Dynamic Rational Expectations Equilibrium with Private Information: How Not to Forecast the Forecasts of Others

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This version: March 1, 2003
(very preliminary)

Abstract

This paper studies dynamic rational expectations equilibria with heterogeneously informed risk averse traders. It is well-known that an infinite regress 'forecasting the forecasts' problem arises unless one assumes that: (*i*) informed traders are infinite in number, with no aggregate noise, or (*ii*) traders' information is only valuable for one period, or (*iii*) traders are hierarchically informationally-ranked from the most to the least informed. All such assumptions are somewhat undesirable.

This paper offers a tractable characterization of the steady-state rational expectations equilibria for the general case of finitely many generally informed trader classes with long-lived information. Our approach involves only *one* probability distribution F that escapes the infinite regression of beliefs in the same way that Harsanyi (1967-68) employed in his development of Bayes-Nash equilibrium.

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1 Introduction

Understanding the forces for price formation and asset trade is the backbone of modern financial economics. In the oldest of adages, people trade because they are different – either informationally or from a risk sharing perspective. The most obvious difference that can explain rapid large price changes is new information. To understand information dynamics, one must have a suitable technical understanding of how individuals trade when they are differentially endowed with information. They must learn from past and current prices, fully recognizing that those prices reflected both the information of other traders and their attempts to learn from others (including themselves). They must literally learn how to 'forecast the forecasts of others'.[1] As the history grows, the nature of this forecast problem becomes overwhelming, seemingly requiring an infinite dimensional belief space. This problem has long proven a major roadblock in solving intertemporal trading models with differential information.

Existing models of intertemporal trading escape this 'infinite regress' problem either by assuming (a) a purely static world with one-shot trading; (b) homogeneously informed traders — i.e. privy solely to public information; (c) or cleverly chosen heterogeneously informed traders. Various clever devices have been employed. First, one may posit that all information is short-lived, e.g., Singleton (1987). As such, forecasts from past prices are clearly payoff-irrelevant. Second, the informed classes of agents might enjoy a hierarchial information structure, so that for any pair of agents A and B, either A knows whatever B knows, or vice versa (e.g., Wang (1993) and Wang (1994)). Third, the number of differentially investors may be assumed infinite in such a way that the Law of Large Numbers eliminates all associated aggregate randomness (see He and Wang (1995)). This obviously circumvents the infinite belief regress problem.[2] Of these approaches, the hierarchical information approach is most often employed, since it yields a nontrivial stochastic price pattern that is deemed most realistic.

This paper offers a tractable way of characterizing the steady state dynamic rational expectations equilibria with heterogeneously informed traders. We assume only finitely many classes of traders; most realistically, they need

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[1] See, e.g., Singleton (1987), Wang (1994), Brunnermeier (2001). Townsend (1983) gives rise to the expression, however, it has been recently shown by Pearlman and Sargent (2002) that Townsend's original model has "too few sources of randomness to put decision makers into a situation where they should form 'higher order beliefs' ".

[2] There is a final approach that leaves the world of risk averse traders. One may introduce a risk-neutral market making sector of the economy, e.g., Vives (1995).

not be informationally ranked, and who possess long-lived information. Our approach involves only *one* probability distribution F. While an infinite regression of beliefs *does exist*, it does not matter, as all are generated by the same parent probability distribution F. This general idea is new in finance. However, Harsanyi (1967-68) adopted the same approach to analyze *static* games of incomplete information with Bayes-Nash equilibrium. He assumed that players are each told their informational 'type', and the joint distribution over types. We introduce his 'type-centered' characterization of the equilibrium, and endow each agent with her 'type' and the joint probability distribution F over all the types in the economy. A trader's type is her private information, F is common knowledge. Agents' types together with the probability distribution F fully characterize the equilibrium.

We present a model of dynamic asset pricing under asymmetric information. We assume the economy is endowed with a given quantity of an equity asset. The asset value evolves in a stochastic mean-reverting fashion. Investors are partitioned into several classes. Every period, each investor class observes a private signal of the asset value. They also commonly observe a dividend, which is stochastically related to the asset value. Since the asset value follows a mean-reverting process, current prices provide signals about future asset values. As is typical, we assume an incomplete markets structure by way of either supply shocks or noise trader net demands, so that the signals do no fully reveal the investors' private signals.

To solve this model, we summarize each trader's private history by a recursively defined weighted sum of her private signals (the 'trader's private type'). We also summarize the common history of publicly observable prices and dividends by a recursively defined statistic Ψ. We show how the statistic Ψ and the investors' private types can be chosen so that the steady state of the economy at date t depends only on the fundamentals at time t — namely, the investors' private types, the public statistic Ψ_t, and noise traders' demand at time t. Moreover, we show that there exists a time-stationary joint probability measure F over the state variables, such that each trader's conditional expectations of the underlying uncertainty of the economy are fully determined by the respective conditionals of F. In this setup, a trader's type at time t reflects her informational advantage over an agent, who possesses only the public information up to and including time t.

The remainder of the paper is organized as follows. We specify the model in Section 2 and solve the equilibrium in section 3. Most of the proofs are appendicized.

2 The Model

We consider a simple economy with a single physical good that can be either consumed or invested, and two publicly traded assets: a riskless bond and a risky asset (‘stock’). The economy is further defined as follows.

Investment opportunities. The riskless bond is assumed to have perfectly elastic supply with a positive constant rate of return r.

Each share of the stock pays a dividend d_t at time t ($t = 0,\ 1,\ 2, ...$). The dividend d_t is governed by the process

$$d_t = v_t + \varepsilon_{d,\,t}, \qquad \varepsilon_{d,\,t} \sim \mathsf{N}(0, \sigma_d^2) \tag{1}$$

where v_t follows an AR(1) process:

$$v_{t+1} = a_v v_t + \varepsilon_{v,\,t}, \quad 0 \leq a_v \leq 1,\ \varepsilon_{v,\,t} \sim \mathsf{N}(0, \sigma_v^2) \tag{2}$$

Here, $\varepsilon_{d,\,t}$ and $\varepsilon_{v,\,t}$ are independent shocks to d_t and v_t respectively; v_t is the persistent component in dividends and $\varepsilon_{d,\,t}$ is the idiosyncratic component; v_t (the ‘fundamental’ of the stock) fully determines the expectation of the future dividends.

Restricting the securities traded in the market makes the market incomplete. This is necessary to prevent the equilibrium prices from being sufficient for all private information. That market incompleteness is not sufficient to prevent fully revealing prices, is well known (see, e.g. Grossman (1976)). As standard in the literature, we assume that u_t, the number of shares available in the market at date t, is random. In particular, u_t follows an AR(1) process:

$$u_{t+1} = a_u u_t + \varepsilon_{u,\,t}, \quad -1 < a_u < 1,\ \varepsilon_{u,\,t} \sim \mathsf{N}(0, \sigma_u^2) \tag{3}$$

The assumption of random supply of the stock is equivalent to the usual noise trading story, that is, liquidity traders have inelastic demands of $\bar{u} - u_t$ shares of stock at t, leaving the remaining u_t shares to the market (assuming that the total number of shares is $\bar{u}$). Changes in liquidity traders’ demands will then generate noise trading and changes in shares supplied to the market. When $a_u = 0$, the amount of noise trading is *i.i.d.* over time, which is likely to happen when the time between the two consecutive dates is very large. When $a_u \to 1$, the incremental changes of noise trading become *i.i.d.* over time. This is likely to happen when the elapsed time between two consecutive trading dates is very small.

Information Structure. There are N+1 ($\mathrm{N} \in \mathbb{N}$) differentially informed classes of investors in the economy, with a continuum of agents in each class. Assuming a continuum of agents within a class justifies investors' price-taking behaviour in a rational expectations equilibrium, and is common in the literature. All investors observe realized dividends d_t and market prices p_t of the stock. In addition, at each date t, investor i ($i = 1, 2, ..., \mathrm{N}+1$) receives a private signal about v_t:

$$s_t^i = v_t + \varepsilon_{s,t}^i, \qquad \varepsilon_{s,t}^i \sim \mathsf{N}(0, \sigma_s^2) \tag{4}$$

We assume noise $\varepsilon_{s,t}^i$ in private signals to be *i.i.d.* across investors.

We assume that all investors have the same prior information about v_0, u_0, and that the prior distributions are normal. Let $\mathcal{F}_t^i = \{d_\tau,\, p_\tau,\, s_\tau^i : \tau \leq t\}$ be the information set of investor i at date t. That $p_t \in \mathcal{F}_t^i$ reflects the perfect foresight nature of a rational expectations equilibrium. We assume all investors to be informed. Introducing a class of uninformed investors is a straightforward extension of the model.

For simplicity, we assume that all the shocks to the economy, ($\varepsilon_{v,t}, \varepsilon_{d,t}, \varepsilon_{u,t}, \varepsilon_{s,t}^i : i = 1, 2, ..., \mathrm{N}+1$) are jointly normal, independent of each other, independent over time, and independent of v_0 and u_0. The structure of the economy is common knowledge.

Preferences. All investors have constant absolute risk aversion (CARA). Investor i ($i = 1, 2, ..., \mathrm{N}+1$) maximizes her conditional expected utility of the following form:

$$\mathsf{E}\left[-\sum_{s=0}^{\infty} \beta^s e^{-\rho\, c_{t+s}} \,|\, \mathcal{F}_t^i\right], \tag{5}$$

where c_{t+s} is her consumption in period $t+s$. All investors are assumed to have the same risk aversion coefficient ρ.

CARA preferences and normality of random variables are standard assumptions in the rational expectations equilibrium literature. They allow for mathematically tractable linear solutions to the model.[3]

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[3] See, e.g., Diamond and Verrecchia (1981), Grossman (1976), Grossman and Stiglitz (1980), and Hellwig (1980), in static settings; Brown and Jennings (1989), Grundy and McNichols (1989), Wang (1993), Wang (1994) and He and Wang (1995) in dynamic settings.

Notation. For future references, we first introduce some notation. Define

- $\mathcal{F}_t^c \equiv$ the common information available to all investors at date t.
- $\mathcal{F}_t^{p,i} \equiv$ the private information available to investor i at date t.
- $\mathcal{F}_t^i \equiv$ the total information available to investor i at date t.

If we introduce the notation $\underline{\mathrm{Z}}_t \equiv (Z_1, Z_2, ..., Z_t)$ for any stochastic process $\{Z_t\}$ (i.e., $\underline{\mathrm{Z}}_t$ represents the history of $\{Z_t\}$ up to and including t), then

$$\mathcal{F}_t^c = \{\mathcal{F}_0, \underline{\mathrm{p}}_t, \underline{\mathrm{d}}_t\},\ \mathcal{F}_t^{p,i} = \{\underline{\mathrm{s}}_t^i\}.\text{ and } \mathcal{F}_t^i = \{\mathcal{F}_0, \underline{\mathrm{p}}_t, \underline{\mathrm{d}}_t\, \underline{\mathrm{s}}_t^i\}.$$

We adopt a standard convention to denote random variables and their realizations by the same letter, whenever the meaning is clear from the context. We will also use $\mathcal{L}[\cdot]$ to denote a general linear relation. For example, $p_t = \mathcal{L}[\underline{\mathrm{Z}}_t]$ means that p_t is a linear function of $\underline{\mathrm{Z}}_t$.

Unless specified otherwise, we will generally use upper-case letters to denote matrices and cumulative probability distributions, and lower-case letters to denote scalars and probability density functions. Matrix scalar entries will generally be denoted by lower-case letters with double subscripts; vector scalar entries will generally be denoted by lower-case letters with single subscripts.[4]

3 Equilibrium

In this section, we solve for the steady state equilibrium of the economy defined in Section 2. The equilibrium concept is that of rational expectations, developed by Lucas (1972), Green (1976), Grossman (1976), and Kreps (1977). The way we obtain an equilibrium is similar in essence to that of Grossman (1976) and others.[5] Namely, conjecture an equilibrium price function; then solve the investors learning and optimization problem, and finally, impose the market clearing to verify the conjectured price function. In contrast to the existing literature, investors in our model *do not* need to forecast the forecasts of others!

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[4] For instance, a (2×2) matrix $\mathrm{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$; a 2-dimensional vector $\mathrm{B} = (\mathrm{b}_1, \mathrm{b}_2)'$.

[5] Also see Hellwig (1980), Diamond and Verrecchia (1981) for the analysis in static settings; Wang (1993), Wang (1994), He and Wang (1995) for the analysis in dynamic models.

'Type-centered' approach. It is standard in rational expectations literature to conjecture the equilibrium price be a function of informed investors' expectations and noise traders' random demands. While this approach works well in static settings, it yields the infinite dimensionality of the state space when applied to dynamic models with differential information.

This paper solves the 'infinite regress' problem by introducing an alternative state space, specifically, that of agents' private types. Loosely, we want a trader's type θ_t^i to be her time-t 'informational advantage' over an agent possessing only public information $\mathcal{F}_t^c$. The equilibrium price at time t should intuitively then be a function of (*i*) all investors' contemporaneous private types (θ_t^i), (*ii*) some statistic Ψ_t for the common information $\mathcal{F}_t^c$ (Ψ_t can be loosely interpreted as the 'uninformed type'), and (*iii*) the noise traders' random demands at time t.

Concretely, we linearly parse a trader's information into private and public 'factors'. Trader i's private factor θ_t^i will then be determined endogenously as part of the rational expectations equilibrium. This approach intuitively should be immune to the 'infinite regress' problem for the very reason that Harsanyi's (1967–68) type construction evades such difficulties with Bayes-Nash equilibrium. For the game theory context, what is done is to introduce Nature as an additional player into the game. In its most general form, Nature chooses the players' payoff-relevant characteristics and sends them private signals thereof. Thus, even though a belief hierarchy exists, and is nontrivial, it does not matter: players simply optimize knowing Nature's strategy. This strategy is formally captured by a joint measure across players' types. Each player then acts solely with knowledge of her own type and the joint measure.

In the current common values financial setting, the *payoff relevant information* consists of the underlying state of the world v_t and the noise traders' demands u_t. But for the same reason as in the strategic setting, each trader i intuitively can act solely with knowledge of the public and private factors Ψ_t and θ_t^i — assuming there is indeed a joint measure over types.

More specifically, assume that all investors share a common prior over all informational factors, and the payoff relevant information. Given his information set $\mathcal{F}_t^i$, investor i is then able to compute the sequence of conditional probability distributions $\{G_t^i \,|\, \mathcal{F}_t^i\}$ over all informational factors (excluding her own private factor). Define a sequence of *joint probability measures* $\{F_t\}$ over the payoff-relevant information, and the public and private factors, such that conditionals $\{F_t \,|\, (\Psi_t,\, \theta_t^i)\}$ coincide with $\{G_t^i \,|\, \mathcal{F}_t^i\}$ for all agents i, for all possible realizations of the private and public factors θ_t^i, Ψ_t. It is absolutely critical to our story that this joint measure be common knowledge among the traders. Investor i's demand at time t is then fully determined by $\{\theta_t^i, \Psi_t; F_t\}$, and an equilibrium can be characterized by $\{\theta_t^1, ..., \theta_t^{\mathrm{N+1}}, \Psi_t; F_t\}$. Observe, in particular, that investor i's first-order expectations are maps from (θ_t^i, Ψ_t); these maps are determined by F_t and are *common knowledge*. Wonderfully, higher-order expectations are not needed in our setup!

In fact, the story becomes simpler. For we have a Markovian system that evolves linearly in the joint measures (F_t); further, this process is a martingale, and thus intuitively a long-run steady state limit joint measure $\bar{F}$ exists. The paper in fact simply focuses on the steady state rational equilibrium of the economy, and exhibits $(\{\theta_t^i\}_{i=1}^{\mathrm{N+1}}, \Psi_t; \bar{F})$ that characterize it.

Stock market Equilibrium. As standard in the rational expectations literature with CARA preferences under normal distributions of payoffs and signals, we only consider the linear equilibria of the economy. In a linear equilibrium, the stock price can be expressed as a linear function of the state variables of the economy. In other words, we have

$$p_t = \mathcal{L}[\Phi_t], \tag{6}$$

where Φ_t is the vector of state variables in the economy. Clearly, one can always define $\Phi_t = (v_t, u_t; \underline{\mathrm{p}}_t, \underline{\mathrm{d}}_t; \{\underline{\mathrm{s}}_t^i\}_{i=1}^N)$. Yet, the definition is rather impractical due to the infinite dimensionality of such state space. We show that by properly choosing the state space, Φ_t can be reduced to a small number of state variables.

In particular, private histories $\{\underline{\mathrm{s}}_t^i\}_{i=1}^N$ can be summarized by traders' private types $\{\theta_t^i\}_{i=1}^N$, and public history $(\underline{\mathrm{p}}_t, \underline{\mathrm{d}}_t)$ can be summarized by some public statistic Ψ_t. Here, θ_t^i are scalars and Ψ_t is a two-dimensional vector. Define $\theta_t \equiv \frac{1}{N+1}\sum_{i=1}^{N}\theta_t^i$ to be the average of time-t private types. Our main result is presented in the following theorem.

Theorem 1 *In a steady-state symmetric linear equilibrium of the economy described above,*

1. *(i) the price function has the following form:*

$$p_t = \lambda_0 + \lambda_\pi \pi_t + \Lambda \Psi_t, \tag{7}$$

where $\pi_t = \theta_t + \lambda_u u_t$*;*

(ii) *private factors* θ^i_t *and public factor* Ψ_t *are defined recursively as follows:*

$$
\begin{aligned}
\theta^i_t &= \alpha\theta^i_{t-1} + \beta s^i_t, \quad i = 1\,..\,\mathrm{N}{+}1 && (8)\\
& && (9)\\
\Psi_t &= \mathrm{L}\Psi_{\mathrm{t}-1} + \mathrm{M}\begin{pmatrix} d_t \\ \pi_t \end{pmatrix}. && (10)
\end{aligned}
$$

Here λ_0, λ_π, λ_u *are constants;* Λ, L, M *are* (2×2) *constant matrices.*

2. *Define* $y_{t+1} \equiv p_{t+1} + d_{t+1} - (1{+}r)p_t$ *be the excess return on one share of the stock. Investor i's value function* J^i_t*, her optimal investment and consumption policies,* q^i_t, c^i_t*, are then given by the following equations:*

$$
\begin{aligned}
J^i_t(w^i_t;\theta^i_t,\pi_t,\Psi_t) &= -\beta^t e^{-\eta w^i_t - \frac{1}{2}\Phi^i_t Z \Phi^{i\prime}_t} && (11)\\
c^i_t &= -\frac{1}{\rho}\ln\left(-\frac{1}{\rho}\frac{\partial J^i}{\partial w}\right), \ \textit{and} && (12)\\
q^i_t &= \frac{1}{\eta}\,\zeta\,\mathsf{E}[y_{t+1}|\,\mathcal{F}^i_t] - \frac{1}{\eta}h(\Phi^i_t), && (13)
\end{aligned}
$$

Here Z *is* (5×5) *constant matrix,* ζ *is a positive constant,* $h(\cdot)$ *is a linear function,* $\eta = r\rho/(1{+}r)$*,* $\Phi^i_t = (1, \theta^i_t, \pi_t, \Psi'_t)$*.*

3. *There exist a probability distribution* $\bar{F}(v_t,\, u_t,\, \theta^1_t, ..., \theta^{\mathrm{N+1}}_t, \Psi_t)$ *such that the equilibrium is fully characterized by* $\{\theta^1_t, ..., \theta^{\mathrm{N+1}}_t, \Psi_t;\ \bar{F};\ \mathcal{F}_0\}$*. Specifically,* $\bar{F}$ *is the steady state distribution of* $(v_t,\, u_t,\, \theta^1_t, ..., \theta^{\mathrm{N+1}}_t, \Psi_t)$*, multivariate normal with mean vector zero and a constant covariance matrix* $\bar{\Sigma}$*.*

Remark 3.1 Note that π_t in the price equation (7) can be interpreted as the 'public price signal' in the sense that the public history of prices and dividends $(\underline{\mathrm{p}}_t, \underline{\mathrm{d}}_t)$ provides the same information about the underlying uncertainty in the economy as $(\underline{\pi}_t, \underline{\mathrm{d}}_t)$. Observe also that π_t is a Gaussian random process. In light of this, it is natural to conjecture statistics θ^i_t, Ψ_t be linear in private signals $\underline{\mathrm{s}}^i_t$ and public signals $(\underline{\pi}_t, \underline{\mathrm{d}}_t)$, respectively. Equations (8-10) support this conjecture.

Expectations, stock demand, and market clearing. We now sketch a four-step proof for Theorem 1. First we discuss some general properties of the linear equilibrium defined by (6). Second, we solve investors' learning problems and define the joint probability distribution $\bar{F}$. Next, we derive investors' optimal investment and consumption. We then show that by imposing the market clearing condition, the equilibrium stock price is given in (7). We will derive solutions to the above learning and optimization problems in a symmetric equilibrium of the economy.

Equilibrium price function. Observe that the price equation (7) can be expressed as $p_t = \mathcal{L}(\pi_t, \Psi_t)$, and further rewritten as

$$p_t = \mathcal{L}(\pi_t,\ d_t,\ \Psi_{t-1}) = \mathcal{L}(u_t,\ d_t,\ \theta_t,\ \Psi_{t-1}). \tag{14}$$

Equation (14) states that the equilibrium price is a linear function of the underlying state variables of the economy (the stock supply u_t and the fundamental of the stock v_t, since $d_t = v_t + \varepsilon_{d,t}$), the average of private types θ_t, and a certain statistic Ψ_{t-1} for all the common information $\mathcal{F}^c_{t-1}$. Thus, to make inferences from the price, a trader will only need first-order expectations about the underlying uncertainty v_t, u_t, and the average of investors' types θ_t.

Compare (14) with a general linear equilibrium as defined by (6). The vector of state variables Φ_t is defined by the whole history of exogenous shocks to the economy, including supply shocks, dividends, and private signals. Our ultimate goal is to prove that in equilibrium, (6) reduces to (7). This is equivalent to showing that Φ_t reduces to $(u_t, v_t, \{\theta^i_t\}_{i=1}^{\mathrm{N+1}}, \Psi_{t-1})$, for appropriately defined private types θ^i_t and the public statistic Ψ_t.

We have assumed noise traders' demands independent across time ($a_u = 0$), and thus further simplified the state space to include only: $v_t, \{\theta^i_t\}_{i=1}^{\mathrm{N+1}}$ and Ψ_{t-1}. Furthermore, define σ_u so that $\lambda_0 u_t$ is a standard normal random variable ($\sigma_u = 1/\lambda_0$). These assumptions are made to simplify the computations and are not crucial to the analysis.

Assuming noise shocks uncorrelated over time implies, in particular, that the joint distribution $\bar{F}(u_t, v_t, \theta^1_t, \theta^2_t, ..., \theta^{\mathrm{N+1}}_t, \Psi_{t-1})$ is multiplicatively separable and can be written as $\bar{F}_u(u_t)\bar{F}_\Phi(v_t, \theta^1_t, \theta^2_t, ..., \theta^{\mathrm{N+1}}_t, \Psi_{t-1})$. To simplify the notation, in what follows, we will use $\bar{F}$ to denote $\bar{F}_\Phi$.

Investors' learning problem. In order to derive each investor's optimal stock demand, we have to solve for the conditional expectations of the 'fundamental' v_t and the average of investors' types θ_t, given her information set.

Define

$$\theta_t^{-i} = \frac{1}{\mathrm{N}+1}\sum_{\substack{k=1\\k\neq i}}^{\mathrm{N}+1} \theta_t^k.$$

Clearly, $\mathsf{E}[(v_t, \theta_t)\,|\,\mathcal{F}_t^i] = \mathsf{E}[(v_t, \theta_t^{-i})\,|\,\mathcal{F}_t^i] + \theta_t^i/(\mathrm{N}+1)$. Thus, it suffices to solve for trader i conditional expectations of v_t and θ_t^{-i}. Since all signals are linear in the state variables, calculating conditional expectations of the state variables is a linear filtering problem. Define $(\hat{v}_t, \hat{\theta}_t^{-i}) \equiv \mathsf{E}[(v_t, \theta_t^{-i})\,|\,\mathcal{F}_{t-1}^i]$ and $(\bar{v}_t, \bar{\theta}_t^{-i}) \equiv \mathsf{E}[(v_t, \theta_t^{-i})\,|\,\mathcal{F}_t^i]$. Note that $(\hat{v}_t, \hat{\theta}_t^{-i})$ and $(\bar{v}_t, \bar{\theta}_t^{-i})$ can be interpreted as trader 2's 'prior' and 'posterior' conditional expectations, respectively. The results of the learning problem are summarized in the following lemma, proof of which can be found in Appendix A.

Lemma 3.1 *Given the linear price function of (7), investor i's 'prior' and 'posterior' conditional probability distributions,* $\hat{G}^i\,|\,\mathcal{F}_t^i$ *and* $\bar{G}^i\,|\,\mathcal{F}_{t+1}^i$*, over* $(v_{t+1},\, \theta_{t+1}^{-i})$ *are bivariate normal with the mean vectors* $\hat{m}_t^i = (\hat{v}_{t+1}, \hat{\theta}_{t+1}^{-i})'$ *and* $\bar{m}_t^i = (\bar{v}_{t+1}, \bar{\theta}_{t+1}^{-i})'$*, respectively, and constant covariance matrices* $\hat{\Gamma}$ *and* $\bar{\Gamma}$*, respectively. The covariance matrices are specified in Appendix A. Mean vectors* $\hat{m}_t^i$ *and* $\bar{m}_t^i$ *are determined recursively by the following linear equations:*

$$\hat{m}_t^i = \begin{pmatrix} \hat{v}_{t+1} \\ \hat{\theta}_{t+1}^{-i} \end{pmatrix} = \begin{pmatrix} a_v & 0 \\ \frac{\beta \mathrm{N}}{\mathrm{N}+1} a_v & \alpha \end{pmatrix} \begin{pmatrix} \bar{v}_t \\ \bar{\theta}_t^{-i} \end{pmatrix}, \tag{15}$$

$$\bar{m}_t^i = \begin{pmatrix} \bar{v}_t \\ \bar{\theta}_t^{-i} \end{pmatrix} = \begin{pmatrix} \hat{v}_t \\ \hat{\theta}_t^{-i} \end{pmatrix} + \mathrm{K} \begin{pmatrix} s_t^i - \hat{v}_t \\ d_t - \hat{v}_t \\ \pi_t^i - \hat{\theta}_t^{-i} \end{pmatrix} \tag{16}$$

where $\pi_t^i = \theta_t^{-i} + \lambda_u u_t$*,* K *is a* (2×3) *constant matrix specified in Appendix A.*

Recall that π_t, the price signal, is equal to $\theta_t + \lambda_0 u_t$. In light of this, $\pi_t^i = \pi_t - \theta_t^i/(\mathrm{N}+1)$ can be interpreted as 'trader i's price signal'.

Given posteriors $(\bar{v}_t, \bar{\theta}_t^{-i}) = \mathsf{E}[(v_t, \theta_t^{-i})\,|\,\mathcal{F}_t^i]$, prior distribution $\hat{G}^i\,|\,\mathcal{F}_t^i$ over period $t+1$ state variables $(v_{t+1}, \theta_{t+1}^{-i})$ is determined deterministically. In what follows, we will restrict attention to solving for the sequence of posterior conditional distributions.

Lemma 3.1 defines conditional distribution $\bar{G}^i\,|\,\mathcal{F}_t^i$. Observe that, through their mean vectors $\bar{m}_t^i$, the $\bar{G}^i$s depend on 'individual price signals' π_t^i. The goal was, however, to specify private types and the public statistic Ψ_t so that $\bar{G}^i$s are fully determined by some joint probability distribution over the state variables: $\bar{F}(v_t, \theta_t^1, \theta_t^2, ..., \theta_t^{\mathrm{N}+1}, \Psi_t)$. For this to be the case, it is necessary that conditional expectations of trader i can be expressed in terms of her private type and Ψ_t. That it can be done, is shown in the following lemma.

Lemma 3.2 *There exist constant* α, β, H, L, M *and* Ψ_0 *such that trader i's posterior conditional expectations of the state variables at time* t *are fully determined by her private type* θ_t^i *and* Ψ_t*:*

$$\begin{pmatrix} \bar{v}_t \\ \bar{\theta}_t^{-i} \end{pmatrix} = \mathrm{H}\theta_\mathrm{t}^\mathrm{i} + \Psi_\mathrm{t} \tag{17}$$

Here α, β *are scalars;* H, L, M *are respectively* (2×1), (2×2) *and* (2×2) *constant matrices.*

The above lemma is proven recursively by assuming (17) to be true at time t and specifying α, β, H, L, M for (17) to be true at time $t+1$. Ψ_0 is defined so that equation (17) holds at $t = 0$. 'Individual price signal' π_t^i is eliminated from the equations, using $\pi_t^i = \pi_t - \theta_t^i/(\mathrm{N}+1)$. The complete proof of the lemma is found in Appendix A.

Joint Measure. Lemmas 3.1, 3.2 show that one can define investors' private types θ_t^i and the public statistic Ψ_t so that trader i's conditional distribution $\bar{G}^i \,|\, \mathcal{F}_t^i$ over (v_t, θ_t^{-i}) is bivariate normal with the mean vector linear in (θ_t^i, Ψ_t), and a constant covariance matrix Γ. It is then possible to find a multivariate normal joint distribution $\bar{F}(v_t, \theta_t^1, \theta_t^2, ..., \theta_t^{\mathrm{N+1}}, \Psi_t)$ such that $\bar{G}^i \,|\, \mathcal{F}_t^i = \bar{F}(v_t, \theta_t^{-i} | \, \theta_t^i, \Psi_t)$ are fully determined by $\bar{F}$. While such $\bar{F}$ is not unique, it can be chosen to be the steady state distribution of the state variables $(v_t, \theta_t^1, \theta_t^2, ..., \theta_t^{\mathrm{N+1}}, \Psi_t)$. This result is summarized in the following lemma.

Lemma 3.3 *Define* $\bar{F}(v_t, \theta_t^1, \theta_t^2, ..., \theta_t^{\mathrm{N+1}}, \Psi_t)$ *to be the steady state distribution of the state variables. For each* i *define the corresponding distribution* $\bar{F}^{-i}(v_t, \theta_t^{-i}, \Psi_t)$*. Then* $\bar{F}^{-i}$ *has the following properties:*

$$\bar{G}^i \,|\, \mathcal{F}_t^i = \bar{F}\left(v_t, \theta_t^{-i} | \, \theta_t^i, \Psi_t\right)$$

Proof sketch. The state variables follow a Markov process, and the existence of the steady state density is guaranteed so long as a_v and α are less the 1 in absolute value. Trader i's private type θ_t^i and the public statistic Ψ_t are sufficient for trader i's private information, thus $\bar{G}^i \,|\, \mathcal{F}_t^i = \tilde{F}\left(v_t, \theta_t^{-i} | \, \theta_t^i, \Psi_t\right)$ for some $\tilde{F}$. Showing that $\tilde{F} = \bar{F}^{-i}$ is an algebraic exercise.

Investors' optimal investment and consumption policies. Given the investors' conditional expectations, we can now derive their optimal investment and consumption policies. Let $y_{t+1} \equiv p_{t+1} + d_{t+1} - (1+r)p_t$ be the excess return on one share of the stock. Solving the optimization problem of the traders, we have the following result.

Lemma 3.4 *Let w_t^i be investor i's wealth at time t, c_t^i her consumption, q_t^i her stock holdings, and J_t^i her value function. Her optimization problem is:*

$$J_t^i \equiv \max_{c_t^i, q_t^i} \beta^t \mathsf{E}\left[-\sum_{s=0}^{\infty} \beta^s e^{-\rho\, c_{t+s}^i} | \mathcal{F}_t^i \right], \tag{18}$$

$$s.t.\ \ w_{t+1}^i = (w_t^i - c_t)(1+r) + q_t^i\, y_{t+1} \tag{19}$$

It has the following solution:

$$\begin{aligned} J_t^i(w_t^i; \theta_t^i, \Psi_t) &= -\beta^t e^{-\eta w_t^i - \frac{1}{2}\Phi_t^{i\prime} \mathrm{Z} \Phi_{\mathrm{t}}^{\mathrm{i}}} & (20) \\ q_t^i &= \frac{1}{\eta}\zeta\, \mathsf{E}[y_{t+1}|\mathcal{F}_t^i] - \frac{1}{\eta} h(\Phi_t^i) & (21) \\ c_t^i &= -\frac{1}{\rho} \ln\left(\frac{1}{\rho\beta^t} \frac{\partial J_t^i}{\partial w} \right). & (22) \end{aligned}$$

Here Z *is* (5×5) *constant matrix,* ζ *is a positive constant,* $h(\cdot)$ *is a linear function,* $\eta = r\rho/(1+r)$*,* $\Phi_t^i = (1, \theta_t^i, \pi_t, \Psi_t')'$.

Proof of Lemma 3.4 is in Appendix B.

It follows from Theorem 1 and the definition of y_{t+1} that $\mathsf{E}\,[\, y_{t+1} \,|\, \mathcal{F}_t^i] = \mathcal{L}\,[\, \theta_t^i, p_t, d_t, \Psi_{t-1}]$. Since Φ_t^i can also be expressed as $\mathcal{L}\,[\, \theta_t^i, p_t, d_t, \Psi_{t-1}]$, a trader's optimal stock holdings q_t^i is a linear function of $(\theta_t^i, p_t, d_t, \Psi_{t-1})$. Imposing the market clearing condition will then result in p_t being linear in the state variables.

Proof of Theorem 1. Now we complete the proof of Theorem 1. We need to show that market clearing of the stock requires the equilibrium price to have the form of equation (7). Express investors' optimal stock holdings, $q_t^i = \frac{1}{\eta}\mathrm{R}\Phi^{\mathrm{i}}_{\mathrm{t}}$, as:

$$q_t^i = \frac{1}{\eta}(p_t + (r_1 - \lambda_0) + r_2\theta_t^i + (r_3 - \lambda_\pi)\pi_t + [(r_4,\, r_5) - \Lambda]\Psi_t). \tag{23}$$

Impose the market clearing condition:

$$\bar{u} = u_t + \frac{1}{\mathrm{N}+1} \sum_{i=1}^{\mathrm{N}+1} q_t^i = \frac{1}{\eta}(p_t + (r_1 - \lambda_0) + r_2\theta_t + (r_3 - \lambda_\pi)\pi_t + [(r_4,\, r_5) - \Lambda]\Psi_t) \tag{24}$$

Impose $\lambda_u = \eta/r_2$, then

$$p_t = (\lambda_0 - r_1 + \eta\bar{u}) + (\lambda_\pi - r_3 - r_2)\pi_t + [\Lambda - (r_4,\, r_5)]\Psi_t. \tag{25}$$

Compare (25) with the conjectured form (7). Imposing individual rationality yields the following system of polynomial equations:

$$\begin{aligned} &r_1 - \eta\bar{u} = 0, \quad r_2 + r_3 = 0, \\ &r_4 = 0, \quad r_5 \;=\; 0, \quad \lambda_u - \eta/r_2 = 0 \end{aligned}$$

This completes the proof of Theorem 1. The current version of the paper if short of an analytical proof of the existence of the solution to the above system of equations.

A Appendix

Proof of Lemma 3.1. To derive the filtering equations (3.1), we use the following lemma, the proof of which can be found, e.g., in Jazwinski (1970).

Lemma A.1 *Assume the discrete linear system is described by the vector difference equation*

$$x_{t+1} = A_t x_t + B_t \varepsilon_{x,\,t+1}, \qquad t = 0, 1, ..,$$

where x_t is the n-vector state at t, A_t is the $n \times n$, non-singular state transition matrix, B_t is $n \times r$, and $\{\varepsilon_{x,t}, t = 1, ..\}$ is an r-vector, white Gaussian sequence, $\varepsilon_{x,t} \sim \mathsf{N}(0, Q_t)$. The discrete linear observations are

$$y_t = C_t x_t + \varepsilon_{y,\,t},$$

where y_t is the m-vector observation, C_t is $m \times n$, $\{\varepsilon_{y,t}, t = 1, ..\}$ is an m-vector, white Gaussian sequence, $\varepsilon_{y,t} \sim \mathsf{N}(0, R_t)$. $x_0, \{\varepsilon_{x,t}\}$, and $\{\varepsilon_{y,t}\}$ are assumed independent.

Then the optimal filter for the discrete system above consists of difference *equations for the conditional mean and covariance matrix. Let*

$$\begin{aligned}
\hat{x}_{t+1} &= \mathsf{E}[x_{t+1} \,|\, y_\tau,\, 0 \le \tau \le t], \\
\hat{\Gamma}_{t+1} &= \mathsf{E}[(x_{t+1} - \hat{x}_{t+1})(x_{t+1} - \hat{x}_{t+1})' \,|\, y_\tau,\, 0 \le \tau \le t] \\
\bar{x}_t &= \mathsf{E}[x_t \,|\, y_\tau,\, 0 \le \tau \le t], \\
\bar{\Gamma}_t &= \mathsf{E}[(x_t - \bar{x}_t)(x_t - \bar{x}_t)' \,|\, y_\tau,\, 0 \le \tau \le t]
\end{aligned}$$

Then

$$\begin{aligned}
\hat{x}_{t+1} &= A_t \bar{x}_t, && (26) \\
\hat{\Gamma}_{t+1} &= A_t \bar{\Gamma}_t A_t' + B_t Q_{t+1} B_t', && (27) \\
\bar{x}_t &= \hat{x}_t + K_t(y_t - C_t \hat{x}_t), && (28) \\
\bar{\Gamma}_t &= \hat{\Gamma}_t - K_t C_t \hat{\Gamma}_t. && (29)
\end{aligned}$$

where $K_t = \hat{\Gamma}_t C_t' [C_t \hat{\Gamma}_t C_t' + R_t]^{-1}$ *is the Kalman gain.*

In our case, the unobserved variables are

$$\begin{aligned}
v_{t+1} &= a_v v_t + \varepsilon_{v,\,t+1} \\
\theta_{t+1}^{-i} &= \alpha \theta_t^{-i} + \beta \frac{1}{\mathrm{N+1}} \sum_{\substack{k=1 \\ k \ne i}}^{\mathrm{N+1}} s_{t+1}^k = \alpha \theta_t^{-i} + \beta \frac{N}{\mathrm{N+1}} v_{t+1} + \beta \frac{1}{\mathrm{N+1}} \sum_{\substack{k=1 \\ k \ne i}}^{\mathrm{N+1}} \varepsilon_{s,\,t+1}^k
\end{aligned}$$

Lemma 3.1 then follows directly from Lemma A.1 by making the following substitutions: $x_t = (v_t, \theta_t^{-i})'$, $y_t = (s_t^i, d_t, \pi_t^i)'$, $\varepsilon_{x,\,t} = (\varepsilon_{v,\,t}, \varepsilon_{s,\,t}^{-i})'$, $\varepsilon_{y,\,t} = (\varepsilon_{s,\,t}^i, \varepsilon_{d,\,t}, \lambda_0 \varepsilon_{u,\,t})'$, where $\varepsilon_{s,\,t}^{-i} \equiv \frac{1}{\mathrm{N+1}} \sum_{\substack{k=1 \\ k \ne i}}^{\mathrm{N+1}} \varepsilon_{s,\,t}^k$; and

$$\mathrm{A} \equiv \mathrm{A_t} = \begin{pmatrix} a_v & 0 \\ \frac{\beta \mathrm{N}}{\mathrm{N+1}} a_v & \alpha \end{pmatrix}, \; \mathrm{B} \equiv \mathrm{B_t} = \begin{pmatrix} 1 & 0 \\ \frac{\beta \mathrm{N}}{\mathrm{N+1}} & \beta \end{pmatrix}, \; \mathrm{C} \equiv \mathrm{C_t} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{Q} \equiv \mathrm{Q_t} = \begin{pmatrix} \sigma_v^2 & 0 \\ 0 & \frac{N}{(\mathrm{N+1})^2} \sigma_s^2 \end{pmatrix}, \; \mathrm{R} \equiv \mathrm{R_t} = \begin{pmatrix} \sigma_s^2 & 0 & 0 \\ 0 & \sigma_d^2 & 0 \\ 0 & 0 & \sigma_\pi^2 \end{pmatrix},$$

where $\sigma_\pi = \lambda_u \sigma_u$. Steady state requires that covariance matrices are constant. They can be found by substituting $\hat{\Gamma}_t = \hat{\Gamma}$ and $\bar{\Gamma}_t = \bar{\Gamma}$ into equations

(27) and (29). We then obtain the following Riccati equation

$$\hat{\Gamma} = \mathrm{A}(\hat{\Gamma} - \mathrm{K}\mathrm{C}\hat{\Gamma})\mathrm{A}' + \mathrm{B}\mathrm{Q}\mathrm{B}', \tag{30}$$

where $\mathrm{K} = \hat{\Gamma}\mathrm{C}'[\mathrm{C}\hat{\Gamma}\mathrm{C}' + \mathrm{R}]^{-1}$. The steady state solution to (30) exists so long as a_v and α are less than 1 in absolute value (see, e.g. Anderson and Moore (1979)).

Proof of Lemma 3.2. The proof is by induction. (i) It is straightforward to define Ψ_0 so that equation (17) holds at time 0. (ii) Assume that (17) holds at time $t-1$. First, use (9), (15), and $\pi_t^i = \pi_t - \theta_t^i/(\mathrm{N}+1)$ to rewrite (16) as follows:

$$\begin{aligned} \begin{pmatrix} \bar{v}_t \\ \bar{\theta}_t \end{pmatrix} &= \mathrm{L}\begin{pmatrix} \bar{v}_{t-1} \\ \bar{\theta}_{t-1} \end{pmatrix} - \theta_{t-1}^i \begin{pmatrix} \frac{\alpha}{\mathrm{N+1}} k_{13} \\ \frac{\alpha}{\mathrm{N+1}} k_{23} \end{pmatrix} \\ &\quad + s_t^i \begin{pmatrix} k_{11} - \frac{\beta}{\mathrm{N+1}} k_{13} \\ k_{21} - \frac{\beta}{\mathrm{N+1}} k_{23} \end{pmatrix} + \mathrm{M}\begin{pmatrix} d_t \\ \pi_t \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \mathrm{L} &= \begin{pmatrix} a_v(1 - k_{11} - k_{12} - \frac{\beta \mathrm{N}}{\mathrm{N+1}} k_{13}) & -\alpha k_{13} \\ a_v(\frac{\beta \mathrm{N}}{\mathrm{N+1}} - k_{21} - k_{22} - \frac{\beta \mathrm{N}}{\mathrm{N+1}} k_{23}) & \alpha(1 - k_{23}) \end{pmatrix} \\ \mathrm{M} &= \begin{pmatrix} k_{12} & k_{13} \\ k_{22} & k_{23} \end{pmatrix}. \end{aligned}$$

Define the law of motion for the public statistic Ψ_t to be

$$\Psi_t = \mathrm{L} \cdot \Psi_{\mathrm{t-1}} + \mathrm{M}\begin{pmatrix} d_t \\ \pi_t \end{pmatrix}.$$

Then, using the induction assumption, (31) can be further rewritten as

$$\begin{pmatrix} \bar{v}_t \\ \bar{\theta}_t^{-i} \end{pmatrix} = \mathrm{L}\,\mathrm{H}\;\theta_{\mathrm{t-1}}^{\mathrm{i}} + \mathrm{K}\begin{pmatrix} s_t^i \\ 0 \\ -\frac{1}{\mathrm{N+1}}(\alpha\theta_{t-1}^i + \beta s_t^i) \end{pmatrix} + \Psi_{\mathrm{t}}$$

To complete the induction proof, we require that

$$\mathrm{H}\,\theta^{\mathrm{i}}_{\mathrm{t}} = \mathrm{L}\,\mathrm{H}\,\theta^{\mathrm{i}}_{\mathrm{t-1}} + \mathrm{K}\left(\begin{array}{c} s^i_t \\ 0 \\ -\frac{1}{\mathrm{N+1}}(\alpha\theta^i_{t-1} + \beta s^i_t) \end{array}\right).$$

Together with the conjectured law of motion for the private type θ^i_t, the above implies:

$$\alpha\mathrm{H} = \mathrm{L}\,\mathrm{H} - \mathrm{K}\left(\begin{array}{c} 0 \\ 0 \\ \frac{\alpha}{\mathrm{N+1}} \end{array}\right), \quad \beta\mathrm{H} = \mathrm{K}\left(\begin{array}{c} 1 \\ 0 \\ -\frac{\beta}{\mathrm{N+1}} \end{array}\right)$$

Solving the last equation for H, we obtain the following "consistency equation" on α and β:

$$\mathrm{K}\left(\begin{array}{c} \alpha \\ 0 \\ 0 \end{array}\right) = \mathrm{L}\,\mathrm{K}\left(\begin{array}{c} 1 \\ 0 \\ -\frac{\beta}{\mathrm{N+1}} \end{array}\right) \tag{31}$$

To prove the existence of α, β, H, L, M of Lemma 3.2, it suffices to show that there exist a solution $(\alpha, \beta, \bar{\gamma}_{11}, \bar{\gamma}_{12}, \bar{\gamma}_{22})$ to the system of 5 non-linear equations (30) – (31). As long as $|\alpha| < 0$, there exists a steady state solution to (30). We believe that the existence of such α and β can be shown using fixed point methods. We are able to obtain solution numerically, however, the current version of the paper is short of an analytical argument.

B Appendix

Proof of Lemma 3.4. The proof closely follows that of Theorems 2 and 3 in Wang (1994).
First, we have the following lemma.

Lemma B.1 *Let* $(w^i_t, \Phi^{i\prime}_t)'$ *be the vector of trader i's state variables. Assume further that* Φ^i_{t+1} *and the excess return on a share of the risky asset* y_{t+1} *can be written in the following form:*

$$\Phi^i_{t+1} = \mathrm{A}_\Phi \Phi^{\mathrm{i}}_{\mathrm{t}} + \mathrm{B}_\Phi \varepsilon^{\mathrm{i}}_{\Phi,\,\mathrm{t+1}} \tag{32}$$

$$y_{t+1} = \mathrm{A}_{\mathrm{y}} \Phi^{\mathrm{i}}_{\mathrm{t}} + \mathrm{B}_{\mathrm{y}} \varepsilon^{\mathrm{i}}_{\Phi,\,\mathrm{t+1}}, \tag{33}$$

where $\mathrm{A_y}, \mathrm{B_y}, \mathrm{A}_\Phi, \mathrm{B}_\Phi$, *are constant matrices of proper order, and* $\varepsilon^i_{\Phi,\,t+1} \,|\, \mathcal{F}^i_t \sim$

$\mathsf{N}(0, \Upsilon)$. *Then the solution to the investor i's optimization problem (18)-(19) is given by*

$$\begin{aligned} J_t^i(w_t^i; \theta_t^i, \Psi_t) &= -\beta^t e^{-\eta w_t^i - \frac{1}{2}\Phi_t^{i\prime} \mathrm{Z} \Phi_t^{\mathrm{i}}} && (34) \\ q_t^i &= \frac{1}{\eta} \mathrm{R} \Phi_{\mathrm{t}}^{\mathrm{i}} && (35) \\ c_t^i &= \bar{c} + \frac{r}{1+r} w_t^i + \frac{1}{2\rho(1+r)} \Phi_t^{i\prime}\, \mathrm{X}\, \Phi_{\mathrm{t}}^{\mathrm{i}}, && (36) \end{aligned}$$

where η, $\bar{c}$ *are constants,* X, R *are constant matrices of proper order.*

The remainder of the appendix is structured as follows. We will first show that trader i's state variables Φ_t^i and the excess return on a share y_{t+1} can be expressed as (32) and (33). We will then proceed to prove Lemma B.1 and show that equations (20)-(22) follow from (34)-(36).

Deriving equations (32)-(33). Define $\Phi_t^i = (1, \theta_t^i, \pi_t, \psi_{1\,t}, \psi_{2\,t})'$.

Let us show that Φ_{t+1}^i can be written as (32). For the remainder of this appendix, we will fix i and write $\hat{Z}_{t+1} \equiv \mathsf{E}[Z_{t+1} \,|\, \mathcal{F}_t^i]$ for any stochastic process Z_t. Use filtering equations (15) and (17) to write

$$\begin{aligned} \hat{v}_{t+1} &= a_v(c_1 \theta_t^i + \psi_{1,\,t}) && (37) \\ \hat{\theta}_{t+1}^{-i} &= \alpha(c_2 \theta_t^i + \psi_{2,\,t}) + \frac{\beta \mathrm{N}}{\mathrm{N}+1} \hat{v}_{t+1} && (38) \end{aligned}$$

Note further that v_{t+1} and θ_{t+1}^{-i} can be expressed as

$$\begin{aligned} v_{t+1} &= \hat{v}_{t+1} + \varepsilon_{v,\,t+1}^i && (39) \\ \theta_{t+1}^{-i} &= \hat{\theta}_{t+1}^{-i} + \varepsilon_{\theta,\,t+1}^i, && (40) \end{aligned}$$

where $(\varepsilon_{v,\,t+1}^i, \varepsilon_{\theta,\,t+1}^i) | \mathcal{F}_t^i \sim \mathsf{N}(0, \Upsilon_{v\theta})$ for some covariance matrix $\Upsilon_{v\theta}$. Use recursive definitions of θ_t^i, π_t and Ψ_t from Theorem 1 to write

$$\begin{aligned} \theta_{t+1}^i &= \alpha \theta_t^i + \beta s_{t+1}^i && (41) \\ \pi_{t+1} &= \theta_{t+1}^{-i} + \frac{1}{\mathrm{N}+1} \theta_{t+1}^i + \varepsilon_{\pi,\,t+1} && (42) \\ \psi_{1,\,t+1} &= l_{11} \psi_{1,\,t} + l_{12} \psi_{2,\,t} + m_{11} d_{t+1} + m_{12} \pi_{t+1} && (43) \\ \psi_{2,\,t+1} &= l_{21} \psi_{1,\,t} + l_{22} \psi_{2,\,t} + m_{21} d_{t+1} + m_{22} \pi_{t+1}, && (44) \end{aligned}$$

where $\varepsilon_{\pi,\,t+1} \equiv \lambda_0 \varepsilon_{u,\,t+1}$.

Define $\varepsilon_{\Phi,\,t+1}^i = (\varepsilon_{v,\,t+1}^i, \varepsilon_{\theta,\,t+1}^i, \varepsilon_{s,\,t+1}^i, \varepsilon_{d,\,t+1}, \varepsilon_{\pi,\,t+1})$. It is easy to show that

$\varepsilon^i_{\Phi,t+1} | \mathcal{F}^i_t \sim \mathsf{N}(0, \Upsilon)$, where

$$\Upsilon = \begin{pmatrix} \hat{\Gamma} & 0 \\ 0 & \Sigma_{sd\pi} \end{pmatrix},$$

where $\hat{\Gamma}$ is the covariance matrix of trader i's prior conditional expectations (see Lemma 3.1), and $\Sigma_{sd\pi} = diag\{\sigma_s^2, \sigma_d^2, \sigma_\pi^2\}$; $\sigma_\pi = \lambda_u \sigma_u$. Use equations (37)-(40) together with definitions of d_t and s^i_t from section 2 to rewrite (41)-(44) in form (32).

We will now show that the excess return on a share of the risky asset can be expressed in form (33). First, rewrite

$$\begin{aligned} y_{t+1} &= p_{t+1} + d_{t+1} - (1+r)p_t = \lambda_\pi \pi_{t+1} + \Lambda \Psi_{t+1} \\ &\quad + d_{t+1} - (1+r)\lambda_\pi \pi_t - (1+r)\Lambda \Psi_t - r\lambda_0 \end{aligned} \tag{45}$$

Now use equations (37)-(40) together with definitions of d_t andπ_t to write (45) in form (33).

Optimal consumption and investment policies. We will now prove Lemmas B.1 and 3.4. The Bellman equation for the problem is given by

$$0 = \max_{c^i_t, q^i_t} \{-\beta^t e^{-\rho c_t} + \mathsf{E}[J^i_{t+1}(w^i_{t+1}; \theta^i_{t+1}, \Psi_{t+1}) | \mathcal{F}^i_t] - J^i_t(w^i_t; \theta^i_t, \Psi_t)\}, \tag{46}$$

$$\text{s.t.} \quad w^i_{t+1} = (w^i_t - c_t)(1+r) + q^i_t \, y^i_{t+1} \tag{47}$$

$$\lim_{s \to \infty} \mathsf{E}_t \left[J^i_{t+s}(w^i_{t+s};\, \theta^i_{t+s},\, \Psi_{t+s})\right] = 0. \tag{48}$$

Consider the following trial solution for the value function:

$$J^i_t(w^i_t; \theta^i_t, \Psi_t) = -\beta^t e^{-\eta w^i_t - \frac{1}{2}\Phi^{i\prime}_t \mathrm{Z} \Phi^{\mathrm{i}}_{\mathrm{t}}}, \tag{49}$$

where η is some constant and Z is a (5×5) symmetric constant matrix. It is then straightforward to show that

$$\mathsf{E}[J^i_{t+1}(w^i_{t+1}; \theta^i_{t+1}, \Psi_{t+1})\} | \mathcal{F}^i_t] = \xi e^{-\eta(w^i_t - c_t)(1+r) - \eta q_t \mathrm{A_y} \Phi_\mathrm{t} + \frac{1}{2}\mathrm{C}'_{\mathrm{q^i_t}} \mathrm{D} \mathrm{C}_{\mathrm{q^i_t}} - \frac{1}{2}(\mathrm{A}_\Phi \Phi_\mathrm{t})' \mathrm{Z} (\mathrm{A}_\Phi \Phi_\mathrm{t})}, \tag{50}$$

where $\xi = \sqrt{\frac{\det \mathrm{D}}{\det \Upsilon}}$, $\mathrm{D} = [\mathrm{B}'_\Phi \mathrm{Z} \mathrm{B}_\Phi + \Upsilon^{-1}]^{-1}$ and $\mathrm{C}_{\mathrm{q^i_t}} = \mathrm{D}[\eta\, \mathrm{q^i_t} \mathrm{B}'_\mathrm{y} + \mathrm{B}'_\Phi \mathrm{Z} \mathrm{A}_\Phi \Phi_\mathrm{t}]$.

The first order conditions for the optimal investment-consumption policy are:

$$q_t^i = \frac{1}{\eta}\mathrm{R}\Phi_{\mathrm{t}}^{\mathrm{i}} \text{ and } \mathrm{c}_{\mathrm{t}}^{\mathrm{i}} = \bar{\mathrm{c}} + \frac{\eta(1{+}\mathrm{r})}{\rho+\eta(1{+}\mathrm{r})}\mathrm{w}_{\mathrm{t}}^{\mathrm{i}} + \frac{1}{2(\rho+\eta(1{+}\mathrm{r}))}\Phi_{\mathrm{t}}^{\mathrm{i}\prime}\mathrm{X}\,\Phi_{\mathrm{t}}^{\mathrm{i}}, \tag{51}$$

where $\mathrm{R} = \zeta[\mathrm{A_y} - \mathrm{B_y D B'_\Phi Z A_\Phi}]$, $\zeta = (\mathrm{B_y D B'_y})^{-1}$, $\bar{c} = \frac{1}{\rho+\eta(1+r)}\ln\left(\frac{\rho}{\xi\beta\eta(1+r)}\right)$,

$$\mathrm{X} = \eta(\mathrm{R'A_y} + \mathrm{A'_y R}) - (\eta\mathrm{B'_y R} + \mathrm{B'_\Phi Z A_\Phi})'\mathrm{D}(\eta\mathrm{B'_y R} + \mathrm{B'_\Phi Z A_\Phi}) + \mathrm{A'_\Phi Z A_\Phi}$$

Define $h(\Phi_t^i) = \mathrm{B_y D B'_\Phi Z A_\Phi}\Phi_{\mathrm{t}}^{\mathrm{i}}$. The optimal stock holding given by (51) can then be rewritten as follows

$$q_t^i = \frac{1}{\eta}\,\zeta\,\mathsf{E}[y_{t+1}|\mathcal{F}_t^i] - \frac{1}{\eta}h(\Phi_t^i),$$

which is precisely the optimal investment policy described in Lemma 3.4. Substituting the optimal consumption-investment policy back into the Bellman equation yields

$$\eta = \frac{\rho r}{1{+}r}, \quad \bar{c} = -\frac{1}{\rho(1{+}r)}\ln(r\beta\xi) \tag{52}$$

$$\exp\left\{-\frac{1}{2}\Phi_t^{i\prime}\left[-\mathrm{Z} + \frac{1}{1{+}\mathrm{r}}\mathrm{X} + 2(\ln\frac{\mathrm{r}}{1{+}\mathrm{r}} + \rho\,\bar{\mathrm{c}})\mathbf{i}_{11}^{(5,5)}\right]\Phi_t^i\right\} = 1 \tag{53}$$

Here $\mathbf{i}_{11}^{(5,5)}$ is a (5×5) index matrix.[6] This leads to the following equation for Z:

$$-\mathrm{Z} + \frac{1}{\mathrm{r}+1}\mathrm{X} + 2\left[\ln\frac{\mathrm{r}}{\mathrm{r}{+}1} + \rho\,\bar{\mathrm{c}}\right]\mathbf{i}_{11}^{(5,5)} = 0 \tag{54}$$

We currently do not have an existence proof. Note, however, since X and $\mathbf{i}_{11}^{(5,5)}$ are symmetric, such Z, if exists, has to be symmetric.

To see that c_t^i is as described in Lemma 3.4, use (52) and (54) to rewrite the first order condition (51) for c_t^i as follows:

$$c_t^i = -\frac{1}{\rho}\ln\frac{\eta}{\rho} + \frac{\eta}{\rho}w_t^i + \frac{1}{2\rho}\Phi_t^{i\prime}\mathrm{Z}\,\Phi_{\mathrm{t}}^{\mathrm{i}}$$

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[6] An index matrix $\mathbf{i}_{kl}^{(m,n)}$ is an $(m\times n)$ matrix with the element $\{k,l\}$ being one and all other elements being zero. For example,

$$\mathbf{i}_{11}^{(2,2)}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

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