Econometrics Prelim

Please answer all four questions as completely as possible. You have four hours to complete the exam. Good luck!
Question 1

All variables are defined relative to their mean.

Consider the following model with $Y_i$ a scalar random variable, $X_i$ a $1 \times K$ vector of regressors, $\beta_0$ a vector of $K$ parameter values, and $\epsilon_i$ a scalar random variable:

$$Y_i = X_i \beta + \epsilon_i$$

for observations $i = 1, \ldots, N$. Assume the $N \times K$ matrix $X$ constructed from stacking the $i$ observations has full rank. Let $Y$ similarly be the $N \times 1$ vector of stacked $Y_i$'s. The object of estimation will be $\mathbb{E}[Y_i | X_i = x_i]$, for different values of $x_i$.

Suppose there exists a set of random variables $Z_j, j = 1 \ldots L$, with $L \geq K$, such that $\mathbb{E}[\epsilon_i Z_{ij}] = 0 \ \forall i, j$. Assume the $N \times L$ matrix $Z$ is full rank.

Consider each of the following estimators for $\mathbb{E}[Y_i | X_i = x_i]$.

a) $x_i \hat{\beta}$, where $\hat{\beta}$ is calculated by multiplying the inverse of $X'X$ times $X'Y$.

b) $x_i \tilde{\beta}$, where $\tilde{\beta}$ is calculated by multiplying the inverse of $Z'X$ times $Z'Y$.

c) $x_i \beta^*$, where $\beta^*$ is calculated by multiplying the inverse of $\hat{X}'\hat{X}$ times $\hat{X}'Y$, with $\hat{X} = Z (Z'Z)^{-1} Z'X$. 

1) What are the conditions under which estimator a) is unbiased? What are the conditions under a) is consistent?

2) What are the conditions under which estimator b) is unbiased? What are the conditions under b) is consistent?

3) What are the conditions under which estimator c) is unbiased? What are the conditions under c) is consistent?

4) Suppose we change the condition to $E[\epsilon_i | Z_{ij}] = 0 \forall i, j$. How does that change your analysis of 1-3 above.
Question 2

In applied work, our variables are frequently measured with error. Also, we may only have access to variables which proxy for the correct exogenous variables suggested by economic theory. Therefore, it is important to consider the case in which our variables are measured with error.

Consider a linear econometric model of the form

\[ y_i = x_i^* \beta + u_i \]

In the above, we assume that \( y_i, x_i^* \) and \( u_i \) are all scalars for \( i = 1, ..., N \). The term \( \beta \) is an unknown parameter that we wish to estimate.

Suppose that \( x_i^* \) is not directly observed by the econometrician. Instead, we observe an error prone measure of \( x_i^* \) which we refer to as \( x_i \)

\[ x_i = x_i^* + v_i \]

Assume \( x_i^* \) has mean zero (e.g. we use deviations from the mean). Also assume that the measurement error \( v_i \) satisfies:

\[ E[v_i|x_i^*] = E[v_i|u_i] = 0 \]

Let \( Y, X, X^*, U \) and \( V \) denote the \( N \) by 1 vectors composed of the individual \( y_i, x_i, x_i^*, u_i \) and \( v_i \). We can write this system in vector notation as:

\[
\begin{align*}
Y &= X^* \beta + U \\
X &= X^* + V
\end{align*}
\]

Since \( X^* \) is unobserved, suppose that we naively attempt to estimate our model using ordinary least squares by regressing \( Y \) on \( X \). The probability limit of \( \hat{\beta} \) must satisfy:

\[
\text{plim} \hat{\beta} = \text{plim}(N^{-1}X'X)^{-1}\text{plim} N^{-1}X'Y
\]

1. First derive the probability limits of \( \text{plim}(N^{-1}X'X)^{-1} \) and \( \text{plim} N^{-1}X'Y \)
2. Second, derive the probability limit of \( \hat{\beta} \) as a function of the variance of \( X^* \) and \( V \)
3. Third, using this formula, interpret cases in which you would expect the measurement error to generate larger versus smaller biases.
4. Fourth, describe a strategy for consistently estimating \( \beta \) and sketch a proof for why this strategy works.
5. Fifth, extend the results from question 2 by deriving a related formula for the case where we have access to panel data and we estimate the model using first differencing. Could it be possible that the biases are larger for panel data than for ordinary least squares?
6. What is the Wald estimator for this model? Under what assumptions is the Wald estimator consistent? Are there any disadvantages associated with the Wald estimator?
7. Suppose that we run the reverse regression of \( x \) on \( y \). How can we use this along with the direct regression of \( y \) on \( x \) to bound the true parameter?
Question 3

Let $\hat{\pi}_n$ be an unrestricted consistent estimator of a $k$-vector parameter $\pi_0$. Suppose $\pi_0$ is a known function of a $d$-vector parameter $\theta_0$ ($d \leq k$)

$$\pi_0 = g(\theta_0).$$

Let $A_n$ be a $k \times k$ random weight matrix. Then, the minimum distance (MD) estimator $\hat{\theta}_n$ is obtained as

$$\hat{\theta}_n = \arg\min_{\theta \in \Theta} Q_n(\theta) \equiv \|A_n(\hat{\pi}_n - g(\theta))\|^2 / 2.$$

Now consider the following problem. We have

$$\pi_{10} = g_1(\theta_{10}) \quad (1)$$
$$\pi_{20} = g_2(\theta_{10}, \theta_{20}) \quad (2)$$

where $g_1(\cdot)$ and $g_2(\cdot)$ are known functions. $\pi_{10}$ is $k_1 \times 1$, $\pi_{20}$ is $k_2 \times 1$, $\theta_{10}$ is $d_1 \times 1$, and $\theta_{20}$ is $d_2 \times 1$ with $d_1 \leq k_1$ and $d_2 \leq k_2$. We also let $\hat{\pi}_{1n}$ and $\hat{\pi}_{2n}$ be unrestricted consistent estimators of $\pi_{10}$ and $\pi_{20}$, respectively.

**Answer the followings:**

1. Construct the two step MD estimation for $\theta_{20}$ (you may let $A_n$ be an identify matrix)

2. Discuss the consistency of the two step MD estimator $\hat{\theta}_{20}$ i.e., provide conditions under which the estimator is consistent.

3. Derive the asymptotic distribution of $\sqrt{n}(\hat{\theta}_{20} - \theta_{20})$. Clearly state your assumptions in the derivation.

4. Suppose $\theta_{10}$ is known and obtain the (infeasible) MD estimator for $\hat{\theta}_{20}$. Note that the estimator is obtained from the MD estimation based on (2), which is now a function of $\theta_{20}$ only.

5. Compare the asymptotic variances of $\hat{\theta}_{20}$ and $\hat{\theta}_{20}$. Can we say that $\hat{\theta}_{20}$ is better than $\hat{\theta}_{20}$ in terms of the asymptotic variance? Justify your answer.
Question 4

Consider the dynamic, linear, cross country, random effects regression model

\[ y_{it} = \alpha + \beta x_{it} + \delta z_{it} + \gamma y_{it-1} + u_i + e_{it}, \quad t = 1, \ldots, 5 \] (\( y_{i0} \) is observed, but \( x_{i0} \) and \( z_{i0} \) are not).

in which \( i \) is a country and \( t \) is a year; \( y_{it} \) is national income per capita, \( z_{it} \) is domestic investment and \( x_{it} \) is a measure of national labor input. You have 100 countries (\( N=100 \)) and 5 years of data (\( T=5 \)).

(a) Is pooled OLS estimator consistent? Is GLS estimator inconsistent? Explain.

(b) Let \( w_{it} = y_{it} - \alpha - \beta x_{it} - \delta z_{it} - \gamma y_{it-1} \). Assume that \( x_{it} \) and \( z_{it} \) are pre-determined. Consider the set of instruments \( f_{it} = (1, x_{it}, z_{it}, x_{it-1}, z_{it-1}) \). Does the strategy of pooling the panel and simply using two stage least squares with \( f \) as the set of instruments produce a consistent estimator of the parameters? Explain.

(c) I propose to use a GMM estimator based on the moment conditions corresponding to \( E[f_{it} w_{it}] = 0 \), \( t=2,3,4,5 \). How many moment conditions do we have in total? Describe how the GMM estimation will proceed.

(d) Now suppose that strict exogeneity holds for \( x \) and \( z \). Write down additional moment conditions that you can exploit for estimation. How does this change the computations in (c)?

(e) Now let’s suppose that the regression model in fact does not satisfy random effects assumptions: Rather, fixed effects model is the right one to use for the problem because \( x_{it} \) and \( z_{it} \) could be correlated with \( u_i \). Are the estimators proposed in (b) and (c) consistent when our model is a fixed effects model?

(g) Continue to assume that FE model is the right one to use for our problem. Propose a way to consistently estimate \( \beta \) in this setting. Clearly specify the assumptions under which your instruments are valid.