Ph.D. Preliminary Examination

MICROECONOMIC THEORY

MAJORS

Spring 2009

The time limit for this exam is $3\frac{1}{4}$ hours.

Answer one question from each part, for a total of four questions.

You may use calculators to make calculations during the examination. However: in answering any question that requires you to justify your answer, if you do use a calculator, you must mention at what point in your answer you obtained results using a calculator, and what it was you were seeking to calculate.

BE SURE you clearly define all **boldfaced/underlined** terms. Also, please be sure to define precisely any notation that you introduce.

NOTE: This examination should have 14 pages including this one (Check to make sure!)
Part I

Answer one question from Part I.
Question I.1

Consider a pure exchange economy with two traders (indexed by subscripts 1 and 2) and \( \ell \) commodities. Suppose that both initial endowment vectors, \( e_1 \) and \( e_2 \), are strictly positive (\( e_i \in \mathbb{R}^{\ell}_{++}, \ i = 1, 2 \)). Let \( F = \{ (x_1, x_2) \in \mathbb{R}^{2\ell}_+ | x_1 + x_2 = e_1 + e_2 \} \) denote the set of feasible allocations (with no free disposal). Suppose that each trader \( i = 1, 2 \) has a preference relation \( \preceq_i \) defined on \( F \) which is representable by a utility function \( u_i: F \to \mathbb{R} \).

(a) Define what it means for \( u_i \) to represent \( \preceq_i \).

(b) If there is a utility \( u_i: F \to \mathbb{R} \) that represents \( \preceq_i \), what assumption must \( \preceq_i \) satisfy?

(c) What additional assumptions (be very precise) are needed for the first welfare theorem to hold in this economy?

(d) What minimal additional assumptions on \( \preceq_1 \) and \( \preceq_2 \) guarantee that \( u_1 \) and \( u_2 \) can always be chosen to represent \( \preceq_1 \) and \( \preceq_2 \) respectively such that at any Pareto optimal allocation \( \hat{x} \in \mathbb{R}^{2\ell}_+ \), \( u_1(\hat{x}) < u_2(\hat{x}) \)?

Can you find an example of \( \preceq_1 \) and \( \preceq_2 \) on \( F \) such that there is a utility \( \bar{u}_1: F \to \mathbb{R} \) representing \( \preceq_1 \) for which there is no \( \bar{u}_2: F \to \mathbb{R} \) representing \( \preceq_2 \) such that \( \bar{u}_1(\hat{x}) < \bar{u}_2(\hat{x}) \) for every Pareto optimal allocation \( \hat{x} \). Hint: Consider linear preferences of \( F \) such that \( x' \sim_1 x'' \) if and only if \( x_{11}' + x_{12}' = x_{11}'' + x_{12}'' \) and \( x' \sim_2 x'' \) if and only if \( x_{21}' + x_{22}' = x_{21}'' + x_{22}'' \) where \( x \in F \) is written as \( x = (x_{11}, x_{12}, x_{21}, x_{22}) \) but suppose that person 1 is indifferent between consuming all of the economy’s resources and consuming nothing.
Question I.2

In general equilibrium theory, changing from a model with a finite number of traders to one with uncountably many—in fact, an atomless continuum of traders—can lead to better results on the existence of competitive equilibrium and its welfare properties. Consider a pure exchange economy in which there are $\ell$ commodities and each trader $i$ has consumption set $X_i \subseteq \mathbb{R}_+^\ell$ and $e_i \in \text{int}(X_i)$, so that $e_i \gg 0$. For each complication below, explain whether changing from a finite number of traders to an atomless continuum leads to either better results or simpler/easier proofs for the existence of competitive equilibrium and the first and second fundamental theorems of welfare economics. Explain your reasoning briefly. You may assume that preferences $\preceq_i$ are strictly monotone complete continuous preorders defined on $X_i$.

(a) Preferences that are convex but not strictly convex.

(b) Nonconvex preferences.

(c) Nonconvex consumption sets that are closed (and bounded from below since $X_i \subseteq \mathbb{R}_+^\ell$).

(d) Consumption externalities.
Part II

Answer one question from Part II
Question II.1

Consider a profit maximizing firm with single output and \( n \) inputs, with production function \( f: \mathbb{R}^n_+ \to \mathbb{R}_+ \) assumed strictly increasing, continuous (but possibly nondifferentiable), and \( f(0) = 0 \). Let \( q \in \mathbb{R}_{++} \) be the price of output and \( w \in \mathbb{R}^n_{++} \) be the vector of prices of inputs. The firm is taxed at rate \( t > 0 \) of its total cost. The firm’s profit maximization problem is

\[
\max_{x \geq 0} [qf(x) - wx - t(wx)].
\]

Let \( x^*(t) \) denote the profit maximizing vector of inputs (assumed unique) as function of tax rate \( t \).

(a) State a definition of production function \( f \) being supermodular. State a criterion for supermodularity of \( f \) under an additional assumption that \( f \) is twice differentiable.

(b) Show that if \( f \) is supermodular, then input demand \( x^* \) is a nonincreasing function of \( t \), that is, if \( t' \geq t \), then \( x^*(t') \leq x^*(t) \). If you use a known mathematical theorem in your proof, make sure that you state that theorem clearly.
Question II.2

Consider an agent whose preferences over state-contingent consumption plans on a finite state space $S$ have an expected utility representation with strictly increasing and twice-differentiable utility function $v: \mathbb{R} \to \mathbb{R}$ and probability measure $\pi$ on $2^S$.

Prove that the agent is risk averse if and only if $E[v(\tilde{z})] \geq E[v(\tilde{y})]$ for every $\tilde{y}: S \to \mathbb{R}_+$ and $\tilde{z}: S \to \mathbb{R}_+$ such that $E(\tilde{y}) = E(\tilde{z})$ and $\tilde{y}$ is more risky than $\tilde{z}$. Expected value $E$ is taken with respect to probability measure $\pi$.

Your definition of more risky should be stated in terms of cumulative distribution functions of $\tilde{y}$ and $\tilde{z}$. You may use the Theorem of Pratt without proving it.
Part III

Answer one question from Part III
An extensive form game (EFG) is said to be *linear* if every information set is crossed at most once by every history.

(a) Give an example of an EFG which is not linear.

(b) Compare linear games and games with perfect recall. Is one of the two a subset of the other? Prove your answer.
Question III.2

Part 1:
Find the Nash equilibria of the following game.

(a) Two players move sequentially, an initial amount of $I$ dollars is paid by a third part into a common fund.

(b) Player 1 and player 2 pick a card out of a set of three cards numbered 1, 2, and 3.

(c) Player 1 has decide whether he bets or folds. If player 1 folds, player 2 gets an amount of $I$ dollars and the game is over. If player 1 bets, he has to pay $B$ dollars to the common fund, and the game goes to the next stage.

(d) Player 2 is informed of the decision of player 1, and has to decide whether he bets or folds. If player 2 folds, player 1 gets the $I$ dollars, and gets the $B$ amount back. If player 2 bets, he has to pay $B$ dollars to the common fund, and the game goes to the next stage.

(e) Both players show their card; the player with the highest number wins the amount in the common fund, that is the $I$ amount and the amount $2 \times B$ paid in the earlier stages.

The utility of monetary amounts is linear. The equilibria may depend on the values of the parameters $I$ and $B$: please specify the set as a function of these parameters.

Part 2:
Find the Nash equilibria of the game described in the first part, where the two players pick each one out of four cards, numbered 1, 2, 3, and 4. In the final stage card 4 beats card 3, which beats card 2, which beats card 1.
Part IV

Answer one question from Part IV
Question IV.1

Suppose there are two urns, each containing a number of balls that are either Red or Blue. Suppose we are to draw one ball from each urn. There are four possibilities: $RR$ (both balls are red), $RB$ (the ball from the first urn is red, and the one from the second is blue), $BR$ (the ball from the first urn is blue, and the one from the second is red), and $BB$ (both balls are blue). Let $\{RR, RB, BR, BB\}$ be the state space.

Consider four acts, denoted $1R, 1B, 2R, 2B$, where $1R$ means betting on a red ball out of the first urn, $2R$ on a red ball out of the second urn, etc. These four acts are summarized by the following matrix:

\[
\begin{array}{cccc}
RR & RB & BR & BB \\
1R & 1 & 1 & 0 & 0 \\
1B & 0 & 0 & 1 & 1 \\
2R & 1 & 0 & 1 & 0 \\
2B & 0 & 1 & 0 & 1 \\
\end{array}
\]

Suppose an agent is indifferent between acts $1R$ and $1B$, and between acts $2R$ and $2B$. But suppose he strictly prefers act $1R$ to act $2R$.

(a) State Savage’s axiom P2 (also known as the Sure Thing Principle).

(b) Explain why the agent’s preference violates Savage’s axiom P2.
Construct an example of a social choice function that is:

(a) strategy-proof,

(b) not Maskin-monotonic,

(c) fully/strongly implementable in dominant strategy equilibrium by some indirect mechanism,

(d) not fully/strongly implementable in dominant strategy equilibrium by the direct revelation mechanism.

Please explain how your example satisfied each of the above four properties.
Question IV.3

Consider a quasilinear environment where two agents are to contribute to a public project. Let $K = \{0, 1\}$ be the possible levels of the project, with 1 meaning that the project is “done”, and 0 “not done”. Agent $i$’s private valuation of the project is denoted by $\theta_i$, which is independently drawn from a uniform distribution on $[0, 1]$. The project costs $c$ to finish, where $c$ is a constant strictly in between 0 and 2. Let $k: \Theta \to K$, where $\Theta = \Theta_1 \times \Theta_2 = [0, 1] \times [0, 1]$, denote the following allocation function:

$$k(\theta_1, \theta_2) = \begin{cases} 1 & \text{if } \theta_1 + \theta_2 \geq c \\ 0 & \text{otherwise}. \end{cases}$$

A transfer rule $t: \Theta \to \mathbb{R}^2$ specifies, for each agent $i$, the amount of monetary transfer received by agent $i$ at each $\theta = (\theta_1, \theta_2) \in \Theta$. Writing $t(\theta)$ as $(t_1(\theta), t_2(\theta))$, we say that the transfer rule $t$ balances the budget if, for any $\theta$,

$$t_1(\theta) + t_2(\theta) = \begin{cases} -c & \text{if } k(\theta) = 1 \\ 0 & \text{otherwise}. \end{cases}$$

Construct a budget-balancing transfer rule $t$ that Bayesian-implements the allocation rule $k$; i.e., construct a budget-balancing transfer rule $t$ such that the social choice function $(k, t)$ is Bayesian incentive compatible.