Ph.D. Preliminary Examination

MICROECONOMIC THEORY

MAJORS

Spring 2011

The time limit for this exam is 3\(\frac{1}{4}\) hours.

**Answer one question from each part, for a total of four questions.**

You may use calculators to make calculations during the examination. However: in answering any question that requires you to justify your answer, if you do use a calculator, you must mention at what point in your answer you obtained results using a calculator, and what it was you were seeking to calculate.

Be sure you clearly define all **boldfaced/underlined** terms. Also, please be sure to define precisely any notation that you introduce.

Note: This examination should have 14 pages including this one (Check to make sure!)
Part I

Answer one question from Part I.
Question I.1

Consider a utility function $u$ on $\mathbb{R}^L_+$ defined by

$$u(x) = \inf_{q \in Q} qx$$

for every $x \in \mathbb{R}^L_+$. The set $Q$ is a closed and convex subset of the unit simplex $\Delta$ in $\mathbb{R}^L$ and such that $Q \subset \mathbb{R}^L_{++}$.

(a) Show that utility function $u$ is locally non-satiated and concave.

(b) Characterize the points $x$ of differentiability of $u$. What is the derivative (or the gradient vector) $Du(x)$ at a point $x$ of differentiability?

You may use any well-known mathematical result pertaining to (b) without proof.
Let \( \tilde{y} \) and \( \tilde{z} \) be two random variables on some state space (i.e. probability space) such that \( E(\tilde{z}) = 0 \).

(a) Give an example of \( \tilde{y} \) and \( \tilde{z} \) with \( \tilde{z} \neq 0 \) such that \( \tilde{y} + \tilde{z} \) is more risky than \( \tilde{y} \). Justify your claim.

(b) Give an example of \( \tilde{y} \) and \( \tilde{z} \) such that \( \tilde{y} + \tilde{z} \) is not more risky than \( \tilde{y} \). Justify your claim.

(c) Show that if \( \tilde{y} + \tilde{z} \) is more risky than \( \tilde{y} \), then \( \tilde{y} + 2\tilde{z} \) is more risky than \( \tilde{y} \), too.

Your definition of more risky should be stated in terms of cumulative distribution functions of random variables. You may use any well-known characterization of more risky without proof, but you need to state it clearly.

Clearly define any **boldfaced/underlined** terms.
Part II

Answer one question from Part II
Question II.1

Consider a pure exchange economy with $\ell$ commodities and $n$ consumers, each having consumption set $IR_+^\ell$, initial endowment vector $e_i \in IR_+^\ell$, and preferences $\preceq_i$ represented by a utility function $u_i : IR_+^\ell \to IR$ which is assumed to be continuous and strictly monotone.

Now consider the following statement about such economies:

There might not exist a competitive equilibrium because the potential lack of at least quasi-concavity of utilities prevents one from applying the Maximum Theorem to obtain continuity of (individual and hence aggregate) excess demand, so that Kakutani’s Fixed Point Theorem cannot be used.

(a) Is this statement true or false? Pick one and draw a box around your answer. Explain your reasoning very carefully.

(b) Give an example of such an economy with $\ell = n = 2$ that does not have a competitive equilibrium.

(c) State the Maximum Theorem.

(d) State Kakutani’s Fixed Point Theorem and define a fixed point of a correspondence.

(e) What assumptions must the preferences $\preceq_i$ in this economy satisfy?

(f) Without reference to the utilities $u_i$, define competitive equilibrium for this economy.
Question II.2

This question asks you to compare and contrast different alternative approaches to general equilibrium models of pure exchange economies. Throughout please use the notation that there are (where \( \ell \) and \( n \) are finite positive integers greater than two) \( \ell \) commodities and \( n \) agents, indicated by subscripts \( i = 1, \ldots, n \), each with initial endowment vector \( e_i \in IR^{\ell}_{++} \).

Economy A, denoted \( \mathcal{E}(A) \) has preferences specified by continuous complete preorders \( \preccurlyeq_i \) on \( IR^{\ell}_{++} \) which are strictly monotone and at least weakly convex. Economy B, denoted \( \mathcal{E}(B) \), has preferences as in \( \mathcal{E}(A) \) represented by continuous utility functions \( u_i : IR^{\ell}_{++} \rightarrow IR \) that satisfy further assumptions (to be specified by you) such that demands are continuous functions on \( \Delta = \{ p \in IR^{\ell}_{++} \mid \sum_{j=1}^{\ell} p_j = 1 \} \). Economy C, denoted \( \mathcal{E}(C) \), is smooth and satisfies further assumptions on the \( u_i \) such that demands are (at least once) continuously differentiable functions on \( \Delta \).

(a) Define continuity of a complete preorder \( \preccurlyeq_i \) on \( IR^{\ell}_{++} \). Define weak convexity of a continuous complete preorder \( \preccurlyeq_i \) on \( IR^{\ell}_{++} \).

(b) What conditions do the utilities \( u_i \) representing \( \preccurlyeq_i \) as in \( \mathcal{E}(A) \) necessarily satisfy? What additional assumption(s) is/are needed to guarantee that demands are continuous functions for economy \( \mathcal{E}(B) \)?

(c) State all additional assumptions that are needed for \( \mathcal{E}(C) \) to have continuously differentiable demand functions.

(d) What can we say about the set of (normalized) equilibrium prices in \( \Delta \) for each of these economies, \( \mathcal{E}(A) \), \( \mathcal{E}(B) \), and \( \mathcal{E}(C) \)? Specifically, is the set nonempty and if so, how large can it be? What tools (theorems) are needed for your argument? Please explain your reasoning but do not write a proof.

(e) What are the advantages and disadvantages of using each model \( \mathcal{E}(A) \), \( \mathcal{E}(B) \), and \( \mathcal{E}(C) \) in microeconomic theory? When and why should we choose one of these models over the other?
Part III

Answer one question from Part III
Question III.1

(a) Find all the Nash equilibria in the following game:

\[
\begin{array}{c|cc}
\text{ } & l & m \\
\hline
T & 1, 10 & 0, 0 \\
M & 1, 2 & 1, 2 \\
B & 3, -10 & 0, 0 \\
\end{array}
\]

(b) Find all the perfect equilibria of the same game.

(c) Take a Nash equilibrium \( \hat{\sigma} \) of any normal form finite game, and remove the action of a player \((i)\) which is not a best response to the strategy of the others, \(\hat{\sigma}^{-i}\). Is the restriction of the strategy profile \(\hat{\sigma}\) to the new game a Nash equilibrium of the new game? Prove your answer.

(d) Take a perfect equilibrium \(\hat{\sigma}\) of any normal form finite game, and remove the action of a player \((i)\) which is not a best response to the strategy of the others, \(\hat{\sigma}^{-i}\). Is the restriction of the strategy profile \(\hat{\sigma}\) to the new game a perfect equilibrium of the new game? Prove your answer.
Consider a normal form game with a set $I$ of players.

Part 1

Show that the set of strategies surviving iterated elimination of weakly dominated strategies is non-empty

Part 2

A **Strong Nash Equilibrium** is a strategy profile $(\sigma_i)_{i \in I}$ such that for no subset $J$ of players there exists a strategy profile $(\tau_j)_{j \in J}$ that gives to each player in the set $J$ a higher payoff than the profile $(\sigma_j)_{j \in J}$, while every player $k \in I \setminus J$ plays the strategy $\sigma_k$.

(a) Show an example of a Strong Nash Equilibrium in a game with two players.

(b) Show an example of a Strong Nash Equilibrium in a game with three or more players.

(c) Does a Strong Nash equilibrium always exist? Prove or disprove with an example.
Part IV

Answer one question from Part IV
Question IV.1

Below you will find several normal form games; these are exactly those for which you found the Nash equilibria in the past. For each of these games find:

(a) The set of Nash equilibria and Nash equilibrium payoffs for the stage game

(b) The set $\text{co}(F^0)$, where $F^0 \equiv \{u(a) : a \in A\}$, and $A$ is the set of action profiles

(c) The vector of minimax values $(v^i)_{i \in I}$

(d) The set of feasible and individually rational payoffs

The games

(a) \[
\begin{array}{ccc}
 & l & r \\
T & 2,2 & -1,-1 \\
B & 0,4 & 0,4
\end{array}
\]

(b) \[
\begin{array}{ccc}
 & l & m & r \\
T & -1,8 & 2,5 & 4,3 \\
B & 1,2 & 2,4 & 1,9
\end{array}
\]

(c) \[
\begin{array}{ccc}
 & l & m & r \\
T & 1,1 & 0,0 & 0,0 \\
M & 0,0 & 4,4 & 8,0 \\
B & 0,0 & 0,10 & 9,9
\end{array}
\]
Question IV.2

Problem IV. 2

Consider a repeated game with imperfect monitoring (the action chosen by the players is not observed) and public monitoring (a public signal, like the quantity produced or the price in an oligopoly model is observed).

Define a perfect equilibrium in public strategies. Prove that the equilibrium corresponds to a subgame perfect equilibrium of the repeated game where players may use private information.
Question IV.3

Consider the following moral hazard problem. There are two effort levels: $e = 0$ and $e = 1$. The agent is an expected-utility maximizer. If he puts in effort $e$, and subsequently receives monetary payment $t$, then his utility will be

$$u(e, t) = \begin{cases} 
  t & \text{if } e = 0 \\
  \frac{1}{2} - \exp(-t) & \text{if } e = 1
\end{cases}$$

Notice that $u(0, t) > u(1, t)$ for all $t$; and that while $u(0, \cdot)$ is linear, $u(1, \cdot)$ is concave. The agent’s reservation utility is 0.

The principal’s utility is $v(e, t) = e - t$ if the agent puts in effort $e$ and she pays the agent monetary payment $t$.

(a) Suppose the agent’s effort is neither observable (to the principal) nor verifiable (to the court). What is the cheapest way for the principal to induce the agent to put in effort $e = 1$?

(b) Suppose the agent’s effort is observable (to the principal) but not verifiable (to the court). What is the cheapest way for the principal to induce the agent to put in effort $e = 1$?