Ph.D. Preliminary Examination

MICROECONOMIC THEORY

MINORS

Spring 2009

The time limit for this exam is $3\frac{1}{4}$ hours. Notation:

\[ \mathbb{IR} \text{ is the set of real numbers} \]
\[ \mathbb{IR}^n_+ = \{x \in \mathbb{IR}^n : x_1 \geq 0 \& \ldots \& x_n \geq 0\} \]
\[ \mathbb{IR}^n_{++} = \{x \in \mathbb{IR}^n : x_1 > 0 \& \ldots \& x_n > 0\} . \]

For vectors \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) in \( \mathbb{IR}^n \):

\[ x \geq y \text{ means } x_1 \geq y_1 \& \ldots \& x_n \geq y_n. \]

Answer Question I.1 (required);

AND

Answer three additional questions, one from each of Parts II, III, and IV. (So the total number of questions to be answered for the exam is four).

You may use calculators to make calculations during the examination. However: in answering any question that requires you to justify your answer, if you do use a calculator, you must mention at what point in your answer you obtained results using a calculator, and what it was you were seeking to calculate.

BE SURE you clearly define all **boldfaced/underlined** terms.

NOTE: This examination should have 16 pages including this one (Check to make sure!)
Part I

Answer Question I.1 from Part I.
Question I.1

Consider a two-consumer, two-good pure exchange economy. Consumer 1 has a preference relation represented by the utility function:

$$U^1(x_1, y_1) = x_1^2 y_1 \quad \text{for } x_1 \geq 0 \text{ and } y_1 \geq 0.$$  

Consumer 2 has a preference relation represented by the utility function:

$$U^2(x_2, y_2) = x_2^2 y_2 \quad \text{for } x_2 \geq 0 \text{ and } y_2 \geq 0.$$  

Consumer 1’s endowments for each good are $x^e_1 = 5$ and $y^e_1 = 15$. Consumer 2’s endowments for each good are $x^e_2 = 15$ and $y^e_2 = 5$.

(a) Construct an Edgeworth box diagram (to scale, on the graph paper provided) showing (and labeling) the endowment allocation and typical indifference contours/sets including the directions of increasing preference for each consumer.

(b) Characterize the set of Pareto Efficient allocations and illustrate them on your diagram.

(c) Identify the competitive (Walras) equilibrium allocation including the equilibrium prices and illustrate it on your diagram.
Part II

Answer one (1) question from Part II
Question II.1

Consider a consumer with the preference relation represented by the utility function

\[ U(x, y) = 2x^{1/2}y^{1/2} \quad \text{for } x \geq 0 \text{ and } y \geq 0. \]

Let \( w > 0 \) be the income the consumer has available to spend on \( x \) and \( y \). Let \( p_x > 0 \) and \( p_y > 0 \) be the competitive market prices for \( x \) and \( y \). Furthermore, suppose the government imposes a unit surcharge of \( t > 0 \) on good \( x \) for every unit in excess of \( \bar{x} \).

(a) Assuming \( \bar{x} \geq x \), set up the consumer’s optimization problem and derive her Marshallian demands for \( x \) and \( y \).

(b) Assuming \( x \geq \bar{x} \), set up the consumer’s optimization problem and derive her Marshallian demands for \( x \) and \( y \).

(c) Use your response to part (a) and (b) to specify the consumer’s Marshallian demand for \( x \) and \( y \) in general.

(d) Are these Marshallian demands homogeneous of degree 0 in \( w, p_x, p_y, \) and \( r \)? Explain.

(e) Do these Marshallian demands satisfy Walras Law? Explain.

(f) Will these Marshallian demands be unique for all \( w, p_x, p_y \) and \( t \)? Explain.
Question II.2

Consider a firm that produces a single output, \( q \geq 0 \), using two factors, \( z_1 \geq 0 \) and \( z_2 \geq 0 \) with an input requirement set that is regular, monotonic, and strictly convex. Assume the firm operates in competitive factor markets. The cost function for the firm is

\[
c(r_1, r_2, q) = r_1^{1/3} r_2^{2/3} q^{4/3}
\]

where \( r_1 > 0 \) and \( r_2 > 0 \) are factor prices.

(a) Derive the firm’s conditional factor demands.

(b) What are four other conditions that a valid cost function must satisfy? Verify that these conditions hold for the cost function above.

(c) Use the conditional factor demands you derived in part (a) to derive the production function that produced this cost function.

(d) Is this production function homothetic? Explain.

(e) Does this production function exhibit increasing, decreasing, or constant returns to scale? Explain.
Question II.3

Consider the regular factor requirement set

\[ Z(q) = \left\{ (z_1, z_2) \in \mathbb{R}^2_+ : \min \left\{ \frac{z_1^{1/2}}{z_1^{1/2}}, \frac{z_2^{1/2}}{z_2^{1/2}} \right\} \geq q \right\} \text{ for } q \geq 0. \]

(a) Verify that this factor requirement set is monotonic.

(b) Verify that this factor requirement set is convex.

(c) Derive the firm’s cost function assuming factor markets are competitive with factor prices \( r_1 > 0 \) and \( r_2 > 0 \) for \( z_1 \) and \( z_2 \).

(d) Use the cost function you derived in part (c) to derive the profit function assuming a competitive output market where the price of output is \( p > 0 \).

(e) Use the profit function you derived in part (d) to derive the unconditional factor demands.

(f) List three other properties that this profit function will satisfy.
Part III

Answer one (1) question from Part III
Question III.1

Consider Cournot duopoly with two firms, $i = 1, 2$, selling a single output good. Both firms produce at constant unit cost $c$. There is uncertainty about the inverse market demand function. Specifically, the function can be

\[ p = a - b(q_1 + q_2) \]

with probability $\pi$ or

\[ p = \bar{a} - b(q_1 + q_2) \]

with probability $1 - \pi$, where $p$ denotes the price of the output good and $q_1$ and $q_2$ denote firms’ output quantities. Suppose that firm 1 knows which market demand function it is, while firm 2 does not know.

(a) State a definition of Bayes-Nash equilibrium in this duopoly game.

(b) Find a Bayes-Nash equilibrium for the following specification of parameters:
\[ \pi = 1/2, \quad a = 3, \quad \bar{a} = 5, \quad b = 1, \quad \text{and} \quad c = 1. \]
Question III.2

Consider the following statement concerning a game \((N, \{S_i\}, \{u_i\})\) with \(N\) players \(i = 1, \ldots, N\) whose (finite) strategy sets are \(S_i\) and payoff functions are \(u_i\):

If \(\hat{m}\) is a **mixed strategy Nash equilibrium**, then \(u_i(\hat{m}_i, \hat{m}_{-i}) = u_i(s_i, \hat{m}_{-i})\) for every \(s_i \in S_i\) such that \(\hat{m}_i(s_i) > 0\).

(a) Prove this statement.

(b) Give an example of a game in which there is a Nash equilibrium \(\hat{m}\) such that \(\hat{m}_i\) is not a pure strategy for at least on player \(i\), and show that the statement holds for \(\hat{m}\) in your example.
Part IV

Answer one (1) question from Part IV
Question IV.1

In this question, there are two states of the world, Heads and Tails, which are equally probable. Write \( \Omega = \{H, T\} \) for convenience. Information means that a decision maker knows whether the state is heads or tails before making his or her choice.

(a) Suppose that a single expected utility maximizer decision maker has a choice set \( C \subseteq \mathbb{R}^n \) which is closed and bounded, and state-dependent (cardinal) utility functions \( U_H: C \to \mathbb{R} \) and \( U_T: C \to \mathbb{R} \) which are both assumed to be continuous. Prove that the value of information cannot be strictly negative. Does your proof require any additional assumptions such as convexity of \( C \) or concavity of \( U_H \) and \( U_T \)? Briefly explain why or why not.

(b) For the decision maker in part (a), let \( x: \Omega \to C \) be a state-dependent allocation and define **incentive compatibility** of \( x \).

(c) Now examine whether the value of information can be negative in strategic situations by considering an extensive form game with incomplete information in which nature (a nonstrategic player receiving no payoff) first chooses Heads or Tails, each with probability one half, and then two players (I and II) play the following extensive form game where \( \lambda > 0 \) is a strictly positive scalar parameter. Assume that everything except whether Heads or Tails has occurred is common knowledge. You may also assume that before II makes a decision, II sees I’s move (and this fact is common knowledge too). Take payoffs to be denominated in dollars (and assume risk neutrality).
Question IV.1 continued:

Extensive form game in state Heads:

Question IV.1 continued on next page
Question IV.1 continued:

Extensive form game in state Tails:

(i) Find the subgame perfect equilibria if player I does not know whether Heads or Tails has occurred (and it is common knowledge that neither player has such information).

(ii) Now assume that both players are informed (and this is common knowledge) and find the subgame perfect equilibria for state Heads or Tails.

(iii) Find the value of information to I.

Question IV.1 continued on next page
Question IV.1 continued:

(iv) Find the value of information to II as a function of $\lambda$.

(v) If $\lambda = 25$, can I increase his or her own payoff (where both players are still informed and this is common knowledge) by threatening to always go down (D)? Explain briefly.

(vi) For what values of $\lambda$ would II be willing to pay I to credibly destroy the information?

(vii) Finally, return to the earlier statement in the question that “you may also assume that, before II makes a decision, II sees I’s move (and this is common knowledge too.)” and explain that this assumption does not affect the subgame perfect equilibrium outcomes but it does affect the subgame perfect equilibrium strategies for some (identify which ones) of the informational situations analyzed above.
Question IV.2

Consider a simplified mortgage lender and borrower in a nonrecourse jurisdiction (which means that, in case of default, the lender receives the house but cannot collect additional money from the borrower). The borrower can pay a downpayment $D \in \mathbb{R}$ and borrow $M \in \mathbb{R}$ to purchase a house at price $P \in \mathbb{R}$, where $P = M + D$ and $0 \leq D \leq P$. In the following period, the final valuation of the house is $V \in \mathbb{R}$ which is observable to all, the borrower receives wealth $W \in \mathbb{R}$ where $W > V$ and $W > M$, and the mortgage requires repayment of $M + I$ where $I \geq 0$ (think of $I \in \mathbb{R}$ as the interest). To simplify, assume that there is no time discounting and that default incurs no penalties on the borrower other than the loss of the house (no credit rating effects, etc.)

(a) Suppose that the valuation $V$ is a random variable drawn after the mortgage is signed (i.e., at the beginning of the final period), where $V = P + \tilde{\epsilon}$ and the “noise” term $\tilde{\epsilon} = +\epsilon$ with probability one-half and $\tilde{\epsilon} = -\epsilon$ with probability one half. Show that, depending on the value of the non-negative scalar $\epsilon \in \mathbb{R}_+$ (specify the conditions) either a rational borrower never defaults, the borrower always defaults, or the borrower defaults with probability one half.

(b) If $D = 0$ (no downpayment) and $I = 0$ (no interest, so that just the principle must be repaid), show that default occurs with probability one-half for all values of $\epsilon > 0$ but riskier distributions of $\tilde{\epsilon}$ (define them) lead to larger expected losses for the lender.

(c) Define moral hazard and adverse selection and explain exactly where they arise in this model, if they do.

(d) Return to part (a) and discuss the effect that higher down payments and higher interest rates have on default for different values of $\epsilon$. 
