Ph.D. Preliminary Examination

MICROECONOMIC THEORY

MAJORS

Fall 2011

The time limit for this exam is $3\frac{1}{4}$ hours.

Answer one question from each part, for a total of four questions.

You may use calculators to make calculations during the examination. However: in answering any question that requires you to justify your answer, if you do use a calculator, you must mention at what point in your answer you obtained results using a calculator, and what it was you were seeking to calculate.

Be sure you clearly define all boldfaced/underlined terms. Also, please be sure to define precisely any notation that you introduce.

Note: This examination should have 15 pages including this one (Check to make sure!)
Part I

Answer one question from Part I.
Question I.1

Consider an agent facing uncertainty described by a finite set of states $S$. The agent’s preferences over state-contingent consumption plans $x \in \mathbb{R}_+^S$ are described by the utility function

$$u(x) = \inf_{\pi \in \mathcal{P}} E_{\pi} x,$$

where $E_{\pi} x = \sum_{s=1}^{S} \pi_s x_s$ denotes the expected value of $x$ and $\mathcal{P}$ is a set of probability measures on $S$. Set $\mathcal{P}$ is a subset of the unit simplex $\Delta$ in $\mathbb{R}^S$ and is assumed closed, convex, and such that $\mathcal{P} \subset \mathbb{R}_+^S$.

(a) Show that utility function $u$ is locally non-satiated and concave.

(b) Characterize the points $x$ of differentiability of $u$. What is the derivative (or the gradient vector) $Du(x)$ at a point $x$ of differentiability?

You may use any well-known mathematical result pertaining to (b) without proof.
Question I.2

Consider two real-valued random variables $\tilde{y}$ and $\tilde{z}$ on some state space (i.e. probability space). Let $F_{\tilde{y}}$ and $F_{\tilde{z}}$ be their cumulative distribution functions, and $E(\tilde{z})$ and $E(\tilde{y})$ their expected values.

(a) State a definition of $\tilde{z}$ first-order stochastically dominating (FSD) $\tilde{y}$. Show that if $\tilde{z}$ FSD $\tilde{y}$, then $E(\tilde{z}) \geq E(\tilde{y})$.

(b) Show that, if $\tilde{z}$ FSD $\tilde{y}$ and $E(\tilde{z}) = E(\tilde{y})$, then $\tilde{y}$ and $\tilde{z}$ have the same distribution, i.e., $F_{\tilde{y}}(t) = F_{\tilde{z}}(t)$ for every $t \in \mathbb{R}$. If you find it convenient, you may assume in your proof that random variables $\tilde{y}$ and $\tilde{z}$ have densities, or alternatively that $\tilde{y}$ and $\tilde{z}$ are discrete random variables (i.e., take finitely many values).

(c) State a definition of $\tilde{z}$ second-order stochastically dominating (SSD) $\tilde{y}$. Show that if $\tilde{z}$ SSD $\tilde{y}$, then $E(\tilde{z}) \geq E(\tilde{y})$.

(d) Show that if $\tilde{z}$ FSD $\tilde{y}$, then $\tilde{z}$ SSD $\tilde{y}$.

(e) State a definition of $\tilde{y}$ being more risky than $\tilde{z}$. Give a brief justification for why it is a sensible definition of more risky.
Part II

Answer one question from Part II
Question II.1

Consider a pure exchange economy with $n$ consumers $i = 1, \ldots, n$, each having initial endowment $e_i \in \mathbb{R}_+^\ell$ and preferences $\preceq_i$ assumed to be complete preorders on $\mathbb{R}_+^\ell$.

(a) Define the core of this economy.

(b) Define the $m$-replica economy for any integer $m \geq 1$. State and prove the Equal Treatment Property for the core of the $m$-replica economy.

(c) State the Debreu-Scarf Theorem and discuss its economic significance.
Question II.2

Let $E$ be a pure exchange economy (with no free disposal) with $\ell$ commodities and $n$ consumers $i = 1, \ldots, n$, each having consumption set $\mathbb{R}^\ell_+$, initial endowment vector $e_i \in \mathbb{R}^\ell_{++}$, and utility function $u_i : \mathbb{R}^\ell_+ \rightarrow \mathbb{R}$ which is assumed to be continuous, strictly monotonically increasing, and strictly concave. Let $S$ be the set of Pareto optimal allocations of $E$.

(a) Consider an allocation $\bar{x}$ that solves the following maximization problem

$$
\max_x \sum_{i=1}^n \lambda_i u_i(x_i)
$$

subject to

$$
\sum_{i=1}^n x_i = \sum_{i=1}^n e_i,
$$

$$
x_i \in \mathbb{R}^\ell_+, \forall i.
$$

for some $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda \geq 0$ and $\lambda \neq 0$. Show that allocation $\bar{x}$ is Pareto optimal.

(b) Prove that the set $S$ of Pareto optimal allocations is non-empty.

(c) Prove that the set $S$ is compact.
Part III

Answer one question from Part III
Question III.1

In a finite game with \{1, \ldots, n\} players let \( A^i \) be the action set of player \( i \), and \( A \) the set of action profiles. A finite automaton (FA) for player \( i \) is \( M^i \equiv (X^i, \tau^i, f^i, x^i_1) \) where \( X^i \) is a finite set, \( \tau^i : X^i \times A \rightarrow X^i \) the transition function on \( X^i \), \( f^i : X^i \rightarrow A^i \), and \( x^i_1 \in X^i \) the initial state.

(a) Prove that an FA induces a unique pure strategy in an infinitely repeated game

(b) What can you say about the converse? Prove your answer, providing a counterexample if appropriate.
Question III.2

(a) Define Backward Induction for a finite extensive form game.

(b) Prove that Backward Induction always produces a non empty set of strategy profiles. Is an element in this set of strategy profiles a **Subgame Perfect Equilibrium**? Is it a **Perfect Equilibrium**?

(c) Give an example of a finite game of **perfect information** in which the Backward Induction strategies are not unique, but the payoff vector is unique.
Part IV

Answer one question from Part IV
Question IV.1

Consider the game which is obtained as a finite repetition for \( N \) periods of the Prisoner’s dilemma stage game

\[
\begin{array}{c|cc}
 & l & r \\ 
T & 2,2 & -1,3 \\ 
B & 3,-1 & 0,0 \\
\end{array}
\]

for a finite number of periods, and with final payoff equal to the sum on the payoffs in every period.

(a) Find the set of the Subgame perfect equilibria

(b) Find the set of Nash equilibria. Prove your answer.

(c) Now note that the PD is a game with the following property (M):

The vector of minimax values is equal to the unique Nash equilibrium payoff.

Consider any two players stage game with finite action set and with property M. Find Subgame perfect and Nash equilibria for any such game.
Question IV.2

Consider the infinitely repeated game, discounted by $\delta \in (0, 1)$, where the stage game is Matching pennies game:

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<td>$B$</td>
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(a) Describe the set of behavioral strategy Nash equilibria of the repeated game.

(b) Describe the set of mixed strategy Nash equilibria of the repeated game.
Question IV.3

Consider the following moral hazard problem. The principal has a business project, but is unsure about its quality, $q$. With probability $\gamma \in (0, 1)$, the project has high quality ($q = h$), and the principal will make a positive profit of $y_h > 0$ if she pursues it. With probability $1 - \gamma$, the project has low quality ($q = l$), and the principal will make a negative profit of $y_l < 0$ if she pursues it.

The agent is an expert who provides the principal with information about the project’s quality. He can observe a signal, $\theta \in \{\theta_h, \theta_l\}$, that is correlated with $q$. How informative the signal is depends on how much effort, $e \geq 0$, he exerts. In particular,

$$\Pr\{\theta = \theta_h | q = h, e\} = \alpha + \beta_h \eta(e) \quad \text{and} \quad \Pr\{\theta = \theta_h | q = l, e\} = \alpha - \beta_l \eta(e),$$

where $\alpha \in (0, 1)$, $\beta_h, \beta_l > 0$, and the function $\eta$ satisfies $\eta(0) = 0$, $\eta'(e) > 0$, $\eta''(e) < 0$ for all $e$, and $\lim_{e \to \infty} \eta(e) = \bar{\eta} < \min\{(1 - \alpha)/\beta_h, \alpha/\beta_l\}$.

Exerting effort, $e$, is costly for the agent, which entails a disutility of $\psi(e)$, where the function $\psi$ satisfies $\psi(0) = 0$, $\psi'(0) = 0$, $\psi'(e) \geq 0$, and $\psi''(e) > 0$ for all $e$.

The agent’s effort is unobservable, but he cannot lie about the signal realization. Therefore, a contract is a pair of numbers, $(w_h, w_l)$, specifying the wages paid to the agent if the signal realization is $\theta_h$ and $\theta_l$, respectively.

Both the principal and the agent are risk neutral. The principal’s payoff equals to her expected profit minus the expected wage paid to the agent. The agent’s payoff equals to the expected wage received minus the disutility of effort.

The agent’s reservation utility is 0. He is protected by limited liability, and thus wages are restricted to be nonnegative.

(a) Set up the optimal contract design problem. (You do not need to solve it.)
(b) Suppose the optimal contract induces a strictly positive effort level (i.e., $e^* > 0$).
Show that, if $\gamma < \beta_l/(\beta_h + \beta_l)$, then the optimal contract must have the property that $w_t^* > 0 = w_h^*$; that is, the agent gets paid only when he tells the principal that her project has low quality. Explain intuitively why the principal seems to be more pleased with bad news in this case.

(c) What is the optimal contract if $\gamma = \beta_l/(\beta_h + \beta_l)$?