Instructions: There are three sections, each divided into two parts. The first part in each section consists of short questions, the second part consists of long questions.

Answer one question from each of the six parts.

The time limit for this exam is 4 hours.

Note: This examination should have 8 pages including this one (Check to make sure!)
Section A
Answer one short question and one long question from Section A.

Question A.I.1: Short

Suppose that a seller runs a second price auction for a single object with a reserve price $r$. The only choice that the seller has prior to the auction is how to set the reserve price. Suppose that all bidders are independently drawn from a common differentiable distribution $F$ on $[0, 1]$ with density $f$ such that $f(x) > 0$ for all $x \in [0, 1]$. $F$ has a monotone hazard rate. Does the optimal reserve price increase, decrease or stay the same as the number $n$ of bidders increases? Explain your answer.

Question A.I.2: Short

Consider a simultaneous ascending auction for two goods with two bidders, Ann and Bob. The initial price for each good is zero and the bid increment is 1. Ann values a single good at 8, but only wants one good. She is indifferent between the two goods. When Ann already has one good, then she gains no additional value from consuming the second good. Bob values each individual good at 10, and values the package containing both goods at 20. Ann and Bob know one another’s values. What is straightforward bidding? Is there a Nash equilibrium in which both bidders bid straightforwardly? Explain your answer.
Question A.II.1: Long

Suppose that there is a single good and two bidders, Ann and Bob. Ann’s value for the good is $v_A$ and Bob’s value for the good is $v_B$. Both bidders may either have a value of 1 or a value of 5. Let $\alpha \in [0, \frac{1}{2}]$. The probability that bidders have any combination of values is given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>$v_A = 1$</th>
<th>$v_A = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_B = 1$</td>
<td>$\alpha$</td>
<td>$(1 - 2\alpha)/2$</td>
</tr>
<tr>
<td>$v_B = 5$</td>
<td>$(1 - 2\alpha)/2$</td>
<td>$\alpha$</td>
</tr>
</tbody>
</table>

For example the probability that $v_A = 1$ and $v_B = 1$ is $\alpha$, and the probability that $v_A = 1$ and $v_B = 5$ is $(1 - 2\alpha)/2$. Recall that a mechanism is (interim) individually rational if for each agent and each type of that agent, the agent receives a non-negative expected utility conditional on her type by participating in the mechanism.

(a) Find a revenue-maximizing incentive compatible and individually rational mechanism when $\alpha = \frac{1}{8}$.

(b) Find a revenue-maximizing incentive compatible and individually rational mechanism when $\alpha = \frac{1}{4}$.

(c) Let $f : [0, \frac{1}{2}] \rightarrow \mathbb{R}$ be the function which maps the probability $\alpha$ (in the above table) to the resulting expected revenue in the revenue-maximizing mechanism. What function is $f$? Is $f$ continuous? Explain your answer. (In answering this question, you may appeal to any theorem about auctions as long as you clearly state it).
Question A.II.2: Long

Consider the VCG mechanism applied to a combinatorial auction with \( n \) goods. Define the following:

(i) The Vickrey payoff vector (This includes each bidder’s Vickrey payoff and the seller’s Vickrey payoff).

(ii) Gross substitutes valuations.

(iii) Submodular valuations.

(iv) The cooperative game associated with a VCG auction as well as the core of that cooperative game.

For each of the following statements find the maximum number \( n \) of goods such that the statement is always true, or say that the statement is true for all \( n \):

(a) Truthful bidding is a dominant strategy in the VCG mechanism.

(b) If all bidders have gross substitutes valuations then the Vickrey payoff vector is in the core.

(c) If all bidders have submodular valuations, then the Vickrey payoff vector is in the core.

(d) The Vickrey payoff vector is in the core.

Provide a proof for each of your answers. In the course of answering these questions you may appeal to the following:

- A theorem relating bidder submodularity to the core (in order to appeal to this you must define bidder submodularity and clearly state the theorem).

- A theorem relating bidder submodularity to gross substitutes valuations (in order to appeal to this theorem, you must clearly state it).
Section B
Answer one short question and one long question from Section B.

Question B.I.1: Short
Find the value of the following TU game $v : 2^I \to \mathbb{R}$ with player set $I = \{1, 2, 3, 4\}$ and $v(\emptyset) = 0$:

$$
\begin{align*}
v(1) &= v(2) = 5 \\
v(3) &= v(4) = 0 \\
v(12) &= 10 \\
v(13) &= v(14) = v(23) = v(24) = 5 \\
v(34) &= 0 \\
v(123) &= v(124) = 10 \\
v(134) &= v(234) = 5 \\
v(I) &= 10.
\end{align*}
$$

Question B.I.2: Short
True or false and carefully explain why:

For a finite TU game $v : 2^I \to \mathbb{R}$ with $v(\emptyset) = 0$, core($v$) exists if and only if $v$ is balanced (result due to Shapley).
Question B.II.1: Long

This question concerns axiomatic (cooperative) bargaining theory.

(a) Define a bargaining problem formally. Be sure to define all notation that you introduce.

(b) Clearly state the axioms for the Nash Bargaining Solution.

(c) State Nash’s theorem on the possible existence and/or uniqueness of the Nash Bargaining Solution.

(d) Discuss the economic applicability of the Nash Bargaining Solution.

(e) Clearly state the axioms for the Utilitarian Bargaining Solution.

(f) State the theorem on the possible existence and/or uniqueness of the Utilitarian Bargaining Solution.

(g) Discuss the economic applicability of the Utilitarian Bargaining Solution.

Question B.II.2: Long

Write an essay about cardinal and ordinal utility representations in cooperative game theory.

(a) For both TU and NTU games, clearly define ordinal and cardinal solution concepts. Be sure to define all notation that you introduce and to specify the assumptions that are needed for your definitions to make sense.

(b) Define cardinal and ordinal strategic equivalence for TU and NTU games.

(c) Give at least one example of a solution concept that is cardinal. Give at least one example of an ordinal solution concept.

(d) Discuss the economic implications of ordinal and cardinal solution concepts. What factors would cause you to choose an ordinal or a cardinal solution concept? Give at least one example (explain) where you would use a cardinal solution concept and one where you would use an ordinal solution concept.
Answer one short question and one long question from Section C.

Question C.I.1: Short

What is strong belief à la Battigalli and Siniscalchi (2002)?

Question C.I.2: Short


Question C.II.1: Long

(a) State the definition of extensive-form rationalizability (EFR).

(b) For each of the following statements, prove that it is true or give a counter-example to show that it is not.

(i) Every EFR outcome is a subgame perfect equilibrium (SPE) outcome.

(ii) Every SPE outcome is an EFR outcome.

(iii) EFR is invariant.
Question C.II.2: Long

For any finite extensive-form game, define a new solution concept called *extensive-form stable equilibrium* as a stable equilibrium (or, more precisely, “a stable set of Nash equilibria”) of the corresponding agent-normal-form game.

Recall that an outcome is called stable if there exists a stable equilibrium such that every strategy profile in that stable equilibrium induces that same outcome. We define *extensive-form stable* outcome similarly.

*Sketch* a proof for the following statement: for generic finite extensive-form game, there exists at least one outcome that is both stable and extensive-form stable.

**Hint:** A complete proof will be very long and tedious, and full of $\epsilon$’s and $\delta$’s. You are asked to provide only a *sketch* of the proof, by enumerating what intermediate lemmas you plan to prove, explaining why these lemmas would lead to the final conclusion, and perhaps using some pictures to explain how you would prove some of these lemmas. At one point, you may find it helpful to recall the definition of *fully stable equilibrium*, which I include below:

Let $G$ be a finite normal-form game. A set $E$ of Nash equilibria of $G$ is called a *fully stable equilibrium* if it is a minimal closed set with the property that: for any $\epsilon > 0$ there exists a $\delta > 0$ such that, whenever each player’s strategy set is restricted to some closed convex polyhedron contained in the interior of his strategy simplex and at distance less than $\delta$ from the (boundary of the) simplex, then the resulting game has a Nash equilibrium $\epsilon$-close to $E$. 