The time limit for this exam is *four* hours.

You should answer *five* questions:

- *TWO* questions from Part I; and
- *THREE* questions from Part II;

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PART I – SHORT QUESTIONS

Answer TWO (2) questions from Part I.
Question I.1

Consider the following statement about competitive equilibrium in (two-dates) security markets with strictly risk-averse investors:

Expected return on every risky security is greater than the risk-free return.

Is this statement true or not? Justify your answer.

Question I.2

Describe (a version of) the Ellsberg paradox. Can the pattern of preferences in the Ellsberg paradox be explained using non-additive (Choquet) expected utility? Justify your answer.

Question I.3

Answer all of the following with true or false (with clarification if necessary).

(a) The market capitalization of U.S. corporations has never been below 0.5 times U.S. GDP.
(b) The market capitalization of U.S. corporations has never been above 1.5 times U.S. GDP.
(c) The U.S. share of world equity market capitalization exceeds the U.S. share of world GDP.
(d) Corporate profits of foreign subsidiaries of U.S. companies is included in U.S. direct investment payments.

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Question I.4

Suppose \( \pi_t \) is a state price process that follows \( d\pi_t = \pi_t(-r_t dt + \sigma'_t dW_t) \) and that correctly prices the \( N \) cumulative returns \( dR_t = \text{diag}(R_t)(\mu_t dt + \Sigma_t dW_t) \). Here \( W_t \) is a \( K \)-vector of independent Brownian Motions, \( K > N \), and \( \Sigma_t \) has full rank.

Show that
\[
\sigma'_t \sigma_t \geq (\mu_t - r_t \iota_N)'(\Sigma_t \Sigma_t')^{-1}(\mu_t - r_t \iota_N).
\]

Question I.5

Let \( \{\pi_t\}_{t=0}^\infty \) be a state price process and let \( Q_t[N] \) be the price of bond at time \( t \) that pays one unit of numeraire at \( t + N \). Define \( R_{t+1}[N] = Q_{t+1}[N-1]/Q_t[N] \) and let \( R_{t+1} \) be the gross return from \( t \) to \( t+1 \) on some other security. Show that
\[
E_t [\ln (R_{t+1}[N])] - E_t [\ln (R_{t+1})] \geq E_t \left[ \ln \left( \frac{E_{t+1}[\pi_{t+1}]}{E_t[\pi_{t+1}]} \right) \right].
\]

If \( \pi_t = \gamma^t x_t \) where \( x_t > 0 \) is stationary with a finite unconditional mean, what does this imply for the expected return on long bonds versus stocks in this economy?
PART II – LONG QUESTIONS

Answer THREE (3) questions from Part II.
Question II.1

Consider security markets with two dates and two states at date 1. There is no consumption at date 0. There are two agents: risk-neutral agent 1 with von Neumann-Morgenstern utility function

\[ v^1(y) = y, \]

and strictly risk-averse agent 2 with von Neumann-Morgenstern utility function

\[ v^2(y) = \ln(y). \]

Endowments at date 1 are \( \omega^1 = (3, 3) \) and \( \omega^2 = (4, 2) \). Probabilities are \( \frac{1}{2} \) for states 1 and 2, the same for both agents. There are two securities with payoffs \( x_1 = (1, 1) \) and \( x_2 = (1, 2) \).

(a) Show that all interior Pareto optimal allocations have risk-free consumption for agent 2.

(b) Find an equilibrium in security markets.

(c) Suppose now that agent 2 has a multiple-prior expected utility instead of the expected utility. Her von Neumann-Morgenstern utility is the same. Her set of multiple priors is \( \{ (\pi_1, \pi_2) : 0.4 \leq \pi_1 \leq 0.6, \ \pi_1 + \pi_2 = 1 \} \). The expected utility of agent 1 remains unchanged. Is the equilibrium of your answer (ii) still an equilibrium for this new specification of utility of agent 2? Justify your answer.
Question II.2

Consider security markets with infinite time-horizon and uncertainty described by an event tree with a finite number of events at every date \( t = 1, 2, \ldots \). There are \( J \) securities with dividends \( x_j(s_t) \geq 0 \) for every \( j \) and every event \( s_t \) at date \( t \geq 1 \). Securities are traded at every date and in every state prior to their expiration date (which may be finite or infinite). There are \( N \) agents and a single good available for consumption at every date. Each agent \( i \) has strictly increasing preferences over infinite-time event-contingent consumption plans, an event-contingent endowment of the good at every date \( t \) (denoted by \( \omega_i^t \)) and an initial portfolio of securities (denoted by \( \alpha^i \in \mathbb{R}^J_+ \)). Consumption is restricted to be positive. You may assume that agents' preferences have discounted expected-utility representation.

Suppose that agents' portfolio holdings are restricted by debt constraints of the form

\[
[p(s_t) + x(s_t)]h(s_{t-1}) \geq -D(s_t), \quad \forall s_t, \quad t \geq 1,
\]

for some sequence of positive bounds \( \{D(s_t)\} \), where \( p(s_t) \) denotes vector of security prices in event \( s_t \) and \( h(s_{t-1}) \) denotes the portfolio from the predecessor event of \( s_t \) at date \( t - 1 \).

(a) State a definition of an equilibrium under the debt constraint.

(b) State a definition of price bubble. Prove that there cannot be a price bubble on any security whose expiration date is finite.
Question II.3

Consider an economy with a representative consumer whose preferences are

$$ E_0 \left[ \int_0^\infty e^{-\rho t} u(c_t) dt \right], $$

where $c_t > 0$, $u(c) = c^{1-\gamma}/(1 - \gamma)$ and $\gamma$ is positive but not equal to 1.

There is a capital stock $k_t$ that evolves according to

$$ dk_t = k_t \left[ -\delta dt + \sigma dW_t \right] + g_t dt, $$

where $W_t$ is a standard Brownian motion, $\delta$ and $\sigma$ are positive parameters, and $g_t$ is the flow of new capital. The resource constraints at time $t$ are

$$ x_t + z_t \leq k_t $$
$$ c_t + a_t z_t \leq A x_t $$
$$ g_t \leq G(z_t, a_t z_t) $$
$$ 0 \leq a_t, c_t, x_t, z_t. $$

Here, $A$ is positive and $G$ is a smooth production function that exhibits constant returns to scale. Note well that there is no joint production of goods and new capital in this economy.

Markets are complete. Write $\pi_t$ for the equilibrium state price process and $q_t$ for the price of capital in units of consumption. Let $v_t$ be the rental price of capital, also in units of consumption. Conjecture that it is optimal, at all times, to produce a strictly positive flow of new capital and choose $c_t = \phi k_t$ for some $\phi \in (0, A)$. Use this conjecture in answering the following questions.

(a) What does the fact that output and new capital are produced at the same time tell you about $v_t$ and $q_t$? Show that $a_t = a$ solves $G(1, a) = D_2 G(1, a)(a + A)$.

(b) Verify that $k_t$ follows a geometric Brownian motion and give an expression for its drift $k_t \mu$. Describe the equilibrium state price process.

(c) Now do a present-value calculation to obtain an alternative expression for $q_t/v_t$. You may use the fact that $y_t = \exp \left( -\frac{1}{2} \sigma^2 t + \sigma W_t \right)$ implies $E_0 [y_t^\alpha] = \exp \left( -\frac{1}{2} \alpha (1 - \alpha) \sigma^2 t \right)$ to simplify your algebra.

(d) Explain how you can now determine the conjectured $\phi$. If time permits, show that

$$ \phi = \frac{a + A}{\gamma G(1, a)} \left( \rho + (1 - \gamma) \left[ \delta - \frac{AG(1, a)}{a + A} \right] + \frac{1}{2} \gamma (1 - \gamma) \sigma^2 \right), $$

where $a$ was determined in (a).
Question II.4

Consider a manager with a project that has cumulative payoffs $Y_t$ that evolve according to

$$dY_t = (\mu - a_t)dt + \sigma dW_t,$$

where $W_t$ is a standard Brownian motion and $a_t \in [0, \infty)$ is the manager’s action at time $t$. Given actions $a_t$, non-decreasing cumulative consumption $C_t$, and a termination date $\tau$, the expected discounted utility at time $t$ of the manager is

$$e^{-\rho t}V_t = E_t \left[ \int_t^\tau e^{-\rho s} \left[ dC_s + \lambda a_s ds \right] + e^{-\rho \tau}U \right],$$

where $\lambda \in (0, 1)$, $\rho > 0$ and $U > 0$. The expected discounted value of a bank receiving the payoffs of the project and paying the manager is

$$e^{-rt}P_t = E_t \left[ \int_t^\tau e^{-rs} \left[ dY_s - C_s \right] + e^{-r\tau}S \right],$$

where $0 < r < \rho$ and $S > 0$. Assume further that $\mu > r(S + U)$.

(a) Explain why $a_t = 0$ and $\tau = \infty$ is optimal in the absence of incentive constraints.

From here on, suppose the bank cannot see $a_t$ and the manager can quit at will. The project payoffs go directly to the bank. Conjecture that, in an optimal contract, the bank lets the manager quit when the manager’s discounted utility $V_t$ reaches $U$ and pays the manager enough to keep $V_t$ from rising above some $B > U$. Inside $(U, B)$, conjecture $a_t = 0$ and $dC_t = c_t dt$.

(b) Explain why it is that $dV_t = [\rho V_t - c_t]dt + g_t \sigma dW_t$ for some $g_t$.

(c) Suppose $c_t = c[V_t]$ and $g_t = g[V_t]$. How can the bank compute the promised utilities $V_t$ given that it only sees $Y_t$? Show that incentive compatibility requires that $g[V] \geq \lambda$ for all $V \in (U, B)$.

(d) Write $P_t = F(V_t)$. Determine the Bellman equation for $F$ inside $(U, B)$ and find the optimal $c[V]$ and $g[V]$. What do you know about $F(U)$?

(e) Given that the bank pays the manager just enough to keep $V_t$ from rising above $B$, what do you know about $DF(B)$?
Question II.5

President Obama is proposing the following changes to U.S tax law:

(i) to raise taxes on qualified dividends and capital gains from 15 percent to 20 percent;
(ii) to increase marginal tax rates on ordinary taxable income (Form 1040);
(iii) to “reform” rules allowing multinationals to defer U.S. tax on profits earned overseas and not yet repatriated.

Using economic theory, explain how these proposed changes would affect corporate capital stocks and valuations.
Question II.6

Some investment in business capital is capitalized and some is expensed. For the most part, investment in tangible capital (like structures and equipment) is capitalized and investment in intangible capital (like R&D and brand equity) is expensed. Using economic theory, explain how tax policies can affect:

(a) the allocation of tangible and intangible capital;

(b) the level of tangible and intangible capital;

(c) the relative price of tangible to intangible capital.

Fill in any details needed to answer these questions.