from (20) that $\mathcal{E}(\cdot, w^*_{-i})$ is constant with value (p^*, x^*) for $w_i \in S_i(w^*)$. The second system of this paper is derived below from this observation. The third system is derived in the next section from the Cartesian product structure of the level sets of $\mathcal{E}(\cdot)$ that is expressed in (20).

The invariance of the equilibrium mapping $\mathcal{E}(\cdot, w^*_{-i})$ on $S_i(w^*)$ implies that the k functions

$$p_1(\cdot, w^*_{-i}), \ldots, p_{k-1}(\cdot, w^*_{-i}), \text{ and } x_{q,t}(\cdot, w^*_{-i}) \tag{21}$$

are constant on each of the submanifolds $S_i(w^*)$ of W_i. The submanifold $S_i(w^*)$ has codimension $k-1$ in W_i, and hence the derivatives of the k functions in (21) with respect to w_i must be dependent. This observation leads to the second system of equations.

Theorem 2 *For $1 \leq q, i \leq n$ and $1 \leq t \leq k$, trader q's equilibrium net trade $x_{q,t}(w)$ in good t satisfies the equation*

$$\det \begin{pmatrix} D_{w_i} p_{-k} \\ D_{w_i} x_{q,t} \end{pmatrix} = 0 \tag{22}$$

on W.

Proof. Equation (22) clearly holds at $w^* \in W$ where the rank condition (4) on $p_{-k}(\cdot)$ is violated, and so (4) is assumed in the remainder of this proof. The solution set of the $k-1$ equations

$$p_1(\cdot, w^*_{-i}) = p^*_1, \ldots, p_{k-1}(\cdot, w^*_{-i}) = p^*_{k-1} \tag{23}$$

is a 1-dimensional submanifold through w^*_i that contains the 1-dimensional submanifold $S_i(w^*)$. These submanifolds must be identical near w^*_i, and so $S_i(w^*)$ is the solution of the $k-1$ equations in (23) near w^*. The additional equation $x_{q,t}\left(\cdot, w^*_{-i}\right) = x^*_{q,t}$ does not restrict this solution set further, and so it follows that (22) holds at w^*. ■

The system defined by (22) is not simply an alternative expression of trader q's budget equation $p(w) \cdot x_q(w) = 0$. Differentiating the budget equation with respect to w_i by the product rule implies

$$x_q \cdot D_{w_i} p + p \cdot D_{w_i} x_q = 0 \Leftrightarrow$$

$$x_q \cdot D_{w_i} p = -p \cdot D_{w_i} x_q. \tag{24}$$

Equation (24) states that a particular linear combination of the rows of $D_{w_i} p$ equals a particular linear combination of the rows of $D_{w_i} x_q$. The kth row $D_{w_i} p_k$ of $D_{w_i} p$ can be omitted from this statement because $D_{w_i} p_k = -\Sigma_{s \neq k} D_{w_i} p_s$. Statement (24) thus implies that a particular linear combination of the rows of

$D_{w_i}p_{-k}$ equals a particular linear combination of the rows of $D_{w_i}x_q$. Equation (22), however, states that each individual row $D_{w_i}x_{q,t}$ of $D_{w_i}x_q$ is a linear combination of the rows of $D_{w_i}p_{-k}$, which is not implied by (24). The system (22) thus binds in way that the budget equation of trader q does not.

Equation (24) does imply, however, that the kth row $D_{w_i}x_{q,k}$ of $D_{w_i}x_q$ is a linear combination of $D_{w_i}p_1, \ldots, D_{w_i}p_{k-1}$ and $D_{w_i}x_{q,1}, \ldots, D_{w_i}x_{q,k-1}$. Equation (24) together with equation (22) holding for goods $1 \le t \le k-1$ therefore insure that (22) holds for good k. For each $1 \le i, q \le n$, there are thus $k-1$ equations defined by (22) that are independent of the budget equation of trader q. Because $x_{n,t} = -\Sigma_{q<n} x_{q,t}$ for each good t, it is also the case that (22) holding for traders $q < n$ implies that is also satisfied by $q = n$. By varying $1 \le i \le n$, $1 \le q < n$, and $1 \le t < k$, the system (22) therefore imposes $n(n-1)(k-1)$ equations on $p(w)$ and $x(w)$ that are independent of the system of n budget equations (3) and the balance equation (2).

5 Integrability Equations on Prices

Equation (20) states that each level set of the competitive equilibrium mapping $\mathcal{E}(w)$ is an n-dimensional C^2 submanifold of W that is a Cartesian product of n 1-dimensional C^2 submanifolds, the ith of which lies within W_i. This fundamentally reflects the fact that $\mathcal{E}(w)$ is realized by a decentralized mechanism. The equations that are derived below reflect this special geometric structure of the level sets of $\mathcal{E}(w)$.

The traders' excess demand functions define n C^2 foliations[4] of the space W of initial endowments,

$$\mathcal{S}_i = \{S_i(w) \times \{w_{-i}\} \,|w \in W\}, \; 1 \le i \le n,$$

where $S_i(w)$ is defined by (19). Let $\mathcal{S}$ denote the foliation of W defined by the level sets of $\mathcal{E}(w)$,

$$\mathcal{S} = \left\{\mathcal{E}^{-1}(p(w), x(w)) \,|w \in W\right\}.$$

The system of equations derived in this section states differential conditions on $p(\cdot)$ for the foliations $\mathcal{S}_1, \ldots, \mathcal{S}_n$ and $\mathcal{S}$ to satisfy (20). The mathematical theory behind this derivation originates in Hurwicz, Reiter and Saari (1978, Ch. IV) and Saari (1984) as part of their effort to develop a differential approach to decentralization.[5]

For any trader i and any C^1 function $f : W \to \mathbb{R}$, let $d_i f$ denote the 1-form

$$d_i f = \sum_{t=1}^{k} \frac{\partial f}{\partial w_{i,t}} dw_{i,t}.$$

[4] A C^2 *foliation* of an open set $Z \subset \mathbb{R}^u$ is a partition $\{T(z) \,|z \in Z\}$ of Z such that (i) each set $T(z)$ is a C^2 submanifold of Z, and (ii) the submanifolds in the partition vary smoothly in the sense that there exists a set of C^1 vectors fields on Z that span the tangent space to $T(z)$ at every $z \in Z$.

[5] See also Williams (2002) for a more comprehensive development of this theory.

Assume for the moment that $p(\cdot)$ satisfies the regularity condition (4) of section 2. The foliation $\mathcal{S}_i$ of W determines a differential ideal I_i that is generated by the 1-forms in the set

$$\{d_i p_l \,|1 \leq l \leq k-1\} \cup \{dw_{j,t} \,|1 \leq j \neq i \leq n, 1 \leq t \leq k\}. \tag{25}$$

The set (25) generates I_i because: (i) $p_l(\cdot)$ is constant on each $S_i(w) \times \{w_{-i}\}$ and so $d_i p_l \in I_i$; (ii) $S_i(w) \subset W_i$ and so $dw_{j,t} \in I_i$; (iii) the submanifolds in $\mathcal{S}_i$ are of codimension $(k-1)+(n-1)k$ in W, which is the number of linearly independent 1-forms in (25). A similar argument using (20) implies that the ideal I that is dual to the foliation $\mathcal{S}$ is generated by the $n(k-1)$ 1-forms[6]

$$\bigcup_{i=1}^{n} \{d_i p_l \,|1 \leq l \leq k-1\}. \tag{26}$$

Because I defines a foliation $\mathcal{S}$ of W, the Frobenius Theorem[7] implies that the differential $d(d_i p_l)$ of each of the 1-forms $d_i p_l$ in the set (26) lies in the set of 2-forms generated by the set (26). This defines a set of integrability conditions on the 1-forms in (26). It requires that there exists for each $d_i p_l$ C^0 1-forms

$$\left\{\rho_{j,t} \,|1 \leq j \leq n, 1 \leq t \leq k-1\right\}$$

such that

$$d\left(d_i p_l\right) = \sum_{j=1}^{n} \sum_{s=1}^{k-1} \rho_{j,s} \wedge d_j p_s. \tag{27}$$

The remainder of this section is devoted to eliminating the variables $\rho_{j,s}$ from (27) to obtain equations directly upon $p(w)$.

Some notation concerning the vector spaces of 1- and 2-forms on W is needed. Let Ψ denote the space of C^0 1-forms on W. For each trader j, let Ψ_j denote the subspace of Ψ generated by the 1-forms in the set

$$\{dw_{j,v} \,|1 \leq v \leq k\}.$$

It is clear that $\Psi = \oplus_{j=1}^{n} \Psi_j$. Let Γ denote the space of all C^0 2-forms on W. For $1 \leq i \neq j \leq n$, let $\Gamma_{i,j}$ denote the space of 2-forms on W spanned by the set

$$\{dw_{i,u} \wedge dw_{j,v} \,|1 \leq u, v \leq k\}.$$

It is clear that

$$\Gamma = \oplus_{1 \leq r \leq q \leq n} \Gamma_{q,r}. \tag{28}$$

[6]This is shown in Hurwicz, Reiter and Saari (1978, Thm. 7) and Saari (1984, Theorem 2.2), in each case as part of a general theory of mechanism construction.

[7]A statement of the Frobenius Theorem in terms of differential ideals can be found in Spivak (1979, Prop. 14, p. 293) or Warner (1971, Prop. 2.30, p. 74).

Applying (28), equation (27) is now rewritten as a set of $n-1$ equations by (i) replacing each side of (27) with its unique representation as a sum of 2-forms in the spaces $\Gamma_{i,j}$ for $j \neq i$ and (ii) equating the elements of $\Gamma_{i,j}$ from each side of the equation. The 2-form $d(d_i p_l)$ expands as

$$d\left(d_i p_l\right) = \sum_{j \neq i} \sigma_{i,j}$$

where

$$\sigma_{i,j} = \sum_{1 \leq t,u \leq k} \frac{\partial^2 p_l}{\partial w_{j,u} \partial w_{i,t}} dw_{j,u} \wedge dw_{i,t} \in \Gamma_{i,j}.$$

Consequently, the right-side of (27) must be expressible as a sum of elements of $\Gamma_{i,j}$ for $j \neq i$, i.e., all 2-forms on the right side of (27) that are not in $\oplus_{j \neq i} \Gamma_{i,j}$ sum to zero. Equation (27) can thus be replaced with the equation

$$\sum_{j \neq i} \sigma_{i,j} = \sum_{j \neq i} \left(\sum_{s=1}^{k-1} \tau_{j,s}^{i} \wedge d_j p_s + \tau_{i,s}^{j} \wedge d_i p_s \right), \tag{29}$$

where each $\tau_{r,s}^{q} \in \Psi_q$. The term in parentheses collects all the elements on the right side of (29) that lie in $\Gamma_{i,j}$, and so (27) holds if and only if

$$\sigma_{i,j} = \sum_{s=1}^{k-1} \tau_{j,s}^{i} \wedge d_j p_s + \tau_{i,s}^{j} \wedge d_i p_s \tag{30}$$

for each $j \neq i$.

The variables $\tau_{r,s}^{q}$ are now eliminated from (30) to obtain conditions directly on $p(w)$. For any good l and for distinct traders i and j, let $D^2_{w_i,w_j} p_l$ denote the $k \times k$ matrix of mixed partials

$$D^2_{w_i,w_j} p_l = \left(\frac{\partial^2 p_l}{\partial w_{i,t} \partial w_{j,s}} \right)_{1 \leq t,s \leq k}.$$

Let Z^t denote the $t \times t$ zero matrix for any $t \in \mathbb{N}$. The equations on $p(w)$ concern the *bordered mixed Hessian* $BMH_{i,j}(p, p_l)$, which is the $(2k-1) \times (2k-1)$ matrix[8]

$$BMH_{i,j}(p, p_l) = \begin{pmatrix} Z^{k-1} & D_{w_j} p_{-k} \\ \left(D_{w_i} p_{-k}\right)^T & D^2_{w_i,w_j} p_l \end{pmatrix}. \tag{31}$$

[8]The bordered Hessian form of the equations in the following theorem are inspired by the bounds on minimal message space dimension derived in Hurwicz (1979), Williams (1984), and Chen (1992).

Theorem 3 *For any good $l < k$ and for each pair of distinct traders i and j, the competitive equilibrium price function $p(w)$ satisfies the equation*

$$\det BMH_{i,j}(p, p_l) = 0, \tag{32}$$

where $BMH_{i,j}(p, p_l)$ is defined in (31).

Interchanging the role of i and j in (32) does not change this equation, and so each pair of distinct traders i and j determines exactly one equation for each good l. By varying i, j and l, (32) thus determines a system of $(k-1)n(n-1)/2$ independent equations on the first and second derivatives of $p(w)$, with the $n(n-1)/2$ term counting the number of sets $\{i, j\}$ with $i \neq j$.

Though it is used in the analysis that precedes the statement of Theorem 3, the regularity condition (4) on $p(\cdot)$ is not required as a hypothesis in this theorem because equation (32) holds trivially if either $D_{w_i} p_{-k}$ or $D_{w_j} p_{-k}$ has rank less than $k-1$. Condition (4) is thus assumed without loss of generality in the proof below, which justifies the use of equations (27) and (30) in the proof.

Proof. Row operations on $BMH_{i,j}(p, p_l)$ involving the rows of $D_{w_j} p_{-k}$ and $D^2_{w_i, w_j} p_l$ correspond to reducing $\sigma_{i,j}$ using the 2-forms in the set

$$\{dw_{i,t} \wedge d_j p_s \,|1 \leq s \leq k-1, 1 \leq t \leq k\},$$

while column operations involving the columns of $(D_{w_i} p_{-k})^T$ and $D^2_{w_i, w_j} p_l$ correspond to reducing $\sigma_{i,j}$ using the 2-forms in the set

$$\{dw_{j,t} \wedge d_i p_s \,|1 \leq s \leq k-1, 1 \leq t \leq k\}.$$

Subtracting the term $\tau^i_{j,s} \wedge d_j p_s$ in (30) from $\sigma_{i,j}$ is thus equivalent to a sequence of row operations on $BMH_{i,j}(p, p_l)$ involving $D^2_{w_i, w_j} p_l$ and row s of $D_{w_j} p_{-k}$ while subtracting the term $\tau^j_{i,s} \wedge d_i p_s$ from $\sigma_{i,j}$ is equivalent to a sequence of column operations on $BMH_{i,j}(p, p_l)$ involving $D^2_{w_i, w_j} p_l$ and column s of $(D_{w_i} p_{-k})^T$. Equation (30) therefore implies that $BMH_{i,j}(p, p_l)$ can be reduced through row and column operations to

$$\begin{pmatrix} Z^{k-1} & D_{w_j} p_{-k} \\ (D_{w_i} p_{-k})^T & Z^k \end{pmatrix}.$$

Given the regularity condition (4), this is equivalent to

$$rank \;\; BMH_{i,j}(p, p_l) = 2k - 2,$$

which implies (32). ■

Alternatives to (32) can be stated in terms of the first and second derivatives of $x(w)$ and $p(w)$ by replacing any of the sets

$$\{d_i p_l \,|1 \leq l \leq k-1\} \tag{33}$$

in (26) with a selection of $k-1$ linearly independent 1-forms selected from the set

$$\{d_i p_l \,|1 \le l \le k\} \cup \{d_i x_{q,t} \,|1 \le q \le n, 1 \le t \le k\}.$$

If the regularity condition (4) holds, however, equation (32) holding for all $1 \le i \ne j \le n$ and $1 \le l \le k-1$ is sufficient to insure that the ideal I is differential, while the system (22) insures that each 1-form $d_i w_{q,t}$ is linearly dependent with respect to the set (33). All alternatives to (32) that may be derived in this way are thus redundant to the systems (25) and (22) when (4) holds for each trader i.

6 Example: k=2 Goods, n=2 Traders with Cobb-Douglas Utility Functions

Preferences for the traders are not specified in this paper and they do not appear in the three systems of equations that are presented in Theorems 1-3. They are effectively variables that are implicit in the equilibrium mapping $\mathcal{E}(w)$ and in the traders' excess demand functions but that are eliminated in deriving the three systems. It is instructive, however, to verify that each of the three systems holds in the case of particular utility functions for which $p\,(w)$ can be explicitly computed. This is accomplished here using a Cobb-Douglas utility function for each trader. Rather than normalizing prices to Δ, it is assumed in this example that good 2 is a numèraire. The equilibrium price of good 1 is denoted simply as $p(w)$ below instead of $p_1(w)$.

Letting $y_i = (y_{i,1}, y_{i,2})$ represent an allocation for trader i, the utility functions of traders 1 and 2 are

$$u_1\,(y_{1,1}, y_{1,2}) = y_{1,1}^{\delta} y_{1,2}^{1-\delta} \textit{ and}$$

$$u_2\,(y_{2,1}, y_{2,2}) = y_{1,1}^{\gamma} y_{1,2}^{1-\gamma},$$

respectively, for fixed values of δ, γ in $(0,1)$. A property of this family of utility functions is that the exponent of $y_{i,1}$ is the proportion of trader i's income that he spends on good 1. Consequently,

$$py_{1,1} = \delta\,(pw_{1,1} + w_{1,2}) \textit{ and} \tag{34}$$

$$py_{2,1} = \gamma\,(pw_{2,1} + w_{2,2}),$$

where $y_{i,1}$ now represents trader i's demand for good 1 at the price p. Summing these two equations produces

$$p\,(y_{1,1} + y_{2,1}) = p\,(\delta w_{1,1} + \gamma w_{2,1}) + (\delta w_{1,2} + \partial w_{2,2})\,. \tag{35}$$

Feasibility of competitive equilibrium requires that $y_{1,1} + y_{2,1} = w_{1,1} + w_{2,1}$. Substitution into (35) and simplifying implies

$$p\left[(1-\delta)\, w_{1,1} + (1-\gamma)\, w_{2,1}\right] = \delta w_{1,2} + \gamma w_{2,2},$$

and so

$$p\,(w) = \frac{\delta w_{1,2} + \gamma w_{2,2}}{(1-\delta)\, w_{1,1} + (1-\gamma)\, w_{2,1}}. \tag{36}$$

Trader 1's equilibrium net trade of good 1 is also needed below. From (34) it is clear that

$$\begin{aligned} x_{1,1}\,(w) &\equiv y_{1,1} - w_{1,1} \\ &= \frac{\delta\,(p\,(w)\, w_{1,1} + w_{1,2})}{p\,(w)} - w_{1,1} \\ &= -\,(1-\delta)\, w_{1,1} + \frac{\delta w_{1,2}}{p\,(w)}. \end{aligned} \tag{37}$$

The Rank Conditions (14) and (15) of Theorem 1. The matrix in the rank condition (14) in this case is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ p(w) & 1 & 0 & 0 \\ \frac{\partial p}{\partial w_{1,1}} & \frac{\partial p}{\partial w_{1,2}} & \frac{\partial p}{\partial w_{2,1}} & \frac{\partial p}{\partial w_{2,2}} \end{pmatrix}. \tag{38}$$

The 3×3 submatrix of (38) defined by its first three rows and columns has nonzero determinant and so (38) has rank at least 3. To confirm (14), it is therefore sufficient to show that the determinant of (38) is zero. Expanding along its first row, this determinant equals

$$\left(\frac{\partial p}{\partial w_{2,1}}\right) + \left(-p(w)\frac{\partial p}{\partial w_{2,2}} + p(w)\frac{\partial p}{\partial w_{1,2}} - \frac{\partial p}{\partial w_{1,1}}\right).$$

This reduces to

$$\frac{-\,(1-\gamma)\, p(w) - \gamma p(w) + \delta p\,(w) + (1-\delta)\, p\,(w)}{(1-\delta)\, w_{1,1} + (1-\gamma)\, w_{2,1}} = 0$$

for $p(w)$ defined in (36), which completes the verification.

As discussed after Theorem 1, the rank condition (15) by varying the trader q and the good t defines $(n-1)(k-1)$ independent rank conditions. Condition (15) is thus verified in this example by considering $q = k = 1$. Applying formula

(37) for $x_{1,1}(w)$, the matrix in (15) in this case is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ p(w) & 1 & 0 & 0 \\ \frac{\partial x_{1,1}}{\partial w_{1,1}}+1 & \frac{\partial x_{1,1}}{\partial w_{1,2}} & \frac{\partial x_{1,1}}{\partial w_{2,1}} & \frac{\partial x_{1,1}}{\partial w_{2,2}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ p(w) & 1 & 0 & 0 \\ \delta - \frac{\delta w_{1,2}}{p(w)^2}\frac{\partial p}{\partial w_{1,1}} & \frac{\delta}{p(w)} - \frac{\delta w_{1,2}}{p(w)^2}\frac{\partial p}{\partial w_{1,2}} & -\frac{\delta w_{1,2}}{p(w)^2}\frac{\partial p}{\partial w_{2,1}} & -\frac{\delta w_{1,2}}{p(w)^2}\frac{\partial p}{\partial w_{2,2}} \end{pmatrix}.$$

Multiplying row 4 by $-p(w)^2/\delta w_{1,2}$ and then adding $p(w)/w_{1,2}$ times row 3 to row 4 reduces this last matrix to the matrix (38), which is shown above to have rank equal to 3. This completes the verification of (15).

Dependence Between the Derivatives of Prices and Net Trades as Expressed in Theorem 2. Applying the budget equations of the two traders together with the balance equation $x_1(w) + x_2(w) = 0$, the system defined by (22) in Theorem 2 reduces to the two equations

$$\det \begin{pmatrix} D_{w_i} p \\ D_{w_i} x_{1,1} \end{pmatrix} = 0 \tag{39}$$

defined by $i = 1, 2$.

Starting first with the case of $i = 2$, substitution of (37) into the matrix in (39) produces

$$\begin{pmatrix} \frac{\partial p}{\partial w_{2,1}} & \frac{\partial p}{\partial w_{2,2}} \\ -\frac{\delta w_{1,2}}{p(w)^2}\frac{\partial p}{\partial w_{2,1}} & -\frac{\delta w_{1,2}}{p(w)^2}\frac{\partial p}{\partial w_{2,2}} \end{pmatrix},$$

and so (39) obviously holds. In the case of $i = 1$, the matrix in (39) is

$$\begin{pmatrix} \frac{\partial p}{\partial w_{1,1}} & \frac{\partial p}{\partial w_{1,2}} \\ -(1-\delta) - \frac{\delta w_{1,2}}{p(w)^2}\frac{\partial p}{\partial w_{1,1}} & \frac{\delta}{p(w)} - \frac{\delta w_{1,2}}{p(w)^2}\frac{\partial p}{\partial w_{1,2}} \end{pmatrix},$$

which reduces to

$$\begin{pmatrix} \frac{\partial p}{\partial w_{1,1}} & \frac{\partial p}{\partial w_{1,2}} \\ -(1-\delta) & \frac{\delta}{p(w)} \end{pmatrix}$$

after a row operation. Applying formula (36) for $p(w)$, this last matrix has determinant equal to zero, which completes the verification of (22) in this example.

The Integrability Equation (32) of Theorem 3. For notational convenience, define

$$B(w_{1,1}, w_{2,1}) \equiv (1-\delta)\, w_{1,1} + (1-\gamma)\, w_{2,1}.$$

The matrix in (32) equals

$$\frac{1}{B(w_{1,1}, w_{2,1})^2} \cdot \begin{pmatrix} 0 & -(1-\gamma)(\delta w_{1,2} + \gamma w_{2,2}) & \gamma B(w_{1,1}, w_{2,1}) \\ -(1-\delta)(\delta w_{1,2} + \gamma w_{2,2}) & 2(1-\delta)(1-\gamma)\, p(w) & -\gamma (1-\delta) \\ \delta B(w_{1,1}, w_{2,1}) & -\delta (1-\gamma) & 0 \end{pmatrix}.$$

Consider the following sequence of operations on the above matrix: (i) multiply column 3 by $1/\gamma$; multiply column 2 by $1/(1-\gamma)$; multiply row 2 by $1/(1-\delta)$; multiply row 3 by $1/\delta$; add $1/B(w_{1,1}, w_{2,1})$ times column 1 to column 2; add $1/B(w_{1,1}, w_{2,1})$ times row 1 to row 2. This sequence of row and column operations reduces this matrix to

$$\begin{pmatrix} 0 & -(\delta w_{1,2} + \gamma w_{2,2}) & (1-\delta)\, w_{1,1} + (1-\gamma)\, w_{2,1} \\ -(\delta w_{1,2} + \gamma w_{2,2}) & 0 & 0 \\ (1-\delta)\, w_{1,1} + (1-\gamma)\, w_{2,1} & 0 & 0 \end{pmatrix}.$$

This matrix clearly has rank equal to 2, and so the price vector $p(w)$ satisfies (32).

7 Conclusion

The three systems of equations derived in this paper describe three distinct aspects of the level sets $S_1(w), \ldots, S_n(w)$ of the traders' excess demand functions:

1. *(Theorem 1)* For a sufficiently small net trade vector m that is balanced and budgetarily feasible for each trader relative to the price vector $p(w)$, the equation

   $$S_i(w+m) = m_i + S_i(w)$$

   holds for each trader i in a neighborhood of w_i.

2. *(Theorem 2)* The C^2 submanifold $S_i(w)$ is of codimension $k-1$ in W_i, and so the gradients of any k functions that are constant on $S_i(w)$ are linearly dependent.

3. *(Theorem 3)* The Cartesian products of the form $\Pi_{i=1}^{n} S_i(w)$ define a foliation of W.

These three features of the level sets of excess demand functions reveal that the three systems of equations that represent these features are mutually independent of one another. Summing across the three theorems, a total of $n(n-1)(k-1)(k+1/2)$ equations on the derivatives of $p(w)$ and $x(w)$ have been derived that are mutually independent and independent of the n budget equations, the balance equation on net trades, and the normalization of prices to the price simplex.

The three systems of equations on $p(w)$ and $x(w)$ pose at least three questions for future research. First, are these equations empirically testable? This question ties this paper back to the research program discussed in the Introduction. Second, what are the discrete analogues of these three systems on the values of $p(w)$ and $x(w)$? Answering this second question may facilitate empirical testing of the work. Third, what additional conditions beyond these equations must a pair of mappings $p(w)$ and $x(w)$ satisfy if they are to represent the equilibrium price and net trade mappings for some choice of the preferences for the n traders? This last question seeks sufficient conditions to insure that mappings $p(w)$ and $x(w)$ represent particular market aggregates. The results of this paper are opposite in theme from the Debreu-Mantel-Sonnenschein Theorem in that they identify structure in two market aggregates. This last question, however, shares the motivation of this famed result in the sense that both address sufficient conditions under which a given mapping represents a particular market aggregate.

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