Ph.D. Preliminary Examination

MICROECONOMIC THEORY

MINORS

Fall 2012

The time limit for this exam is $3\frac{1}{4}$ hours.

Answer one question from each part, for a total of four questions.

You may use calculators to make calculations during the examination. However: in answering any question that requires you to justify your answer, if you do use a calculator, you must mention at what point in your answer you obtained results using a calculator, and what it was you were seeking to calculate.

Be sure you clearly define all boldfaced/underlined terms. Also, please be sure to define precisely any notation that you introduce.

Note: This examination should have 13 pages including this one (Check to make sure!)
Part I

Answer one question from Part I.
Question I.1

Consider a consumer with utility function on the consumption set $\mathbb{R}_+^n$ of $n$ goods given by $u : \mathbb{R}_+^n \to \mathbb{R}$. Utility function $u$ is assumed continuous and strictly increasing.

(a) State definitions of Walrasian (Marshallian) demand and Hicksian (compensated) demand for the consumer.

(b) State a theorem asserting that if consumption bundle $x^*$ lies in the Walrasian demand at strictly positive price vector $p$ and income $m > 0$, then $x^*$ must lie in the Hicksian demand at $p$ and suitably chosen utility level (which you need to specify). Prove your stated theorem.

(c) Show by means of an example that the result of (b) may fail if utility function $u$ is continuous but not strictly increasing.
Question I.2

Consider an agent whose preferences under uncertainty have expected utility representation with the von Neumann-Morgenstern utility function

\[ v(c) = \ln(c) \]

for \( c > 0 \), where \( c \) denotes consumption.

(a) Consider risky gamble \( \tilde{z} \) which may take either one of two values, \( g \) or \(-g\), with equal probabilities. Find this agent’s risk compensation \( \rho(w, \tilde{z}) \) as an explicit function of \( g \) and deterministic wealth \( w \). Is risk compensation an increasing function of \( w \)?

(b) Suppose that there are two assets: a risk-free asset with (per-dollar, or gross) return \( \bar{r} \) and a risky asset with return \( \tilde{r} \) which may take either one of two values, \( r_1 \) or \( r_2 \), with equal probabilities. Suppose that \( r_1 > \bar{r} > r_2 \). Find this agent’s optimal investment in the risky asset as a function of initial wealth \( w \) and returns \( \bar{r} \) and \( r_1, r_2 \). Verify whether the optimal investment is an increasing function of wealth.
Part II

Answer one question from Part II
Question II.1

For a pure exchange economy with $\ell$ goods and $n$ consumers, indexed by $i = 1, 2, \ldots, n$, each having consumption set $\mathbb{R}_+^\ell$, initial endowment $e_i \in \mathbb{R}_+^\ell$ and continuous utility function $u_i : \mathbb{R}_+^\ell \to \mathbb{R}$ which is assumed to be strictly monotone and strictly concave, answer the following questions:

(a) Using the above notation, formally define the sets of weakly and strongly Pareto optimal allocations.

(b) Characterize the sets in (a) in terms of an expression involving a summation using utilities.

(c) State the first welfare theorem.

(d) Prove the first welfare theorem.

(e) Discuss the economic significance of the first and second welfare theorems.

(f) Is it possible to have a Pareto optimal allocation in which agent 1 has an allocation that is twice as much (of each commodity) as agent 2, but agent 2 gets twice as much utility as agent 1 (at the Pareto optimal allocation)? Can this happen at a competitive equilibrium allocation? Can this happen at a competitive equilibrium allocation when each agent’s initial endowment is exactly one unit of each commodity? Explain each answer. You may take $n = \ell = 2$ for part (f).
Consider a pure exchange economy with two commodities and three traders (indexed by subscripts 1, 2, and 3), each having the same consumption sets $\mathbb{R}^2_+$ and the same initial endowment vector $e_1 = e_2 = e_3 = (1, 1)$. For $c \in \mathbb{R}_+$ (so that the scalar $c \geq 0$ is a nonnegative constant), the first agent’s indifference curves are given by
\[
\{(x, y) \in \mathbb{R}^2_+ | x = c \text{ and } y \geq c \text{ or } x \geq c \text{ and } y = c\}
\] while the indifference curves of agents 2 and 3 are given by $\{(x, y) \in \mathbb{R}^2_+ | x + y = c\}$. For all agents, it should be understood that larger values of $c$ correspond to higher (better) indifference curves.

(a) Are the preferences of these traders monotone? Strictly monotone? Convex? Strictly convex?

(b) Find a utility representation for each trader’s preferences.

(c) Define competitive equilibrium in this economy. (Be sure to define any notation that you introduce.)

(d) Find all competitive equilibrium allocations in this economy.

(e) Find all competitive equilibrium price vectors in this economy.

(f) State a theorem on the existence of competitive equilibrium and explain why it does or does not apply to this economy. (I.e., does this economy satisfy the assumptions for the theorem you stated? Why or why not?)
Part III

Answer one question from Part III
Question III.1

Give the definition of subgame and subgame perfect equilibrium. Then do the following:

(a) In a finite extensive form game, let $Y$ be a non-empty subset of the set of nodes of the tree $X$, with the partial order induced by the restriction of the order of the tree. Prove that there is an element in $Y$ with no immediate successor.

(b) Prove that in an extensive form game of perfect information every node defines a sub-game.
Question III.2

Define the set of correlated equilibria of a normal form game. Then prove the following:

(a) The set of correlated equilibrium payoffs that can be obtained by a publicly observed signal is the convex hull of the set of the Nash equilibrium payoffs;

(b) The set of correlated equilibrium payoffs that can be obtained with private correlated signals may be strictly larger than the convex hull of the set of the Nash equilibrium payoffs.
Part IV

Answer one question from Part IV
Question IV.1

A seller is designing a selling mechanism to sell an indivisible product to a buyer. She can choose both the quality and the price of her product. Let \( q \in \mathbb{R}_+ \) denote the quality. The higher is \( q \), the higher is the production cost. In particular, the production cost will be \( \frac{1}{2}q^2 \) if she decides that her product should have quality \( q \). The buyer has quasilinear preference: his utility is \( u(q|t) = v(q|t) - p \), where \( v(q|t) = qt \), if he buys a product with quality \( q \) at a price \( p \), and if his quality-sensitivity is \( t \). His quality-sensitivity, \( t \), is known only to him, and the seller believes that \( t \) is drawn from a uniform distribution over \([2, 3]\). Without loss of generality, the seller is to choose a direct mechanism \((q, p): [2, 3] \to \mathbb{R}_+ \times \mathbb{R}\) to maximize

\[
\mathbb{E}_t \left[ p(t) - \frac{1}{2}q(t)^2 \right]
\]

subject to the incentive compatibility constraint that \((q, p)\) is strategy-proof, and the individual rationality constraints that

\[
\forall t \in [2, 3], \quad q(t)t - p(t) \geq 0.
\]

Solve for the optimal selling mechanism.
Question IV.2

Consider the following mechanism-design environments. There is only one agent, who has quasilinear preference. Her type space is \( T = \{1, 2, \ldots, 10\} \), and the outcome space is \( \Gamma = \mathbb{R} \). Her valuation function is \( v(x|t) = xt \), where \( x \in \Gamma \) and \( t \in T \), and her utility function is \( u(x|t) = v(x|t) - p \), where \( p \in \mathbb{R} \) is the monetary transfer she pays.

Consider the following Revenue-Equivalence statement: “For any allocation rule, \( g : T \rightarrow \Gamma \), that is truthfully implementable in dominant strategies, and any two payment rules, \( p, p' : T \rightarrow \mathbb{R} \), that implement it, \( p \) and \( p' \) differ only by a constant; i.e., \( \exists c \in \mathbb{R} \) such that

\[
\forall t \in T, \quad p(t) = c + p'(t).
\]

Determine whether the statement is true for the above mentioned environment. If you think it is true, prove it. If you think it is not true, provide a counterexample.