Ph.D. Preliminary Examination

FINANCIAL ECONOMICS

Spring 2014

Answer 1 question from Section I and 3 questions from Section II.
Section I

I.1 Consider infinite-time security markets with uncertainty described by an event-tree. Assume that security prices are positive and there exist strictly positive event prices. State a definition of price bubbles. Show that price bubbles are non-negative. [You may restrict attention to date-0 price bubbles]. Further, prove that price bubbles on any finitely-lived security (that is, security with finite maturity) must be zero.

I.2 Consider two-date security markets with $N$ securities and $S$ states. Suppose that there are short-sales restrictions on some but not all securities. That is, for a proper subset of $N$ securities, holdings of these securities cannot be lower than some negative lower bounds.

State a necessary and sufficient condition for security prices to exclude arbitrage under short-sales restrictions. Your condition should be stated in terms of state prices. Prove that the condition is sufficient. Give an example of security payoffs and prices such that there is an arbitrage with no restrictions on short-sales, but there is no arbitrage under short-sales restrictions. [Short-sales restrictions in your example may apply to all securities.]

I.3 Consider a discrete-time economy with a finite state space $\mathcal{I}$ and a representative agent with additively separable preferences whose marginal utility of consumption in state $i \in \mathcal{I}$ is $m_i$. The Markov transition probability from state $i$ to state $j$ is $P_{i,j} > 0$. Complete sets of markets in all the states $i \in \mathcal{I}$ reveal the contingent claims prices $Q_{i,j} = \beta P_{i,j} m_j / m_i$ for all $i$ and $j$ in $\mathcal{I}$. Explain how you can use knowledge of these contingent claims prices to infer $\{m_i : i \in \mathcal{I}\}$.

I.4 Let $R_{N,t+1}$ be the gross return from $t$ to $t+1$ on an $N$-period discount bond, and let $R_{t+1}$ be the gross return on any other asset over the same period. Suppose $\pi_t$ is a state price process. Show that

$$E_t \left[ \ln \left( R_{N,t+1} \right) \right] - E_t \left[ \ln \left( R_{t+1} \right) \right] \geq E_t \left[ \ln \left( \frac{E_{t+1}[\pi_{t+N}]}{E_t[\pi_{t+N}]} \right) \right] .$$

If $\pi_t = \gamma^t x_t$ for some positive constant $\gamma$ and a positive stationary process $x_t$ with a finite mean, what does this say about expected logarithmic returns in this economy?
Section II

II.1 Consider the following model of security markets with asymmetric information. There is a risk-free security and a single risky security with payoff \( \tilde{v} \). There are \( N \) agents whose preferences over end-of-period consumption have expected utility representation with (possible different) constant absolute risk aversion. Each agent receives a private signal which communicates the true payoff of the risky security with noise. The signal of agent \( i \) takes the form \( \tilde{s}_i = \tilde{v} + \tilde{\epsilon}_i \), where \( \tilde{\epsilon}_i \) has zero mean. Supply of the risky security is a random variable \( \tilde{z} \). Assume that \( (\tilde{v}, \tilde{z}, \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_N) \) are mutually independent and have multivariate normal distribution.

(i) State a definition of a rational expectations (R.E.) equilibrium in this model.

(ii) State a definition of a linear R.E. equilibrium. Derive agents’ demands for the risky security in a linear R.E. equilibrium as functions of signals and supply.

(iii) Suppose that the supply of the risky security is deterministic and equal to zero. Will there be trade in a linear R.E. equilibrium? Justify your answer.
II.2 Consider security markets with $S$ states of nature, $I$ agents, and $J$ securities. Agents’ preferences over state-contingent consumption plans have expected utility representation with (possible different) quadratic utilities and common probabilities of states. Agents consume only at date 1 when states are realized, and trade securities at date 0. Security markets may be incomplete, but the risk-free payoff lies in the asset span. Further, agents’ date-1 endowments lie in the asset span.

(i) Derive the Capital Asset Pricing Model. In particular, show that the market return (i.e., return on the aggregate endowment) lies on the mean-variance frontier. Using this, show that the equation of the security market line must hold.

(ii) Is the equilibrium consumption allocation in this model Pareto optimal? Justify your answer.
II.3 Consider an asset with a cumulative return process $R_t$ that satisfies $d\ln(R_t) = \mu(x_{t-})dt + \sigma(x_{t-}, x_t)dJ_t$, where $x_t$ is a continuous-time Markov chain with a finite number of states and transition probabilities $\Pr[x_{t+} = j|x_t = i] = \mathcal{P}_\Delta(i, j)$, and $J_t$ is a jump process that increases by 1 whenever $x_{t-} \neq x_t$. Assume that $\sigma(i, i) = 0$ for all $i$. Let $\mathcal{M}$ be the diagonal matrix with diagonal elements $\mathcal{M}(i, i) = \mu(i)$ and write $\mathcal{S}_{i,j} = \exp(\sigma(i, j))$. Let $\mathcal{A} = \lim_{\Delta \to 0}(\mathcal{P}_\Delta - I)/\Delta$.

For any function $f$ of the Markov state, define

$$[\mathcal{G}_\Delta f](x) = \frac{1}{R_0} \mathbb{E}[R_\Delta f(x_\Delta)|x_0 = x]$$

and let $B f = \lim_{\Delta \to 0}[\mathcal{G}_\Delta f - f]/\Delta$.

a. Write down an expression for $[\mathcal{G}_\Delta f - f](x)$ and use the fact that the probability of more than one jump in $R_t$ is of order $\Delta^2$ to explain why $B = \mathcal{M} + A \circ \mathcal{S}$, where $[A \circ B](i, j) = A(i, j)B(i, j)$.

b. Express the instantaneous conditionally expected return on this asset in terms of $B$.

c. Explain why there is a $\delta$ so that $\rho(x) = \lim_{\Delta \to -\infty} e^{-\delta \Delta} \mathbb{E}[R_{t+\Delta}/R_t|x_t = x] \in (0, \infty)$ is well defined and relate $\rho(x)$ to the properties of $B$.

d. Determine the logarithmic long-run expected return $\lim_{\Delta \to -\infty} \frac{\ln(\mathbb{E}[R_{t+\Delta}/R_t|x_t = x])}{\Delta}$.
Define
\[ f(c, V) = \frac{\rho}{1 - \gamma} \left( [(1 - \alpha)V]^{1 - \frac{1-\gamma}{1-\alpha}} c^{1 - \gamma} - (1 - \alpha)V \right), \]
where \( \rho > 0, \alpha > 1 \) and \( \gamma \in (0, 1) \). Using a bit of algebra, which you may take for granted, one can verify that
\[ F(c, r) = \inf_{V} \{ f(c, V) + rV \} = \frac{\rho c^{1 - \alpha}}{1 - \alpha} \left( \frac{1 - \frac{1-\gamma}{1-\alpha} r}{1 - \frac{1-\gamma}{1-\alpha}} \right)^{1 - \frac{1-\gamma}{1-\alpha}}. \]

Consider an exchange economy with aggregate consumption endowments \( y_t \) that satisfy
\[ dy_t = y_t \left[ \mu_{y,t} dt + \sigma_{y,t} dW_t \right], \]
where \( W_t \) is a vector of independent standard Brownian motions. At time \( t \), the representative consumer assigns utility \( V_t \) to the consumption process \( f(c_s, V_s) \), where
\[ V_t = \sup_{\{r_s\}_{s \geq t}} \mathbb{E}_t \left[ \int_t^\infty \exp \left( - \int_t^s r_a da \right) F(c_s, r_s) ds \right]. \tag{1} \]
All stochastic processes are adapted to the filtration generated by \( \{W_t\}_{t \geq 0} \).

a. Explain why one can take state prices for this economy to be
\[ \pi_t = \exp \left( \int_0^t D_c f(y_s, V_s) ds \right) D_V f(y_t, V_t) \]
where \( V_t \) is given by (1) evaluated at \( c_t = y_t \).

b. Suppose \( U : \mathbb{R}^{N \times +} \rightarrow \mathbb{R} \) is a standard utility function that is homogeneous of degree \( \eta \) and consider an economy in which the representative consumer consumes \( y \in \mathbb{R}^{N \times +} \). Show that aggregate wealth in this \( N \)-good economy equals \( \eta U(y)/D_1 U(y) \) in units of good 1. Use this to argue by analogy that aggregate wealth at time \( t \) equals \( w_t = (1 - \alpha) V_t / D_c f(y_t, V_t) \) in units of consumption at time \( t \), where \( V_t \) is given by (1) evaluated at \( c_t = y_t \).

c. Show that aggregate wealth satisfies
\[ dw_t = w_t \left[ \mu_{w,t} dt + \sigma'_{w,t} dW_t \right] \]
with
\[ \sigma_{w,t} = \gamma \sigma_{y,t} + \left( \frac{1 - \gamma}{1 - \alpha} \right) \sigma_{V,t}, \]
where (1) evaluated at \( c_t = y_t \) satisfies
\[ dV_t = V_t \left[ \mu_{V,t} dt + \sigma'_{V,t} dW_t \right]. \]

d. Write \( d\pi_t = \pi_t \left[ \mu_{\pi,t} dt + \sigma'_{\pi,t} dW_t \right] \) and determine \( -\sigma_{\pi,t} \) in terms of \( \sigma_{y,t} \) and \( \sigma_{w,t} \). Verify that your answer reduces to the familiar result for additively separable preferences when \( \alpha = \gamma \).