Ph.D. Preliminary Examination

MICROECONOMIC THEORY

MAJORS

Fall 2009

The time limit for this exam is 3$\frac{1}{4}$ hours.

**Answer one question from each part, for a total of four questions.**

You may use calculators to make calculations during the examination. However: in answering any question that requires you to justify your answer, if you do use a calculator, you must mention at what point in your answer you obtained results using a calculator, and what it was you were seeking to calculate.

BE SURE you clearly define all **boldfaced/underlined** terms. Also, please be sure to define precisely any notation that you introduce.

NOTE: This examination should have 14 pages including this one (Check to make sure!)
Part I

Answer one question from Part I.
Question I.1

Consider a profit maximizing firm with single output and \( n \) inputs, with production function \( f : \mathbb{R}_+^n \to \mathbb{R}_+ \) assumed strictly increasing, continuous (but possibly nondifferentiable), and \( f(0) = 0 \). Let \( q \in \mathbb{R}_{++} \) be the price of output and \( w \in \mathbb{R}_+^n \) be the vector of prices of inputs. The firm’s profit maximization problem is

\[
\max_{x \geq 0} [qf(x) - wx].
\]

Let \( x^*(q) \) denote the profit maximizing vector of inputs (assumed unique) as function of output price \( q \).

(a) State a definition of production function \( f \) being supermodular. Show that the Cobb-Douglas production function \( f(x) = x_1^{\alpha_1}x_2^{\alpha_2} \ldots x_n^{\alpha_n} \), where \( \alpha_i > 0 \) for all \( i \), and \( \sum_{i=1}^n \alpha_i < 1 \), is supermodular.

(b) Show that if \( f \) is supermodular, then input demand \( x^* \) is a nondecreasing function of \( q \). If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.
Question I.2

Consider an agent whose preferences over real-valued random variables (or state-contingent consumption plans) are represented by an expected utility function with strictly increasing and twice-differentiable von Neumann-Morgenstern (or Bernoulli) utility $v: \mathbb{R} \rightarrow \mathbb{R}$. Let $\rho(w, \tilde{z})$ denote the risk compensation for random variable $\tilde{z}$ with $E(\tilde{z}) = 0$ at risk-free initial wealth $w$. Let $A(w)$ denote the Arrow-Pratt measure of risk aversion at $w$.

(a) Prove that $A$ is an increasing function of $w$ if and only if risk compensation $\rho$ is an increasing function of $w$ for every $\tilde{z}$ with $E(\tilde{z}) = 0$ and $\tilde{z} \neq 0$.

(b) Derive an explicit expression for risk compensation for quadratic utility $v(x) = -(\alpha - x)^2$, where $\alpha > 0$. Prove that this quadratic utility is, up to an increasing linear transformation, the only utility function with risk compensation of the form you derived.

If you use the Theorem of Pratt, you need to state it clearly, but you are not asked to prove it.
Part II

Answer one question from Part II
Let $\mathcal{E}$ be a pure exchange economy with two commodities and $n$ traders ($i = 1, \ldots, n$) each specified by a consumption set $\mathbb{R}^2_+$, an initial endowment $e_i \in \mathbb{R}^2_+$ and a preference relation $\preceq_i$ assumed to be a complete continuous preorder on $\mathbb{R}^2_+$.

(a) Define **continuity** of a complete preorder $\preceq$ on $\mathbb{R}^\ell_+$.

(b) Define **completeness** of a preorder $\preceq$ on $\mathbb{R}^2_+$.

(c) Define **competitive equilibrium** in $\mathcal{E}$.

(d) Define **weak Pareto optimality** and **strong Pareto optimality** of an allocation in $\mathcal{E}$.

(e) In $\mathcal{E}$, is it true that every competitive equilibrium allocation is weakly Pareto optimal?

(f) If you answer to part (e) is yes, prove your result. If your answer to part (e) is no,

   (i) provide a counterexample and

   (ii) indicate where the standard proof of the welfare theorem would go wrong for $\mathcal{E}$.

(g) For a continuous complete preorder $\preceq$ on $\mathbb{R}^2_+$, define **weak convexity**, **convexity**, and **strong convexity**.

(h) Would your answer to part (e) change if each $\preceq_i$ is also assumed to be weakly convex? Briefly explain your answer.

(i) Would your answer to part (e) change if each $\preceq_i$ is also assumed to be strongly convex? Briefly explain your answer.
Question II.2

This question concerns the characterization of aggregate excess demand in pure exchange economies with $\ell$ commodities in which $n$ consumers, $i = 1, 2, \ldots, n$ have continuous ordinal utility functions $u_i$ defined on their consumption sets $\mathbb{R}_+^\ell$ and initial endowment vectors $e_i \in \mathbb{R}_+^\ell$. Let $p \in \mathbb{R}_+^\ell$ denote a price vector and write for $i$’s individual excess demand

$$z(\cdot; u_i, e_i) : \mathbb{R}_+^\ell \to \mathbb{R}^\ell$$

and

$$Z(\cdot; u_1, e_1, \ldots, u_n, e_n) : \mathbb{R}_+^\ell \to \mathbb{R}$$

for aggregate excess demand, where

$$Z(p; u_1, e_1, \ldots, u_n, e_n) = \sum_{i=1}^n z(p; u_i, e_i).$$

(a) Briefly explain why

$$Z(p; u_1, e_1, \ldots, u_n, e_n) = Z(\lambda p; u_1, e_1, \ldots, u_n, e_n)$$

for any strictly positive scalar $\lambda > 0$.

(b) Briefly explain why

$$Z(p; \lambda u_1, e_1, \ldots, \lambda u_n, e_n) = Z(p; u_1, e_1, \ldots, u_n, e_n)$$

for any strictly positive scalar $\lambda > 0$.

(c) Is it true that

$$Z(p; u_1, \lambda e_1, \ldots, u_n, \lambda e_n) = \lambda Z(p; u_1, e_1, \ldots, u_n, e_n)$$

for all strictly positive scalars $\lambda > 0$? Briefly explain why or why not.

Question II.2 continues on the next page.
Question II.2 continued:

(d) Find an example of $e_1$ and $e_2$ such that

$$z(p; u_1, e_1) + z(p; u_2, e_2) = z(p; u_1 + u_2, e_1 + e_2)$$

for all $u_1$ and $u_2$ and for all $p \in IR_{++}^\ell$.

(e) Find an example of $u_1$ and $u_2$ such that

$$z(p; u_1, e_1) + z(p; u_2, e_2) = z(p; u_1 + u_2, e_1 + e_2)$$

for all $p \in IR_{++}^\ell$ and all $e_1, e_2 \in IR_{++}^\ell$.

(f) For what price vectors (specify a necessary and sufficient condition) $p_1, p_2 \in IR_{++}^\ell$ is it true that

$$2z\left(\frac{(p_1 + p_2)}{2}; u, e\right) = z(p_1; u, e) + z(p_2; u, e)$$

for all $u$ and all $e \in IR_{++}^\ell$?

(g) State the theorem of Sonnenschein, Debreu, Mantel, Mas-Colell, McFadden and Richter characterizing the aggregate excess demand functions that can be generated by pure exchange economies satisfying certain assumptions (state these assumptions precisely).

(h) Suppose that we restrict all endowments $e_1 = \cdots = e_n = (1, \ldots, 1) \in IR^\ell$ and all utilities to be Cobb-Douglas with possibly different parameters ($u_i = x_1^{\alpha_{1i}} x_2^{\alpha_{2i}} \cdots x_\ell^{\alpha_{\ell i}}$, where $\alpha_{ji} > 0$ and $\sum_{j=1}^\ell \alpha_{ji} = 1$ for all $i = 1 \ldots, n$). Characterize the resulting aggregate excess functions that can arise in such economies.
Part III

Answer one question from Part III
Question III.1

(a) Give an example of a game where the set of correlated equilibrium payoffs is the set of Nash equilibrium payoffs.

(b) A game with public communication is an extensive form game in which first players observe a public signal, and then they play a normal form game. Prove that the set of equilibrium payoffs is a subset of the set of correlated equilibrium payoffs. Give an example to show that the inclusion may be strict.
Question III.2

(a) Find the Nash equilibria of the following game.

(i) Two players move sequentially, an initial amount of \( I \) dollars is paid by a third part into a common fund.

(ii) Player 1 and player 2 pick a card out of a set of three cards numbered 1, 2, and 3.

(iii) Player 1 has to decide whether he bets or folds. If player 1 folds, player 2 gets an amount of \( I \) dollars and the game is over. If player 1 bets, he has to pay \( B \) dollars to the common fund, and the game goes to the next stage.

(iv) Player 2 is informed of the decision of player 1, and has to decide whether he bets or folds. If player 2 folds, player 1 gets the \( I \) dollars, and gets the \( B \) amount back. If player 2 bets, he has to pay \( B \) dollars to the common fund, and the game goes to the next stage.

(v) Both players show their card; the player with the highest number wins the amount in the common fund, that is the \( I \) amount and the amount \( 2 \times B \) paid in the earlier stages.

The utility of monetary amounts is linear. The equilibria may depend on the values of the parameters \( I \) and \( B \): please specify the equilibrium set as a function of these parameters.

(b) Find the Nash equilibria of the game described in the first part, where the two players pick each one out of four cards, numbered 1, 2, 3, and 4. In the final stage card 4 beats card 3, which beats card 2, which beats card 1.
Part IV

Answer one question from Part IV
Question IV.1

A 0-normalized 2-person bargaining game is a subset $S \subset \mathbb{R}_+^2$ that satisfies the following three conditions:

(a) $S$ is convex and compact.

(b) $S$ is comprehensive; i.e., if $x \in S$, $y \in \mathbb{R}_+^2$, and $y \leq x$, then $y \in S$.

(c) There is an $x \in S$ such that $x_i > 0$ for $i = 1, 2$.

Let $\beta$ denote the set of all games. A bargaining solution is a function $\mu : \beta \rightarrow \mathbb{R}_+^2$ such that $\mu(S) \in S$ for every $S \in \beta$.

Consider the following four axioms:

- Weak Pareto Optimality: There is no $x \in S$ such that $x_i > (\mu(S))_i$ for $i = 1, 2$.
- Homogeneity: $\mu(cS) = c\mu(S)$ for every $c > 0$.
- Strong Individual Rationality: $(\mu(S))_i > 0$ for $i = 1, 2$.
- Monotonicity: If $S \subseteq T$, then $(\mu(S))_i \leq (\mu(T))_i$ for $i = 1, 2$.

We say that a solution $\mu$ is proportional if there are strictly positive constants, $p_1$ and $p_2$, such that for every $S \in \beta$, we have $\mu(S) = \lambda(S)p$, where $p = (p_1, p_2)$, and

$$\lambda(S) = \max\{t : tp \in S\}.$$ 

Prove that a solution is weakly Pareto optimal, homogeneous, strongly individually rational, and monotonic if and only if it is proportional.

Hint: Let $\mu$ be any solution that satisfies the four axioms. Let $\Delta$ be the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$. Define $p$ to be the vector $\mu(\Delta)$. 

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Question IV.2

Consider an independent private-value auction with two bidders and one indivisible object. Both bidders have quasi-linear preferences, with each bidder i’s valuation, $v_i$, being independently and uniformly distributed between 0 and 1.

Consider the following allocation rule: bidder 2 gets the object when $v_2 > v_1 + 1/2$, and bidder 1 gets the object otherwise.

In the following questions, “mechanism” can be either direct or indirect, and “implementation” means truthful implementation.

(a) Construct a mechanism that implements the above allocation rule in dominant strategy equilibrium.

(b) Prove that there is no budget-balanced mechanism that can implement the above allocation rule in dominant strategy equilibrium.

(c) Construct a budget-balanced mechanism that implements the above allocation rule in Bayesian Nash equilibrium.