Ph.D. Preliminary Examination

MICROECONOMIC THEORY

MAJORS

Fall 2014

The time limit for this exam is 3\(\frac{1}{4}\) hours.

**Answer one question from each part, for a total of four questions.**

You may use calculators to make calculations during the examination. However: in answering any question that requires you to justify your answer, if you do use a calculator, you must mention at what point in your answer you obtained results using a calculator, and what it was you were seeking to calculate.

Note: This examination should have 14 pages including this one (Check to make sure!)
Answer one question from Part I.
Question I.1

There are two conditions often used to define continuity of a preference relation $\succeq$ on consumption set $X = \mathbb{R}_+^L$:

(A) for every sequences $\{x^n\}$ and $\{y^n\}$ in $X$ such that $\lim_n x^n = x$, $\lim_n y^n = y$, and $x^n \succeq y^n$, it holds $x \succeq y$.

(B) For every $x \in X$, the preferred-to-$x$ set $\{y \in X : y \succeq x\}$, and the lower contour set $\{y \in X : x \succeq y\}$ are closed.

(i) Assuming that $\succeq$ is transitive and complete, prove that conditions (A) and (B) are equivalent.

(ii) Give an example of a transitive and complete preference relation on $\mathbb{R}_+^2$ for which the preferred-to-$x$ sets are closed for all $x$, but the lower contour sets are not closed for some $x$. 

Let $\tilde{y}$ and $\tilde{z}$ be arbitrary random variables on some finite state space. Let $E(\tilde{y}) = E(\tilde{z})$ where $E$ denotes expected value under the given probabilities on states.

(i) State a definition of $\tilde{y}$ being more risky than $\tilde{z}$ in terms of cumulative distribution functions of $\tilde{y}$ and $\tilde{z}$. Prove that if $\tilde{y}$ is more risky than $\tilde{z}$, then

$$E[v(\tilde{z})] \geq E[v(\tilde{y})]$$

for every non-decreasing, continuous and concave utility function $v : \mathbb{R} \rightarrow \mathbb{R}$.

(ii) Show that, if $\tilde{y}$ is more risky than $\tilde{z}$, then $\text{var}(y) \geq \text{var}(z)$, where $\text{var}$ denotes variance.

(iii) Show that, if $\tilde{y}$ is more risky than $\tilde{z}$, then $w + \tilde{y}$ is more risky than $w + \tilde{z}$, for every deterministic $w$. Here, you may use without proof the fact that the necessary condition for more risky from part (i) is also sufficient.
Question II.1

Suppose that a pure exchange economy has \( n \) traders and let \( N = \{1, 2, \ldots, n\} \) be the set of traders. Each consumer \( i \in N \) has consumption set \( \mathbb{R}^\ell_+ \), initial endowment \( e_i \in \mathbb{R}^\ell_+ \) and a utility function \( u_i : \mathbb{R}^\ell_+ \to \mathbb{R} \).

(a) State a set of minimal assumptions needed to define competitive equilibrium and Pareto optimality. Briefly explain why each assumption is needed or, if no assumption is needed, say so and briefly explain.

(b) Define competitive equilibrium in this economy using this notation.

(c) Define weak Pareto optimality and strong Pareto optimality in this economy using this notation.

(d) Again for this economy using this notation, state the first welfare theorem.

(e) Prove the first welfare theorem (as you have stated it).

(f) Now suppose that each \( u_i \) is an expected utility for an economy with contingent commodity contracts. To simplify suppose that there are two states of the world, called \( H \) and \( T \) and occurring with known probabilities \( \pi(H) \) and \( \pi(T) \) where \( 0 \leq \pi(H) \leq 1 \), \( 0 \leq \pi(T) \leq 1 \), and \( \pi(H) + \pi(T) = 1 \). Assume that \( \ell \) is an even (why?) strictly positive integer and interpret the first \( \ell/2 \) commodities as those delivered in state \( H \) with the remaining goods delivered in state \( T \). Then write state-dependent utilities as

\[
v_{i,H} : \mathbb{R}^{\ell/2}_+ \to \mathbb{R}, \quad \text{and} \quad v_{i,T} : \mathbb{R}^{\ell/2}_+ \to \mathbb{R},
\]

so that we have (for \( x_i \in \mathbb{R}^{\ell}_+ \))

\[
u_i(x_i) = \pi(H)v_{i,H}(x_{i,1}, x_{i,2}, \ldots, x_{i,\ell/2}) + \pi(T)v_{i,T}(x_{i,\ell/2+1}, \ldots, x_{i,\ell}).
\]

Under what assumptions (if any) on \( \pi(H) \), \( \pi(T) \), and the \( v_{i,H} \) and \( v_{i,T} \) functions does the first welfare theorem apply? Explain your answer.
Question II.2

In microeconomic theory, we tend to make some assumptions purely for convenience (to simplify specifications of the model or to shorten proofs) and others because we know them to be necessary and sufficient conditions for the desired result. Most fall into the middle ground of being somewhat helpful for the model, proof, etc. but we know they’re not really needed, yet they seem somewhat inconsequential because we expect the assumption either to be satisfied in the situation of interest or to be viewed as something that isn’t closely related to our main focus. This question first asks you to explore some cases where the presence of a few “weird” agents do not destroy our standard results on existence of competitive equilibrium and welfare theorems. Assume that the economy has only two commodities with consumption sets being $\mathbb{R}_+^2$. Initial endowments will be denoted by $e_i \in \mathbb{R}_+^2$. It’s easiest to answer these questions with clearly drawn pictures of endowments and indifference curves.

(a) Give an example of a preference relation $\preceq_i$ which is convex but not strictly convex, and an initial endowment vector $e_i \in \mathbb{R}_+^2$ such that the consumer’s demand is single valued for all nonnegative price vectors in $\mathbb{R}_+^2$. In this case, demand will be a continuous (and in fact smooth) function so that the use of the Easy Existence Theorem can still be valid. In fact, identify the class of preference relations for which this “trick” works for the initial endowment vector you specified, and explain briefly.

(b) Now assume that $e_i \in \mathbb{R}_+^2$. Find an example with monotone and convex but not strictly convex preferences so that demand is a single-valued continuous function for all strictly positive price vectors $p \in \mathbb{R}_+^2$.

(c) Again take $e_i \in \mathbb{R}_+^2$ and find a preference relation which is ℓ.n.s. (but not monotone) and convex (but not strictly convex) such that again demand is a single valued continuous function for all nonnegative price vectors $p \in \mathbb{R}_+^2$. [Hint: You can modify an example that works for part (b).]

(d) For $e_i \in \mathbb{R}_+^2$ and preferences that are weakly convex (but not convex)
and not even ℓ.n.s., find a preference relation such that demand is set
valued for all nonnegative prices but there is a continuous single-valued
selection from the agent’s demand correspondence.

(e) Take a standard pure exchange economy with \( n \) traders such that
\( e_i \in \mathbb{R}_+^2 \) for all \( i \) and each trader’s preference relation \( \succeq_i \) is a continuous
complete preorder which is strictly monotone and strictly convex. Now
add \( m \) traders to this economy with \( e_i \in \mathbb{R}_+^2 \) having the property that
the initial endowment vector belongs to the individual demand
correspondence for all strictly positive prices. Show (explain your
reasoning clearly) that the Easy Existence Theorem can be used to prove
that the augmented economy has at least one competitive equilibrium.
What can you say about the comparison between competitive equilibria in
the original and the augmented economy? Explain briefly.

(f) State the Sonnenschein conjecture on the characterization of aggregate
excess demand in pure exchange economies with preferences satisfying
strict convexity and some other assumptions (state them). How can you
modify the theorem if preferences are convex but not strictly convex?
Discuss whether the result for convex but not strictly convex preferences
should be considered to be weaker or stronger.
Part III

Answer one question from Part III
Question III.1

A two players finite game is called strictly competitive if for every pair $a$ and $b$ of action profiles

$$u^1(a) > u^1(b) \text{ if and only if } u^2(a) < u^2(b)$$

(a) Give an example of a strictly competitive game that is not constant sum game.

(b) Let for each player $i$ $\maxmin_i \equiv \max_{a_i} \min_{a_{-i}} u^i(a^i, a_{-i})$, and $\minmax_i \equiv \min_{a_{-i}} \max_{a_i} u^i(a^i, a_{-i})$. Prove the following:

For a strictly competitive game

(a) If for $i = 1, 2$ we have $\maxmin_i = \minmax_i$, then $G$ has a Nash equilibrium.

(b) If $G$ has a Nash equilibrium, then for for $i = 1, 2$ we have $\maxmin_i = \minmax_i$. 
Question III.2

(a) State precisely Kuhn’s theorem. Please provide definitions of the strategy sets you are using. If you use a notion of equivalence, state its definition.

(b) Provide an example to show why the restriction on the games for which the conclusion of the theorem holds cannot be dispensed with.

(c) Consider an extensive form game with two players who alternate in the choice of moves (player 1 moves in even periods, players 2 moves in odd periods); each player has two actions in every period.

(a) State Kuhn’s theorem in this case.

(b) Prove the theorem you state.
Part IV

Answer one question from Part IV
Question IV.1

There are $n$ individuals considering whether or not to place a sculpture (already sculpted, so it costs nothing) in the park. Each individual $i$ has utility function

$$u_i(k, t_i) = v_i k + t_i,$$

for some number $v_i \in \mathbb{R}$, where $k \in \{0, 1\}$ denotes whether (1) or not (0) the statue is placed in the park. The number $v_i$ reflects the taste of individual $i$ for the sculpture which in principle can range anywhere in $\mathbb{R}$, and the quantity $t_i \in \mathbb{R}$ is a monetary transfer.

(a) Describe a Vickrey-Clarke-Groves (VCG) mechanism for this economy formally and intuitively.

(b) Show that every VCG mechanism fails to satisfy budget balance, that is, there exists a profile $v = (v_1, \ldots, v_n)$ at which the monetary transfers $t = (t_1, \ldots, t_n)$ satisfy

$$\sum_{i=1}^{n} t_i(v) \neq 0.$$

(c) Let $g(v) = \max\{\sum_i v_i, 0\}$ be the maximum possible welfare from optimally placing the sculpture or not when everyone’s utility is given by $v$, and $k(v) = 1_{\{\sum_i v_i > 0\}}(v)$ the optimal decision, that is, the indicator function of whether or not the sum of utilities is positive. Furthermore, let $g_i(v, v'_i) = k(v'_i, v_{-i}) \sum_i v_i$ be the welfare when individual $i$ misreports his utility to be $v'_i$ instead of $v_i$ and the optimal choice is made for reported—rather than actual—utilities.

(a) Find a VCG mechanism $(\tilde{k}, \tilde{t})$ such that

$$u_i(\tilde{k}(v'_i, v_{-i}), \tilde{t}_i(v'_i, v_{-i})) = g_i(v, v'_i)$$

for all $i$, $v_i$, $v'_i$ and $v_{-i}$.

(b) Show that there is no mechanism $(\hat{k}, \hat{t})$ such that

$$u_i(\hat{k}(v'_i, v_{-i}), \hat{t}_i(v'_i, v_{-i})) = g_i(v, v'_i)/n.$$
for all $i$, $v_i$, $v_i'$ and $v_{-i}$.

(c) Is there another way of splitting $g$ (other than equally as considered in part (b) above) to recover budget balance?
Question IV.2

Consider the Prisoners’ Dilemma with imperfect public monitoring.

\[
\begin{array}{c|cc}
 & C & D \\ \hline
C & 1,1 & -1,2 \\
D & 2,-1 & 0,0 \\
\end{array}
\]

\[
\begin{array}{c|cc}
 & C & D \\ \hline
C & p_2 & p_1 \\
D & p_1 & p_0 \\
\end{array}
\]

Flow Payoffs \qquad \text{Pr}(g)

Every period, players observe a binary signal \( \omega \in \Omega = \{g, b\} \) that is IID conditional on players’ actions. The bi-matrix on the left describes flow payoffs, and the matrix on the right describes the conditional probability of \( g \), where \( 0 < p_0 < p_1 < p_2 < 1 \). Players share a common discount factor \( \delta < 1 \).

Assume that players have access to a public randomization device.

(a) Find the maximal strongly symmetric pure-strategy equilibrium payoff as \( \delta \to 1 \).

(b) Under what conditions on the parameters of the problem is this payoff feasible?

(c) Let \( \delta = e^{-r \Delta t} \), where \( r > 0 \) is a discount rate and \( \Delta t > 0 \) is the length of time between interactions. Given \( x \) such that \( x_0 < x_1 < x_2 \), let

\[
p_k = \frac{1}{2} [1 + x_k \sqrt{\Delta t}] 
\]

for each \( k \) (therefore, the signals define a random walk with drift \( x \) that tends to Brownian motion as \( \Delta t \to 0 \)), and assume that \( \Delta t \) is small enough that \( 0 < p_k < 1 \) for all \( k \). Redo parts (a) and (b) above

(i) as \( r \to 0 \) with \( \Delta t \) fixed, and

(ii) as \( \Delta t \to 0 \) with \( r \) fixed.

Discuss.