Majors and Minors : Answer ALL FOUR parts.

Please make your answers neat and concise. Make whatever assumptions you need to answer the questions. Be sure to state your assumptions clearly.
Part I

Answer the question in Part I.
Question I.1

Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital as well as labor. The consumer’s utility function is

$$\sum_{t=0}^{\infty} \beta^t u(c_t, x_t).$$

The consumer is endowed with 1 unit of labor in each period, some of which can be consumed as leisure, $x_t$, and some of which is supplied as labor, $\ell_t$. The consumer is also endowed with $k_0$ units of capital in period 0. Feasible allocations satisfy

$$c_t + k_{t+1} - (1 - \delta)k_1 \leq f(k_t, \ell_t).$$

(a) Formulate the problem of maximizing the representative consumer’s utility subject to feasibility conditions as a dynamic programing problem. Write down the appropriate Bellman’s equation. Provide assumptions on the parameters $\beta$ and $\delta$ and on the functions $u$ and $f$ that guarantee a solution to Bellman’s equation. Cite any results that you need, but do not prove anything.

(b) Define an Arrow-Debreu equilibrium for this economy. Suppose that you have solved the dynamic programming problem in part (a). Provide additional assumptions on the parameters $\beta$ and $\delta$ and on the functions $u$ and $f$ that allow you to turn the solution into an Arrow-Debreu equilibrium. Explain carefully how to calculate the Arrow-Debreu equilibrium. Once again, cite any results that you need, but do not prove anything.

(c) Define a sequential markets equilibrium for this economy. Suppose that you have calculated the Arrow-Debreu equilibrium in part (b). Explain carefully how to calculate the sequential markets equilibrium.
Part II

Answer both questions in Part II.
Question II.1: Asset accumulation

Consider a consumer that solves the following problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^\infty} \mathbb{E}_0 \sum_{t=0}^\infty \beta^t \log (c_t)$$

s.t.

$$c_t + a_{t+1} = (1 + r) a_t + y_t$$
$$a_0 = 0, \text{, No Ponzi}$$

and faces an income process given by

$$y_t = \varepsilon_t,$$
$$\varepsilon_t \rightarrow N(0, \sigma),$$

(a) Assume $\beta(1 + r) > 1$. Use the super-martingale convergence theorem to show that assets of the consumers will diverge to infinity.

(b) Write the problem in recursive form and argue that you need only one state variable.

(c) Assume now $\beta(1 + r) = 1$. Are the assets of the consumer bounded above in this case? Sketch a proof of your claim.
Question II.2: No trade equilibria

Consider a closed economy with a continuum of infinitely lived households. Each household $i$ solves the following standard problem

$$\max E \sum_{t=0}^{\infty} \beta^t \frac{c_{it}^{1-\gamma}}{1-\gamma}$$

s.t.

$$y_{it} + a_{it}(1 + r_t) = a_{it+1} + c_{it}$$

$$a_{it+1} \geq -\bar{a} < 0$$

$$a_{i0} = 0 \text{ for all } i$$

and assume that the idiosyncratic income process is lognormal as follows

$$\log y_{it} = \log y_{it-1} + \varepsilon_{it}$$

$$\varepsilon_t \sim N(0, \sigma^2)$$

(a) Define an equilibrium for this economy

(b) Show that there exist an equilibrium in which the interest rate $r_t$ is constant over time and all agents are in autarky i.e. $c_{it} = y_{it}$ for every $i$ and every $t$. Solve for the constant equilibrium interest rate in this autarkic equilibrium

(c) For this autarkic equilibrium characterize the time path of inequality as measured by the cross sectional variance of log income

(d) Show that if the income process were given by

$$\log y_{it} = \rho \log y_{it-1} + \varepsilon_{it}, 0 < \rho < 1$$

then autarky would not be an equilibrium.
Part III

Answer the question in Part III.
Question III.1

Vintage Capital and Aggregation

Consider an economy with vintages of machines. Over time, newer machines are exogenously better than older machines in the sense that it is possible to produce the same amount of output with less labor. Formally, let $K_{s,t}$ denote the number of machines of vintage $s$ used in production in period $t$. (That is, these are machines produced in period $s$.) The output of the single consumption good $Y_{s,t}$ produced from machines of vintage $s$ is

$$Y_{s,t} = \min\{K_{s,t}, \gamma^s L_{s,t}\}$$

where $\gamma > 1$ is the quantity of vintage $s$ machines and $L_{s,t}$ is the amount of labor used with machines of vintage $s$. The machines do not depreciate. The aggregate amount of labor is inelastically supplied and is equal to 1. Aggregate output is given by

$$Y_t = \sum_{s=t-1}^{\infty} Y_{s,t}.$$  

The resource constraint at date $t$ is

$$c_t + K_{t,t} \leq Y_t$$

where $c_t$ denotes consumption. The stand-in household has preferences of the form

$$\sum_{t=0}^{\infty} \beta^t \ln c_t$$

The stock of machines of past vintages at date zero is given.

1. Consider a social planner who wishes to maximize the utility of the stand in household. Set up this planner’s maximization problem. Does this economy necessarily exhibit full employment in the sense that all available labor is used?

Question III.1 continues on the next page.
Question III.1 continued:

2. Show that there is an initial distribution of capital stocks such that the solution to the planner’s problem exhibits balanced growth in the sense that consumption and output grow at a constant rate. What is this rate? Does the balanced growth path necessarily exhibit full employment in the sense that all available labor is used? In this balanced growth path, show that not all vintages are used in production. Describe how the age of the oldest vintage varies across balanced growth paths as we vary the parameters $\beta$ and $\gamma$.

3. Interpret the disappearance from use of old vintages as a form of depreciation. Show that the balanced growth path of this economy is identical to that of a model with a single aggregate kind of capital which exhibits technological progress in capital accumulation.

4. Now consider a competitive equilibrium for this economy. Let $w_t$ denote the wage rate for this economy in units of period $t$ consumption. Let $q_t$ denote the price of period $t$ consumption in units of period 0 consumption. Let $r_{s,t}$ denote the rental rate in period $t$ of period $s$ machines. Assume that in each period competitive firms produce machines and obtain profits by renting out machines in the future. Define a competitive equilibrium. Show that the allocations in this equilibrium coincide with the social planner’s problem in part 1.
Part IV: Stuff Related to the Fourth Mini

In the following there are 10 questions for 70 points. Answer questions for a total value of 50 points. Be as BRIEF as you can and good luck.
Recursive Equilibria (Production with land)

There is an economy with many identical consumers and infinite time. Consumers have preferences

$$E\left\{ \sum_{t=0}^{\infty} \beta^t [u^m(c^m_t) + u^a(c^a_t, 1 - n^a_t, \ell^a_t)] \right\}$$

where \(c^m_t\) is own consumption at in the morning at time \(t\), \(c^a_t\) is own consumption at in the afternoon at time \(t\), and \(n^m_t\), \(n^a_t\) and \(\ell^m_t\) and \(\ell^a_t\) are the fraction of time worked by the agent at time \(t\) in the morning and afternoon respectively, and \(\ell^m_t\ell^a_t\) are the land that is used for the enjoyment of vacation.

Consumption can be produced with labor and land according to a standard (CRS) neoclassical production function both in the morning and in the afternoon and can be used indistinctively both in the morning and the afternoon

$$C^m_t + C^a_t = z^m_t F(L^M_t, N^m_t) + F(L^a_t, N^a_t)$$

where \(L_t\) is land used for production purposes. Shocks to productivity \(z^m_t\) only occur in the morning and have two possible values that follow a Markov chain with transition matrix \(\Gamma\).

The total amount of land is 1.

1. (5 points) Characterize the steady state of the deterministic economy for the very special case that the utility is the same in the morning and the afternoon (agents do not care about either leisure or vacations).

2. (10 points) Define a recursive competitive equilibrium where agents trade units of land in a real state market and they rent it both to output producing firms and to vacationers. They also sell labor services to firms.

3. (5 points) Describe as much as possible the properties of the equilibrium allocation (what are wages equal to, what is the rental price of land, what is the relative price of morning and afternoon consumption, etc.)

Part IV continues on the next page.
Part IV continued:

4. (5 points) Provide a sufficient condition for firms to be closed in the afternoon.

5. (10 points) Provide a formula for the price of land early in the morning.

Search
Assume risk neutral works and firms in a one period world where there is a measure 1 of each. There is a Cobb-Douglas matching function with parameter $\alpha$. If matched output is 1, if not workers get .1 and firms get zero.

6. (10 points) Imagine that there is no search effort on the part of the worker. What is the wage under Nash bargaining when the worker has a weight of 90%?

7. (5 points) What happens under the same bargaining protocol if workers to find a job have to extend a search effort worth .5 units of consumption?

8. (5 points) What happens under competitive search?

Monopolistic Competition
Imagine that preferences of a representative consumer in a static world are given by

$$u(\{c(i)\}_{i \in [0, A]}) = \left( \int_0^A c(i)^\gamma di \right)^{\theta/\gamma} (1 - n)^{1-\theta}$$

where $1 - n$ is leisure and $n$ is time spent working.

9. (10 points) Give an expression for the time worked as a function of the (identical) price of the goods, the wage and some initial wealth that the households may have.

10. (10 points) Define the equilibrium of the economy where households own the firm. Provide two equations whose solutions yield the equilibrium where you have normalized the wage to be one. The unknowns are the wealth of the households and the price of the good.