Ph.D. Preliminary Examination

MICROECONOMIC THEORY

MAJORS

Fall 2010

The time limit for this exam is $3\frac{1}{4}$ hours.

Answer one question from each part, for a total of four questions.

You may use calculators to make calculations during the examination. However: in answering any question that requires you to justify your answer, if you do use a calculator, you must mention at what point in your answer you obtained results using a calculator, and what it was you were seeking to calculate.

Also, please be sure to define precisely any notation that you introduce.

Note: This examination should have 13 pages including this one (Check to make sure!)
Part I

Answer one question from Part I.
Question I.1

Consider a firm with production function \( f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \). The firm maximizes its profit at prices \( w \in \mathbb{R}_+^n \) for inputs and \( q \in \mathbb{R}_+^n \) for output. The firm’s profit maximization problem is

\[
\max_{x \geq 0} qf(x) - wx.
\]

Let \( x^*(q, w) \) denote the solution (input demand) and assume that \( x^* \) is a (single-valued) function of \( q \) and \( w \).

(i) State a definition of production function \( f \) being supermodular and briefly explain the economic meaning of this definition. Is the production function of two inputs \( f(x_1, x_2) = x_1^\alpha x_2^\beta \), where \( 0 < \alpha, \beta < 1 \) and \( \alpha + \beta = 1 \), supermodular? Justify your answer.

(ii) Show that if production function \( f \) (of \( n \) inputs) is supermodular, then firm’s input demand \( x^* \) is monotone nondecreasing in \( q \). Make any assumptions you like, but state those assumptions clearly.

You may use any well-known mathematical result provided that you state the result clearly and verify that its assumptions are satisfied.
Question I.2

Consider an agent with expected utility function $E[v(\cdot)]$, where the von Neumann-Morgenstern utility function $v$ is strictly increasing.

(i) State a definition of risk compensation $\rho(w, \tilde{z})$ for risky gamble $\tilde{z}$ with $E(\tilde{z}) = 0$ at deterministic initial wealth $w$.

Consider risk compensation $\rho(w, t\tilde{z})$ as a function of scale factor $t$ for arbitrary $t \in IR_+$.

(ii) Show that $\rho(w, t\tilde{z})$ is a strictly increasing function of $t$ that takes zero value at $t = 0$, for every $w$ and $\tilde{z}$ with $E(\tilde{z}) = 0$, if and only if the agent is strictly risk averse.
Part II

Answer one question from Part II
Question II.1

For this questions let $\mathcal{E}$ be a pure exchange economy with $l$ commodities and $n$ traders ($i = 1, \ldots, n$) each having consumption set $\mathbb{R}_+^l$, initial endowment vector $e_i \in \mathbb{R}_+^l$, and preferences $\preceq_i$ on $\mathbb{R}_+^l$, which are assumed to be continuous complete preorders that are strictly monotone.

(a) Define competitive equilibrium in $\mathcal{E}$.

(b) True or false; clearly and precisely explain your answer.

The economy $\mathcal{E}$ might not have a competitive equilibrium because its excess demand fails to be continuous, so that the usual fixed point argument cannot be correctly made.

(c) What is the relationship between competitive equilibrium allocations and Pareto optimal allocations in this economy? Precisely state any theorem(s) that apply and explain why the theorem/proof applies. If a standard result does not hold, clearly explain why not (which hypotheses fail and why the standard proof strategy cannot be correctly used in this situation) and give a simple counter-example.
Question II.2

Let \( E \) be a pure exchange economy (with no free disposal) with \( \ell \) commodities and \( n \) consumers \( i = 1, \ldots, n \), each having consumption set \( \mathbb{R}_+^{\ell} \), initial endowment vector \( e_i \in \mathbb{R}_+^{\ell} \), and utility function \( u_i : \mathbb{R}_+^{\ell} \to \mathbb{R} \) which is assumed to be continuous, strictly monotonically increasing, and strictly concave. The Negishi characterization says that the set \( S \) of Pareto optimal allocations can be expressed as

\[
S = \left\{ x \in \mathbb{R}_+^{\ell n} \mid \text{there exists } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n \text{ with } \sum_{i=1}^n \lambda_i = 1 \text{ such that} \right. \\
\left. x \in \arg \max \left\{ \sum_{i=1}^n \lambda_i u_i(x_i) \mid x \in \mathbb{R}_+^{\ell n} \text{ and } \sum_{i=1}^n x_i = \sum_{i=1}^n e_i \right\} \right\}.
\]

(a) Use this to show that the set of Pareto optimal allocations is nonempty.

(b) Use the Maximum Theorem (state precisely the version that you use) and the Negishi characterization to prove that the set of Pareto optimal allocations is compact under the assumptions stated above.

(c) Suppose that utility functions change to \( \bar{u}_i : \mathbb{R}_+^{\ell n} \to \mathbb{R} \) defined by \( \bar{u}_i(x) = \frac{1}{n} \sum_{i=1}^n u_i(x_i) \) for all \( i = 1, \ldots, n \). Does this change the set of Pareto optimal allocations from the Pareto optimal set \( S \)? If so, how? Prove your answer. What can you say to characterize this set? Explain.

(d) Discuss the economic interpretation of the utility function \( \bar{u}_i \) in part (c).
Part III

Answer one question from Part III
Question III.1

(a) Prove that a Subgame perfect equilibrium induces a subgame perfect equilibrium in every subgame of the original game.

(b) Consider a subgame $G'$ of a game $G$, starting at a node $x$ of $G$. Let $\beta$ a profile of behavioral strategies, which is a Nash equilibrium of $G'$. Consider the truncated game obtained by replacing in $G$ the subgame $G'$ with a terminal node, and assign to this node the payoff induced by $\beta$, and let also $\gamma$ be an equilibrium of such truncated game. Finally consider the behavioral strategy profile in $G$ that is given by $\gamma$ in all the nodes in $G$ that are not in $G'$, and by $\beta$ in $G'$.

Prove that this behavioral strategy profile is a Nash equilibrium of $G$. 
Question III.2

(a) Define a normal form game and its mixed extension.

(b) Give an interpretation of payoffs in the mixed strategy extension.

(c) Define two games to be equivalent if they have the same set of players, the same action set for each player, and the same best response correspondence for every player. What is the class of transformations of the payoffs in a normal form game that give an equivalent game? Prove your answer.
Part IV

Answer one question from Part IV
Question IV.1

Consider the following adverse selection problem. Both the principal and the agent are risk neutral. The principal relies on an agent to produce $q$ units of a good, where $q \in \mathbb{R}_+$ is a contractible quantity. Her utility function is $V = S(q) - t$, where $t \in \mathbb{R}$ is monetary transfer from her to the agent, and $S(\cdot)$ is a gross profit function that satisfies $S' > 0$, $S'' < 0$, $S(0) = 0$, $\lim_{q \to 0} S'(q) = \infty$, and $\lim_{q \to \infty} S'(q) = 0$.

The agent’s total production cost is linear in $q$. Let $\theta$ be his marginal cost. His utility function is $U = t - \theta q$. His reservation utility is 0.

The marginal cost $\theta$ is the agent’s private information, but is unobservable to the principal. The principal believes that $\theta = \bar{\theta}$ (respectively $\theta = \tilde{\theta}$) with probability $\nu$ (respectively $1 - \nu$). We assume that $\bar{\theta} > \theta > 0$.

Suppose, on top of all these standard elements, the principal can also demand (in a contract) that the agent takes a specific test, and contingent the monetary transfer $t$ on the test result. The test costs nothing, and the test result can be either “pass” or “fail”. The probability that an agent passes the test depends on his $\theta$, and is equal to $\pi(\theta)$.

(a) Suppose the agent cannot decrease his probability of passing the test by faking failure, and suppose $\pi(\bar{\theta}) > \pi(\theta)$. What is the principal’s optimal contract?

(b) Suppose the agent can artificially decrease his probability of passing the test by faking failure. In particular, suppose an agent with marginal cost $\theta$ can costlessly choose any passing probability in $[0, \pi(\theta)]$. Suppose $\pi(\bar{\theta}) > \pi(\tilde{\theta})$. What is the principal’s optimal contract?

(c) Repeat parts (a) and (b) using the alternative assumption that $\pi(\bar{\theta}) < \pi(\tilde{\theta})$. 
Question IV.2

Consider the following 2-period moral hazard model. In the first period, the agent chooses a consumption level \( c \in \mathbb{R} \). In the second period, he chooses an effort level \( e \in \{0, 1\} \), with effort cost \( \Psi(e) \), where \( \Psi(0) = 0 \) and \( \Psi(1) = \Psi > 0 \). Neither \( c \) nor \( e \) is observable to the principal. Output \( q \in \{\bar{q}, \bar{q}\} \) is observed (and is verifiable) at the end of the second period. Probability that \( q = \bar{q} \) is \( \pi_e \), with \( 1 > \pi_1 > \pi_0 > 0 \). Assume that the principal wants to induce effort \( e = 1 \), and tries to minimize the expected wage she pays the agent. If the wage contract is \( t(q) \), and if the agent consumes \( c \) in the first period and exerts effort \( e \) in the second, then his utility is

\[
 u(c) + \pi_e u(t(\bar{q}) - c) + (1 - \pi_e)u(t(\bar{q}) - c) - \Psi(e),
\]

where \( u(\cdot) \) is strictly increasing and concave. The agent’s reservation utility is 0.

(i) Assume that the principal can commit to any wage contract she offers to the agent at the beginning of the first period (i.e., before the agent chooses \( c \)). Set up the principal’s problem (as a constrained optimization problem).

(ii) Let \( t^*(q) \) be the solution to your problem in part (i). Let \( \bar{c} \) be the agent’s optimal choice of consumption if he is offered the contract \( t^*(q) \) and if he is constrained to choose effort \( e = 1 \). Similarly, let \( \underline{c} \) be the agent’s optimal choice of consumption if he is offered the contract \( t^*(q) \) and if he is constrained to choose effort \( e = 0 \). Prove that \( \bar{c} > \underline{c} \).

(iii) Use your result in part (ii) to prove that the contract \( t^*(q) \) is not renegotiation-proof. In particular, suppose you were the principal, and suppose you had an opportunity to renegotiate with the agent at the end of the first period (i.e., right after he chose \( c \)), and suppose you’re sure that he had just chosen \( c = \bar{c} \). Explain how you can propose to modify the contract \( t^*(q) \) to make both you and the agent strictly better off from that moment on.