Information Aggregation, Equilibrium Multiplicity and Market Volatility:

Morris-Shin Meets Grossman-Stiglitz

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Abstract

This paper argues that adding endogenous information aggregation to situations where coordination is important – such as currency crises, bank runs and riots – yields novel insights into the multiplicity of equilibria and, more generally, market volatility. Morris and Shin (1998) have shown that with exogenous information multiplicity collapses when individuals observe fundamentals with small enough idiosyncratic noise. In the spirit of Grossman and Stiglitz (1976), we endogenize public information by allowing individuals to observe financial prices or other noisy indicators of aggregate activity. We show that endogenous information typically reverses the limit result: multiplicity is *ensured* when individuals observe fundamentals with small enough idiosyncratic noise. Moreover, multiplicity may emerge with respect to assets prices in addition to regime outcomes. Finally, when the equilibrium is unique market volatilty may rise as the exogenous noise in information is reduced.

JEL Codes: D8, E5, F3, G1.

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1 Introduction

It's a love-hate relationship, economists are at once fascinated and uncomfortable with multiple equilibria. On the one hand, a variety of phenomena seem characterized by large and abrupt changes in outcomes not obviously triggered by commensurate changes in fundamentals. Commentators often attribute these changes to arbitrary changes in 'market sentiments' or 'animal spirits'. Models with multiple equilibria may formally capture these ideas. Prominent examples include self-fulfilling bank runs, currency attacks, debt crises, financial crashes, riots and political regime changes.[1] In these models multiplicity arises due to a coordination problem: for intermediate values of the fundamentals attacking a 'regime' – such as a currency peg – is beneficial if and only if enough agents are also expected to attack.

On the other hand, models with multiple equilibria are sometimes viewed as incomplete theories that should ultimately be extended in some dimension to resolve the indeterminacy. Recently, Morris and Shin (1998, 2000)[2] have contributed to this perspective by enriching the information structure away from common knowledge and showing that a unique equilibrium survives when individuals observe the underlying fundamentals with small enough idiosyncratic noise.

Their main result can be illustrated using Figure 1, where σ_x and σ_z denote the standard deviations of the noise in private and public information regarding the fundamental.[3] In the coordination environments they consider there is multiplicity of equilibria when there is common-knowledge of fundamentals so that either $\sigma_x = 0$ or $\sigma_z = 0$. However, for positive levels of noise as σ_x becomes smaller, holding fixed σ_z, a unique equilibrium survives. Indeed, in the limit as $\sigma_x \to 0$ we approach the common-knowledge case where agents are perfectly informed yet there is a unique equilibrium. One intuition for this uniqueness result is that as idiosyncratic private noise decreases agents condition their actions more and more heavily on their own private information making it harder for agents to coordinate on multiple courses of action. The dispersion of useful information is crucial: small positive amounts of idiosyncratic noise make agents differentially informed *and* imply that this information is valuable and thus used.

The contribution of Morris-Shin is particularly attractive because it can be viewed as a small perturbation around the original common-knowledge model and therefore as a selection

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[1] For example, Diamond and Dybvig (1983), Obstfeld (1986, 1996), Velasco (1996), Calvo (1988), Cooper and John (1988), Cole and Kehoe (1996).

[2] Morris and Shin (1998) build on Carlson and van Damme (1993) who developed the Global Games approach for two-player two-action games. See also Morris and Shin (1999, 2001, 2003, 2004).

[3] Section 2 reviews the setup that leads to this figure.

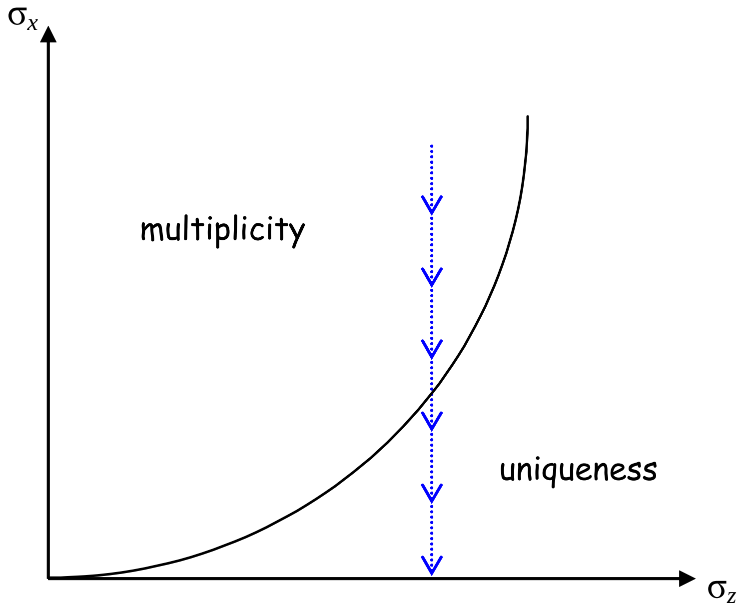

Figure 1: The Morris-Shin benchmark. σ_x and σ_z measure the noise in private and public information (both exogenous). Uniqueness is ensured if σ_x is sufficiently small.

criterion for the latter: a unique equilibrium survives if one introduces small idiosyncratic noise in the observation of the fundamentals. More generally, Morris-Shin provides a useful framework for studying how the information structure affects the determinacy and characterization of equilibria. As discussed above, the dispersion of valuable information regarding fundamentals plays a critical role for the uniqueness result. An earlier literature dealt with the transmission and aggregation of disperse information in rational-expectations equilibria. In particular, Green (1973) and Grossman (1981) highlight that prices may be excellent aggregators of dispersed information by showing that in some cases they can be fully revealing, yielding common knowledge of economic fundamentals.

Morris-Shin abstracts from the role of financial prices and other indicators as endogenous aggregators of information. Taking the information structure as exogenous is a useful first step and helps isolate the critical role played by disperse information. However, when financial prices convey information about the underlying fundamentals the dispersion of information is determined endogenously in equilibrium. Information aggregation can thus play an important role in determining whether multiple equilibria arise. Indeed, Atkeson (2000) notes that multiple equilibria may survive in the extreme case that financial prices are fully revealing and restore common knowledge.

For many applications of these models it seems most natural to allow for public sources of information that may aggregate private information, such as financial prices or other indicators of economic activity. Indeed, in situations where coordination is important individuals would find such information particularly valuable and might be expected to avidly seek them. Also, in many economic crisis the role played by information aggregators seems to have featured prominently. For example, the Argentine crisis in late 2002 involved the abandonment of a currency peg, sovereign-debt default, and the suspension of bank payments. Indeed, bank deposits and the peso-forward rate deteriorated steadily throughout that year. These variables where prominently reported in media and investor reports and it is hard to imagine that they did not play a role in the decision economic agents were making. Similarly, during riots and social unrest the level of others participation is surely observed to some extent and frequently reported in the public media.

This paper investigates the implications of endogenous information aggregation in coordination economies. In all cases, we avoid perfect revelation of fundamentals by allowing enough 'noise' in the aggregation process, as in Grossman and Stiglitz (1976, 1980). Thus, none of our results are driven by restoring perfect information, the main theme in Atkeson's comments. On the contrary, given the informational noise in the price or other indicator of aggregate activity, a reasonable conjecture might be that uniqueness survives again when private information is sufficiently precise. This conjecture turns out to be wrong, however, because it ignores the endogeneity of the information structure.

When the agents' private information regarding fundamentals becomes more precise, their actions become more sensitive to their information. Aggregation implies that the public indicator of others' actions or the financial price becomes more sensitive to variations in the underlying fundamentals. As a result, the precision of the *endogenous* public information is increasing in the precision of the *exogenous* private information.

For the issue of multiplicity or uniqueness of equilibria a horse-race between private and public information emerges. An increase in the precision of private information directly increases the dispersion of information and therefore makes coordination harder ceteris paribus. However, the indirect increase in the precision of public information makes coordination easier. We show that typically the latter effect dominates the former so that endogenous information reverses the Morris-Shin limiting result: multiplicity is *ensured* when individuals observe fundamentals with small enough idiosyncratic noise. Indeed, in this environment uniqueness cannot be viewed as a small perturbation around common knowledge.

To illustrate our results, let σ_x and σ_z measure again the noise in private and public information of fundamentals and let σ_ε measure the exogenous noise introduced in the aggregation process. The essential difference from the Morris-Shin case is that σ_z is now endogenous and depends on the exogenous parameters σ_x and σ_ε. Indeed, σ_z increases with

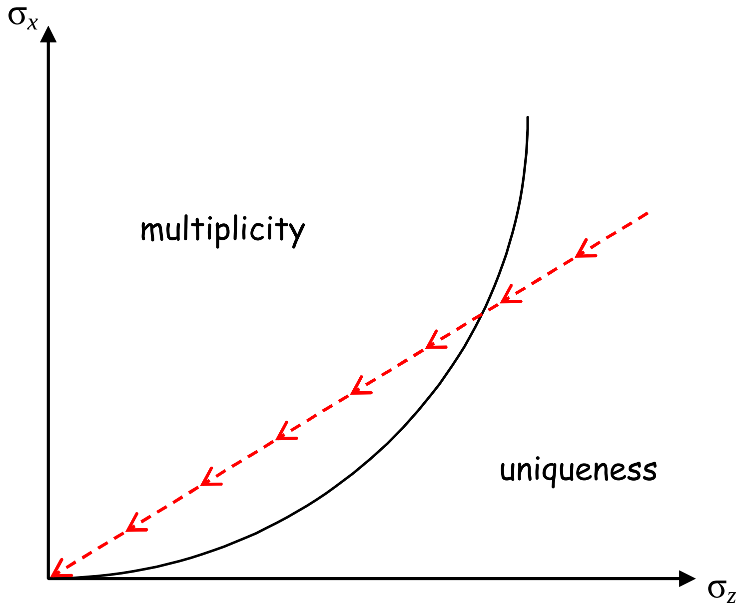

Figure 2: Endogenous public information. In equilibrium, σ_z is proportional to σ_x. As a result, multiplicity is ensured for sufficiently small σ_x.

both σ_ε and σ_x. Figure 2 considers a similar exercise as that in Figure 1 of increasing the precision of private information holding constant the exogenous source of precision of public information, σ_ε. Now that public information is endogenous, as σ_x falls, σ_z also falls, and eventually leaves the uniqueness region to enter the multiplicity region. The limit result of Morris-Shin is reversed: multiplicity reemerges and the common-knowledge outcomes are recovered as $\sigma_x \to 0$.

Figure 3 illustrates our results in terms of the exogenous information parameters $(\sigma_x, \sigma_\varepsilon)$: uniqueness is ensured if both σ_x and σ_ε are sufficiently large. Multiple equilibria instead exist if either the precision of private information or the quality of the indicator is sufficiently good. It follows that uniqueness cannot be seen as the outcome of a small perturbation of a common-knowledge environment: multiplicity survives if one introduces small exogenous noise. Moreover, for parameter values for which the equilibrium is unique, we find that a reduction in the exogenous noise in the economy may actually lead to an increase in volatility of endogenous variables. By introducing sunspots, the multiplicity results may thus be seen as an extreme reincarnation of a this comparative static regarding volatility.

As mentioned above, we consider a variety of endogenous information structures, including indicators other than financial prices. We begin with a model where the indicator is a noisy signal about others' actions, a situation that seems relevant in many applications. For

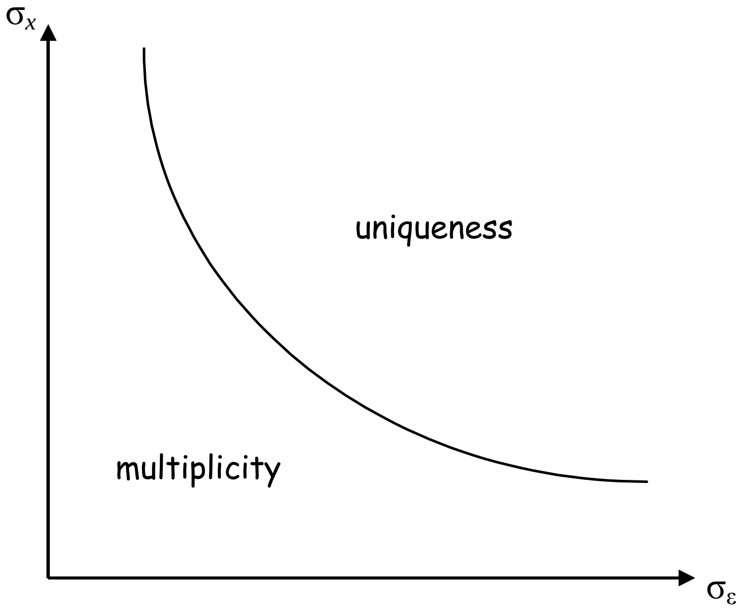

Figure 3: Endogenous public information. σ_x measures the exogenous noise in private information and σ_ε the exogenous noise in the macroeconomic indicator. Multiplicity is ensured if either type of noise is small.

instance, during riots or bank runs, an important part of the story is that people are actively watching what others are doing – ignoring this aspect may be missing an important piece of the puzzle. Second, this model parsimoniously highlights aspects of the problem that recur when financial prices are the instruments for aggregating information.

We next study environments where individuals cannot observe directly a signal about others' actions but instead can trade a financial asset. The asset market opens after agents have received their private information but prior to choosing whether to attack the regime. The rational-expectations equilibrium in the asset market generates imperfect public information that agents can use, in addition to their private information, when deciding whether or not to attack. This framework opens up new modeling choices regarding the specification of the asset's payoff and the agents' preferences over risky returns and we consider four different specifications that can be solved in closed form. In all the cases considered the quality of the public information generated by the price increases with the quality of the private information available to the agents. This parallels the case where the public indicator was a direct signal of others actions. Studying the asset-market environments, however, provides novel insights.

First, we qualify our multiplicity results by presenting a case where uniqueness still obtains in the limit as private information becomes infinitely precise. Thus, the winner in the horse-race between private and public information may depend on the details of the aggregation process. In particular, we show the importance of the sensitivity of asset demands with respect to expected returns and the whether the asset's dividends depends only on exogenous fundamentals or also on endogenous variables of the coordination game.

Second, we find that multiplicity may emerge not only with respect to the probability of regime change, but also with respect to the asset's demand and/or price function. This occurs only when the dividend is allowed to depend on endogenous variables of the coordination stage and is thus a reflection of the coordination problem. Interestingly, individual's decisions to attack or not are then uniquely pinned down as a function of the asset price and their private information. Thus, in these cases, the financial-market multiplicity is at the center stage of the equilibrium multiplicity.

Finally, we consider some interesting implications for asset price volatility when the equilibrium is unique. When the dividend depends only on economic fundamentals the volatility of the price necessarily decreases with a reduction in either source of noise, as in Grossman-Stiglitz. But when the dividend depends on endogenous variables from the coordination stage a reduction in noise may increase the endogenous volatility of the coordination outcome. This increases the volatility of the dividend and may consequently increase price volatility. Indeed, price multiplicity may be viewed as a more extreme version of this phenomena.

Related Literature

We have already discussed the relation of our work with that of Morris-Shin. Despite the difference in results, we view our paper as underscoring the general theme emphasized by Morris-Shin, that multiplicity or uniqueness may depend on details of the information structure and that these are worth exploring. In this respect, we stand on similar methodological grounds as Angeletos, Pavan and Hellwig (2003, 2004), which also endogenize the information structure, but in very different ways than the present paper – the one by considering policy interventions and the other by considering the dynamics of coordination.

The paper most closely related to ours is Mukherji, Tsyvinski and Hellwig (2004). They consider a currency-crises game in which financial prices directly affect the speculator's cost of attacking and the central bank's decision to devalue in addition to aggregating information. They focus, in particular, on how multiplicity depends on different institutional assumptions, namely whether devaluation is triggered by large reserve losses or high interest rates. Nevertheless, their results regarding information aggregation alone mirror and complement our analysis in Section 4. Also, related is Tarashev (2003), who endogenizes interest rates in the currency-crises model of Morris and Shin (1998) but does not investigate the implications for equilibrium multiplicity.

The rest of the paper is organized as follows. Section 2 introduces the basic model and reviews the Morris-Shin benchmark with exogenous public information. Section 3 introduces endogenous public information with a signal on aggregate actions. Section 4 studies the role of financial prices as endogenous aggregators of information. Section 5 recaps on the previous models and focuses on their implications for market volatility.

2 The Basic Model

We present an abstract general formulation of the basic model and then briefly discuss the various interpretations available in the literature.

Actions, Outcomes and Payoffs. There are two possible regimes, the status quo and an alternative. There is a measure-one continuum of agents, indexed by $i \in [0,1]$. Each agent can choose between an action that is favorable to the alternative regime and an action is favorable to the status quo. We call these actions, respectively, "attack" and "not attack". All agents move simultaneously.

We denote the regime outcome with $R \in \{0,1\}$, where $R = 0$ represents survival of the status quo and $R = 1$ represents collapse. We similarly denote the action of an agent with $a_i \in \{0,1\}$, where $a_i = 0$ represents "not attack" and $a_i = 1$ represents "attack".

The payoff from not attacking is normalized to zero. The payoff from attacking is $b > 0$ if the status quo is abandoned and $-c < 0$ otherwise. Hence, the utility of agent i is

$$U(a_i, R) = a_i(bR - c).$$

Finally, the status quo is abandoned $(R = 1)$ if and only if

$$A \geq \theta,$$

where $A \equiv \int a_i di \in [0,1]$ denotes the mass of agents attacking and $\theta \in \mathbb{R}$ parameterizes the exogenous strength of the status quo (or the quality of the economic fundamentals).

Note that the actions of the agents are strategic complements, since it pays for an individual to attack if and only if the status quo collapses and, in turn, the status quo collapses if and only if a sufficiently large fraction of the agents attacks. This coordination problem is the heart of the model.

To see the role of coordination most clearly, suppose for a moment that θ were commonly known by all agents and let $\underline{\theta} \equiv 0$ and $\overline{\theta} \equiv 1$. For $\theta \leq \underline{\theta}$, the regime is doomed with certainty and the unique equilibrium is every agent attacking. For $\theta > \overline{\theta}$, the regime can survive an attack of any size and the unique equilibrium is every agent not attacking. For $\theta \in (\underline{\theta}, \overline{\theta}]$, however, the regime is sound but vulnerable to a sufficiently large attack and therefore there are multiple equilibria sustained by self-fulfilling expectations. In one equilibrium, individuals expect everyone else to attack, they then find it individually optimal to attack, the status quo is abandoned, and expectations are vindicated. In another, individuals expect no one else to attack, they then find it individually optimal not to attack, the status quo is spared, and expectations are again fulfilled. The interval $(\underline{\theta}, \overline{\theta}]$ thus represents the set of "critical fundamentals" for which agents can coordinate on multiple courses of action under common knowledge.

Interpretations. This simple model can capture the role of coordination and multiplicity of equilibria in a variety of interesting applications.[4] For instance, in models of self-fulfilling currency crises (Obstfeld, 1986, 1996; Morris and Shin, 1998), there is a central bank interested in maintaining a currency peg and a large number of speculators, with finite wealth, deciding whether to attack the currency or not. In this context, a "regime change" occurs when a sufficiently large mass of speculators attacks the currency, forcing the central bank to abandon the peg. In models of self-fulling bank runs, on the other hand, a "regime change" occurs once a sufficiently large number of depositors decide to withdraw their deposits, relative to liquid resources available to the system, forcing the bank to suspend its payments. Other related interpretations include debt crises and financial crashes (Calvo, 1988; Cole and Kehoe, 1996; Morris and Shin, 2003, 2004). Finally, Atkeson (2000) interprets the model as describing riots: the potential rioters may or may not overwhelm the police force in charge of containing social unrest depending on the number of the rioters and the strength of the police force.

Information. When θ is common knowledge, individuals can perfectly forecast each other actions in equilibrium and can therefore perfectly coordinate on multiple courses of action. Following Morris and Shin (1998), we assume that θ is never common knowledge and that individuals instead have noisy private information about θ. Private information serves as an anchor for individual's actions that limits the ability to forecast each others' actions and may therefore brake the possibility of coordinating on multiple equilibria.

Initially, nature draws θ from a distribution which constitutes the agents' common prior about θ. For simplicity, we let this prior be a (degenerate) uniform over the entire real line.[5]

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[4] For a discussion of the defining properties and applications of regime-change games, see Angeletos, Hellwig and Pavan (2003, 2004).

[5] This assumption is without any serious loss of generality. At the cost of more notation and more algebra, we could easily extend the model to the case of a Normal prior. In fact, by letting the prior be uninformative, we have biased our results *against* multiplicity.

Agent i then observes a private signal

$$x_i = \theta + \xi_i,$$

where the idiosyncratic noise ξ_i is $\mathcal{N}(0, \sigma_x^2)$ with $\sigma_x > 0$ and is independent of θ. The signal x_i is thus a sufficient statistic for the private information of an agent.

Note that because there is a large number of agents and the private noise is independent across agents, the information contained by the entire economy, $(x_i)_{i \in [0,1]}$, is enough to infer the fundamental θ. However, this information is dispersed throughout the population, which is the key feature of the Morris-Shin framework. Finally, agents may also have access to some public information. We start by reviewing the Morris-Shin benchmark, where public information is *exogenous*. We then turn to our contribution, where public information is *endogenous*.

2.1 The Morris-Shin Benchmark: Exogenous Information

Before examining endogenous information aggregation, it is useful to review the case that public information is *exogenous*, as in Morris and Shin (1999, 2000, 2001) and Hellwig (2002).

Suppose for a moment that, in addition to their private signals, agents observe a public signal $z = \theta + v$, where $v \sim \mathcal{N}(0, \sigma_z^2)$ is common noise, distributed independently of both the fundamental θ and the private noise ξ. The exogenous information structure is then parametrized by σ_x and σ_z, the standard deviations of the private and the public noise. The equilibrium turns out to be unique if and only if the noise in private information is sufficiently small.

Proposition 1 (Morris-Shin) *Suppose agents observe exogenous public and private signals. An equilibrium always exists and it is unique if and only if* $\sigma_x / \sigma_z^2 \leq \sqrt{2\pi}$.

In general, the regime outcome depends on the particular realization of the public signal z, even when the equilibrium is unique. As the precision of private information becomes larger, however, this dependence diminishes. In the limit, the regime outcome is pinned down by fundamentals alone.

Proposition 2 (Morris-Shin limit) *In the limit as* $\sigma_x \to 0$ *or* $\sigma_z \to \infty$, *there is a unique equilibrium in which the regime changes if and only if* $\theta \leq \widehat{\theta}$, *where* $\widehat{\theta} = 1 - c/b \in (\underline{\theta}, \overline{\theta})$.

Note that, as $\sigma_x \to 0$, agents become perfectly informed about the fundamentals, like in the case of common knowledge. Nevertheless, the equilibrium outcome is unique and determined only by the fundamentals. This result, to which we refer to as the *Morris-Shin limit result*, is very attractive, because it suggests that a small perturbation around common knowledge obtains a unique equilibrium.

3 Endogenous Information I: Observable Actions

We now study the case where public information is *endogenous*. Agents no longer receive the public signal z as assumed in section 2.1. Instead, individuals are able to observe a public noisy signal of the aggregate activity of other agents.

We study two versions of such a model. In the first version, contemporaneous actions are observed with noise. Thus, our equilibrium concept is novel and unavoidably at the crossroads of rational-expectations and game theory.

The second version, in Section 3.4, has non-simultaneous moves by dividing the population into two groups, 'early' and 'late' movers. Individuals in the early group make their decisions to attack or not based solely on their private information. Individuals in the late group move and are able to observe a noisy signal of the early group's aggregate action. This non-simultaneous version only requires standard game-theoretic equilibrium concepts. We show that the equilibria of this second version converge to that of the first as the size of the early movers vanishes.

3.1 Set up and equilibrium definition

We assume that, in addition to their private information, agents can condition their behavior on a noisy indicator of the aggregate attack:

$$y = s(A, \varepsilon)$$

where $s : [0,1] \times \mathbb{R} \to \mathbb{R}$ and ε is noise $\mathcal{N}(0, \sigma_\varepsilon^2)$, distributed independently of the fundamentals θ and the idiosyncratic noise ξ. The information structure is then parameterized by the pair of standard deviations $(\sigma_x, \sigma_\varepsilon)$.

In any symmetric equilibrium agents are distinguished solely by their information, summarized by their observation of the private signal x_i and the public signal y. Let $a(x_i, y)$ denote the action chosen by such an agent. The equilibrium concept is a hybrid of a perfect Bayesian equilibrium and a rational-expectations equilibrium:

A *symmetric rational-expectations equilibrium* consists of an endogenous signal $y =$

$Y(\theta, \varepsilon)$, an individual attack strategy $a(x, y)$, and an aggregate attack $A(\theta, y)$, that satisfy:

$$\begin{aligned} a(x,y) &= \arg\max_{a\in[0,1]} \mathbb{E}\left[\, U(a, R(\theta,y)) \mid x,\ y \,\right] & (1) \\ A(\theta,y) &= \int_x a(x,y) d\Phi\left(\frac{x-\theta}{\sigma_x}\right) & (2) \\ y &= s(A(\theta,y),\varepsilon) & (3) \end{aligned}$$

for all $(\theta, \varepsilon, x, y) \in \mathbb{R}^4$, where $R(\theta, y) = 1$ if $A(\theta, y) \geq \theta$ and $R(\theta, y) = 0$ otherwise.

Condition (1) means that $a(x, y)$ is the optimal strategy for the agent given that regime change occurs if and only if $A(\theta, y) \geq \theta$, whereas condition (2) means that $A(\theta, y)$ is simply the aggregate across agents. Of course, the aggregate public signal y must be consistent with individual actions which gives condition (3). This is the rational-expectations feature in our equilibrium concept.[6]

For reasons of tractability, we specify the signal function s as

$$s(A, \varepsilon) = \Phi^{-1}(A) + \varepsilon.$$

As we will see, this specification allows the equilibrium to preserve normality of the information structure, which in turn permits closed-form solutions. This convenient specification was introduced by Dasgupta (2001) in a different setup. Finally, we focus on symmetric equilibria where the information structure is normally distributed and the strategy of the agents is monotone in private information.[7] We refer to such equilibria simply as *monotone equilibria*.

3.2 Equilibrium Analysis

We now study the equilibrium conditions (1)-(3). In monotone equilibria, for any realization of y, there exist thresholds $x^*(y)$ and $\theta^*(y)$ such that an agent attacks if and only if $x \leq x^*(y)$ and the regime changes if and only if $\theta \leq \theta^*(y)$. A monotone equilibrium is thus identified with the triplet of mappings x^*, θ^* and Y.

We construct the set of monotone equilibria in four steps. In Step 1, we start with an arbitrary x^* used by the agents and use conditions (2) and (3) to characterize the implied

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[6] In Section 3.4, we show that our results survive in a variant of the model which avoids the simultaneity in the signal and allows for a standard game-theoretic equilibrium concept.

[7] As we shall see 3.4, normality of the information structure is an implication when the signal is non-simultaneous.

aggregate attack A, the resulting θ^* and the possible public signals Y. In Step 2, we take θ^* and Y as given and use condition (1) to compute the threshold x^{**} that is individually optimal. In Step 3, we study the fixed point $x^* = x^{**}$. Finally, in Step 4, we consider the determinacy of Y.

Step 1. In a monotone equilibrium, $a(x, y) = 1$ if and only if $x \leq x^*(y)$, for some function x^*. The aggregate attack is then

$$A(\theta, y) = \Phi\left(\sqrt{\alpha_x}(x^*(y) - \theta)\right), \tag{4}$$

where $\alpha_x = \sigma_x^{-2}$. Note that $A(\theta, y)$ is decreasing in θ so there exists a function $\theta^*(y)$ such that $A(\theta, y) \geq \theta$ if and only if $\theta \leq \theta^*(y)$. The threshold $\theta^*(y)$ solves $A(\theta^*(y), y) = \theta^*(y)$, or equivalently

$$x^*(y) = \theta^*(y) + \frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(y)). \tag{5}$$

Equilibrium condition (3) implies that the signal must satisfy,

$$y = \sqrt{\alpha_x}\left[x^*(y) - \theta\right] + \varepsilon,$$

or equivalently

$$x^*(y) - \sigma_x y = \theta - \sigma_x \varepsilon. \tag{6}$$

For any $(\theta, \varepsilon) \in \mathbb{R}^2$, let $z = \widetilde{Z}(\theta, \varepsilon) \equiv \theta - \sigma_x \varepsilon$ and note that (6) is a relation between y and z. Define the correspondence

$$\mathcal{Y}(z) = \left\{\ y \in \mathbb{R} \mid x^*(y) - \sigma_x y = z\ \right\}. \tag{7}$$

In Step 4, we show that $\mathcal{Y}(z)$ is non-empty and examine when it is single- or multi-valued.

Now take any function $\tilde{Y}(z)$ that is a selection from this correspondence, i.e., such that $\tilde{Y}(z) \in \mathcal{Y}(z)$ for all z, and let the signal be $Y(\theta, \varepsilon) = \tilde{Y}(\widetilde{Z}(\theta, \varepsilon)) = \tilde{Y}(\theta - \sigma_x \varepsilon)$. As we shall see any such selection preserves normality of the information structure.

Step 2. We now map θ^* and Y to x^{**}. Given that regime change occurs if and only if $\theta \leq \theta^*(y)$, the expected payoff for the agent is given by

$$\mathbb{E}\left[\ U(a, R(\theta, \varepsilon)) \mid x, y\ \right] = a\left\{b \Pr\left[\ \theta \leq \theta^*(y) \mid x, y\ \right] - c\right\}.$$

We thus need to consider the determination of the posterior probability $\Pr\left[\ \theta \leq \theta^*(y) \mid x, y\ \right]$.

The observation of $y = Y(\theta, \varepsilon) = \tilde{Y}(z)$ is equivalent to the observation of

$$Z(y) \equiv x^*(y) - \sigma_x y = \theta - \sigma_x \varepsilon = z$$

That is, it is as if the agents observe a public signal z about θ, with noise $\mathcal{N}(0, \sigma_x^2 \sigma_\varepsilon^2)$. Recall that each agent also observes a private signal about θ, with idiosyncratic noise $\mathcal{N}(0, \sigma_x^2)$. Let $\alpha_x \equiv \sigma_x^{-2}$, $\alpha_\varepsilon \equiv \sigma_\varepsilon^{-2}$, and $\alpha_z \equiv \alpha_x \alpha_\varepsilon \equiv (\sigma_x \sigma_\varepsilon)^{-2}$. Combining the two sources of information we conclude that the posterior of an agent is

$$\theta \mid x, y \ \sim \ \mathcal{N}\left(\ \delta x + (1-\delta) Z(y)\ ,\ \alpha^{-1}\ \right),$$

where $\alpha = \alpha_x + \alpha_z$ is the total precision of information and $\delta = \alpha_x / (\alpha_x + \alpha_z)$ is the precision of x relative to the total.

It follows that

$$\Pr\left[\theta \le \theta^*(y) \mid x, y\right] = 1 - \Phi\left(\sqrt{\alpha}\left(\delta x + (1-\delta)\, Z(y) - \theta^*(y)\right)\right).$$

Note that the above is monotonic in x so the agent attacks if and only if $x \le x^{**}(y)$, where $x^{**}(y)$ solves

$$b \Pr\left[\ \theta \le \theta^*(y) \mid x^{**}(y), y\ \right] = c.$$

Combining the above two conditions and substituting $Z(y) = x^*(y) - y/\sqrt{\alpha_x}$, we find that $x^{**}(y)$ must solve

$$\Phi\left(\sqrt{\alpha}\left(\delta x^{**}(y) + (1-\delta)\left(x^*(y) - \frac{1}{\sqrt{\alpha_x}} y\right) - \theta^*(y)\right)\right) = \frac{b-c}{b}. \tag{8}$$

Step 3. In equilibrium, $x^{**} = x^*$ and (8) reduces to

$$\Phi\left(\sqrt{\alpha}\left(x^*(y) - \theta^*(y) - \frac{1-\delta}{\sqrt{\alpha_x}} y\right)\right) = \frac{b-c}{b}.$$

Combining the above with (5), using $\delta = \alpha_x/(\alpha_x + \alpha_z)$ and $\alpha = \alpha_x + \alpha_z$, and rearranging, we obtain:

$$\begin{aligned} \theta^*(y) &= \Phi\left(\frac{\alpha_z}{\alpha_x + \alpha_z} y + \sqrt{\frac{\alpha_x}{\alpha_x + \alpha_z}} \Phi^{-1}\left(\frac{b-c}{b}\right)\right), && (9) \\ x^*(y) &= \theta^*(y) + \frac{1}{\sqrt{\alpha_x}} \Phi^{-1}\left(\theta^*(y)\right), && (10) \end{aligned}$$

where $\alpha_z = \alpha_x \alpha_\varepsilon$. Hence, for all σ_x and σ_ε, the equilibrium x^* and θ^* are determined uniquely and irrespectively of the selected equilibrium signal Y. Moreover, both $\theta^*(y)$ and $x^*(y)$ are increasing in y. Finally, $\theta^*(y)$ does not depend on σ_x, $\theta^*(y) \to 1 - c/b$ as $\sigma_\varepsilon \to \infty$ and $\theta^*(y) \to \Phi(y)$ as $\sigma_\varepsilon \to 0$.

Step 4. We finally need to consider the equilibrium correspondence $\mathcal{Y}(z)$. Recall that this is given by the set of solutions to

$$x^*(y) - \frac{1}{\sqrt{\alpha_x}} y = z.$$

Using (9) and (10), the above reduces to

$$F(y) \equiv \Phi\left(\frac{\alpha_z}{\alpha_x + \alpha_z} y + \Lambda\right) + \frac{1}{\sqrt{\alpha_x}}\left(-\frac{\alpha_x}{\alpha_x + \alpha_z} y + \Lambda\right) = z, \tag{11}$$

where $\Lambda \equiv \sqrt{\alpha_x / \left(\alpha_x + \alpha_z\right)}\Phi^{-1}(1 - c/b)$. Equilibrium existence and uniqueness thus reduces to existence and uniqueness of solution to (11).

Note that $F(y)$ is continuous in y, and $F(y) \to -\infty$ as $y \to +\infty$, and $F(y) \to +\infty$ as $y \to -\infty$. Thus, the correspondence $\mathcal{Y}(z)$ is non-empty and an equilibrium always exist. To examine uniqueness, we ask whether $\mathcal{Y}(z)$ is single-valued for all z, which is true if and only if F is monotonic in y. Differentiating F we obtain

$$F'(y) = -\frac{\sqrt{\alpha_x}}{\alpha_x + \alpha_z}\left(1 - \frac{\alpha_z}{\sqrt{\alpha_x}} \phi\left(\frac{\alpha_z}{\alpha_x + \alpha_z} y + \Lambda\right)\right).$$

It follows that the determinacy of equilibrium hinges on the ratio $\alpha_z / \sqrt{\alpha_x}$, like in the Morris-Shin benchmark. Since $\max_{w \in \mathbb{R}} \phi(w) = 1/\sqrt{2\pi}$, we have that F is decreasing in y, and therefore $\mathcal{Y}(z)$ is single-valued for all z, if and only if $\alpha_z / \sqrt{\alpha_x} \leq \sqrt{2\pi}$. If instead $\alpha_z / \sqrt{\alpha_x} > \sqrt{2\pi}$, there are thresholds $\underline{z}$ and $\overline{z}$ such that $\mathcal{Y}(z)$ takes one value for $z \notin (\underline{z}, \overline{z})$ but three values for $z \in (\underline{z}, \overline{z})$. These thresholds are given by $\underline{z} = F(\underline{y})$ and $\overline{z} = F(\overline{y})$, where $\underline{y}$ and $\overline{y}$ are, respectively, the lowest and highest solution with $F'(y) = 0$. Unlike the Morris-Shin benchmark, however, the ratio $\alpha_z / \sqrt{\alpha_x}$ is endogenous. Using $\alpha_z = \alpha_\varepsilon \alpha_x$, we conclude that the equilibrium is unique if and only if $\alpha_\varepsilon \sqrt{\alpha_x} \leq \sqrt{2\pi}$.

The following proposition summarizes these results.

Proposition 3 (Morris-Shin meet Grossman-Stiglitz) *A monotone equilibrium is characterized by a triplet of mappings* (Y, x^*, θ^*) *such that the endogenous signal satisfies* $y = Y(\theta, \varepsilon)$*, a agent attacks if and only if* $x \leq x^*(y)$*, and regime change occurs if and only if* $\theta \leq \theta^*(y)$*.*

A monotone equilibrium exists for all $(\sigma_x, \sigma_\varepsilon)$ *and is unique if and only if* $\sigma_\varepsilon^2 \sigma_x \geq 1/\sqrt{2\pi}$. *If* $\sigma_\varepsilon^2 \sigma_x < 1/\sqrt{2\pi}$, *the equilibrium signal function* Y *is indeterminate, but the equilibrium threshold functions* x^* *and* θ^* *remain unique and independent of the selected* Y.

We conclude that the equilibrium is unique only if there is enough noise in both sources of information, the exogenous information of the agents and the endogenous signal about aggregate activity. Multiple equilibria survive as long as either source of information is sufficiently precise.

Interestingly, when multiplicity arises, it is with respect to aggregate outcomes but not with respect to individual behavior. To understand this result, consider the common-knowledge limit ($\sigma_x = \sigma_\varepsilon = 0$), in which case $x = \theta$ and $y = \Phi^{-1}(A)$, so that the agent learns θ perfectly by observing x and learns A by observing y. The agent then finds it optimal to attack if and only if $A \geq \theta$, or equivalently $x \leq \Phi(y)$. Here, the equilibrium strategy $a(x, y)$ for the agent is uniquely determined with $x^*(y) = \Phi(y)$. However, the equilibrium values of A and y are not uniquely determined. Instead, for every $\theta \in (\underline{\theta}, \overline{\theta}]$, both $A = 0$ and $A = 1$ can be sustained in equilibrium.[8] When σ_x and σ_ε are non-zero, the same nature of indeterminacy remains. The equilibrium behavior of the agent is uniquely determined for any given observation x and y, but there can be multiple equilibrium values of A and y for any given realization of θ and ε.

Finally, since our endogenous-information economy is different from the exogenous information economy of Morris-Shin, it is interesting that the determinacy of equilibrium in both cases hinges on exactly the same ratio $\alpha_z/\sqrt{\alpha_x}$. However, note that, in equilibrium, the information generated by y is equivalent to the information generated by $z = Z(y) = \theta - \sigma_x \varepsilon$. Our endogenous-information economy is thus related to an exogenous-information economy with precision of public information given by $\alpha_z = \alpha_y = \alpha_\varepsilon \alpha_x$. Indeed, substituting this expression into the criterion for multiplicity from proposition, **??** that $\sigma_x/\sigma_z^2 > \sqrt{2\pi}$, yields the criterion for multiplicity in proposition 3, that $\sigma_\varepsilon^2 \sigma_x < 1/\sqrt{2\pi}$. As the precision of private information becomes infinite so does the precision of public information.

3.3 Information Limits

Proposition 3 establishes that, for any given level of noise in the agents' private information, multiple equilibria exist if and only if the noise in the macroeconomic indicator is sufficiently

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[8] If $A = 0$, then $y = -\infty$ and $x^*(y) = \Phi^{-1}(-\infty) = \underline{\theta}$, in which case all agents attack whenever $\theta \leq \underline{\theta}$ and no agent attacks whenever $\theta > \underline{\theta}$. If instead $A = 1$, then $y = +\infty$ and $x^*(y) = \Phi^{-1}(+\infty) = \overline{\theta}$, in which case all agents attack whenever $\theta \leq \overline{\theta}$ and no agent attacks whenever $\theta > \overline{\theta}$. In the former case, a regime change is triggered if and only if $\theta \leq \underline{\theta}$; in the latter, if and only if $\theta \leq \overline{\theta}$.

small. Intuitively, an increase in σ_ε reduces the public information generated by the observation of y and thus reduces the ability of the market to coordinate on multiple courses of action.

Proposition 4 (Limit $\sigma_\varepsilon \to \infty$) *Fix σ_x and let $\sigma_y \to \infty$. In the limit, the regime changes if and only if $\theta \leq \widehat{\theta}$, where $\widehat{\theta} \equiv 1 - c/b \in (\underline{\theta}, \overline{\theta})$.*

The Morris-Shin outcome is obtained as the noise in the observation of aggregate activity becomes arbitrarily large. This is intuitive, for in this case no information is generated by the observation of y and the endogeneity of public information is of no importance.

Consider next the limit of the precision of agents' private information, for given level of noise in y. Proposition 3 establishes that, for given σ_ε, multiple equilibria exist if and only if σ_x is sufficiently small. The interval $(\underline{z}, \overline{z})$ represents the region of multiplicity and a reduction in σ_x reduces $\underline{z}$ and increases $\overline{z}$ making larger the multiplicity region. Indeed, as $\sigma_x \to 0$ we can show that any outcome is possible for all $\theta \in (\underline{\theta}, \overline{\theta})$.

Proposition 5 (Limit $\sigma_x \to 0$) *Fix σ_ε and let $\sigma_x \to 0$. For every $\theta \in (\underline{\theta}, \overline{\theta})$, there exists an equilibrium in which the probability of regime change converges to zero, as well as an equilibrium in which the probability of regime change converges to one.*

This result stands in stark contrast to the Morris-Shin limit result in Proposition 2. With exogenous public information a unique equilibrium survives as the noise in private information vanishes. With endogenous public information as modeled here the multiplicity present with common knowledge obtains as the noise in private information vanishes. The reason is once again the endogeneity of public information: As the precision of private information increases, the precision of public information also increases, and indeed at the same rate, so that common knowledge is recovered in the limit.

3.4 Non-Simultaneous Signal

The analysis so far has assumed that agents can condition their decision to attack on a noisy indicator of *contemporaneous* aggregate behavior. Our rational-expectations equilibrium thus required that agents choose optimally given the observed signal and that this signal be generated by the aggregation of individual choices. We now show that our results are robust to a perturbation that breaks the simultaneity in the signal by introducing some simple dynamics and therefore allows for a standard game-theoretic equilibrium concept.

The population is divided into two groups, 'early' and 'late' agents. Early agents move first, on the basis of their private information alone. Late agents move second, on the basis of their private information as well as a noisy public signal about the aggregate activity of early agents. Neither group can observe contemporaneous activity, but late agents can condition their behavior on the activity of early agents.

Let $\mu \in (0,1)$ denote the fraction of early agents, A_1 the aggregate activity of early agents, and A_2 the aggregate activity of late agents. The regime changes if and only if $\mu A_1 + (1-\mu)A_2 \geq \theta$. The signal generated by early agents and observed only by late agents is given by

$$y_1 = \Phi^{-1}(A_1) + \varepsilon, \tag{12}$$

where $\varepsilon \sim \mathcal{N}(0, \sigma_\varepsilon^2)$ is independent of θ and ξ.

Early agents can condition their actions only on their private information, whereas late agents can condition their actions also on y_1. An equilibrium is thus characterized by a threshold x_1^* and a pair of threshold functions $x_2^*(y_1)$ and $\theta^*(y_1)$ such that: an early agent attacks if and only if $x \leq x_1^*$; a late agent attacks if and only if $x \leq x_2^*(y_1)$; and the regime is abandoned if and only if $\theta \leq \theta^*(y_1)$. Like in the case that the signal was simultaneous, the aggregate size of the attack and the regime outcome depend on the particular realization of the signal, but there is no more a fixed-point relation between the signal and the size of the attack. As a result, only standard game-theoretic concepts are need to define the equilibrium.

Most importantly, the main insight of the previous analysis survives: As the precision of the private information of (early) agents increases, the precision of the public information available to late agents also increases, indeed at the same rate. As a result, multiplicity is once again bound to obtain for sufficiently small private noise.

Unfortunately, characterizing the equilibria of the game with non-simultaneous signal is complicated by the fact that early agents face a double forecast problem, as they are uncertain about both the fundamental θ and the signal y_1 upon which late agents will condition their behavior. Nevertheless, as $\mu \to 0$, the impact of early agents on the size of the attack vanishes. In the Appendix we actually show that any equilibrium of the benchmark simultaneous-signal game can be approximated by an equilibrium of the non-simultaneous-signal game as $\mu \to 0$. We conclude that our multiplicity results are robust to non-simultaneity of the signal.

4 Endogenous Information II: Financial Prices

The analysis so far has assumed that agents can condition their decision to attack on a noisy indicator of aggregate behavior. Instead, we now allow agents to observe a *financial price* that is determined earlier in a competitively asset market.

We modify the environment as follows. There are two stages and we refer to stage 1 as the 'asset market'. All individuals begin with the same endowment of wealth w and an exogenous private signal $x_i = \theta + \xi_i$, where the noise ξ_i is as before.

In the first stage, agents trade a financial asset that has a dividend that depends on the underlying fundamentals, either directly or indirectly. In the second stage, agents decide whether to attack the regime or not as in the basic model. The return of the asset and the regime outcome are realized at the end of the second stage. Here, in deciding whether to attack agents can no longer condition their choice on a direct signal of the aggregate attack as in Section 3. However, in addition to their private information, they can use information revealed by the equilibrium price from the first stage.

This framework opens up new modeling choices regarding the specification of the financial asset's payoff and the preferences over risky payoffs. Following Grossman-Stiglitz, we guide our choice with an eye towards tractability. The four specifications we solve below are designed so that they preserve normality of the information structure and are solvable in closed-form.

We first introduce some general notation and state the equilibrium conditions. We denote by p the price of the asset or more generally some measure of the terms of trade. Let f represent the dividend paid by the asset and k_i the investment agent i makes in this asset. For all cases we consider we can express the indirect utility the agent enjoys from his portfolio choice by a function $V(k, f, p)$ so that the total payoff is

$$u_i = V(k_i, f, p) + U(a_i, R),$$

where the utility from attacking U is just as in Section 2. We consider two alternative specifications for the dividend of the financial asset, one in which the dividend depends on the exogenous fundamentals and another in which it depends on the endogenous outcome of the second stage. We thus let $f = f(\theta, A)$.

Let aggregate demand for the asset be $K = \int k_i di$. We assume there is a shock ε to the exogenous net supply of the asset. One interpretation of the net supply shock is that it results from a shock to the demand of other 'noisy' traders. This shock is not observed by individuals and we assume that it is $\mathcal{N}(0, \sigma_\varepsilon^2)$ and independent of both the fundamentals and the private noise. Market clearing requires $K = \varepsilon$, which determines an equilibrium price function $p = P(\theta, \varepsilon)$. As in Grossman and Stiglitz (1976, 1980), the role of the shock ε is to introduce noise in the information revealed by financial prices about fundamentals. Since the price function is a function of both θ and ε the observation of p does not reveal θ

perfectly, leading to a signal-extraction problem.

A *rational-expectations equilibrium* is a price function, $p = P(\theta, \varepsilon)$, individual strategies for investment and attacking, $k(x, p)$ and $a(x, p)$, and aggregate investment and attack functions, $K(\theta, p)$ and $A(\theta, \varepsilon)$, such that, in the first stage:

$$k(x,p) = \arg\max_{k\in\mathbb{R}} \mathbb{E}\left[\ V(k, f(\theta, A), p) \mid x, p\ \right] \tag{13}$$

$$K(\theta,p) = \int_x k(x,p) d\Phi\left(\frac{x-\theta}{\sigma_x}\right) \tag{14}$$

$$K\left(\theta, P(\theta,\varepsilon)\right) = \varepsilon \tag{15}$$

and in the second stage:

$$a(x,p) = \arg\max_{a\in[0,1]} \mathbb{E}\left[\ U(a,R) \mid x, p\ \right], \tag{16}$$

$$A(\theta,p) = \int_x a(x,p) d\Phi\left(\frac{x-\theta}{\sigma_x}\right), \tag{17}$$

where $R = 1$ if $A(\theta, p) \geq \theta$ and $R = 0$ otherwise.

The information an agent has consists of the privately observed signal x and the publicly observed price p. In this sense, the price p takes the place that the signal y had in the observable action model of Section 3. Condition (15) imposes market clearing in the asset market. Condition (13) and (17) requires that an agent's investment take into account the information contained in their private information and prices. Note that the equilibrium in the first stage is a standard

The four specifications we consider below generate the following information structure. There is a strictly monotone function $Z(y)$ and a random variable v that is $\mathcal{N}(0, Q(\alpha_x, \alpha_\varepsilon)^{-1})$ and independent of θ and ξ such that $Z(P(\theta, \varepsilon)) = \theta + v$ for every realization of θ and ε. Thus, the observation of an equilibrium price realization p is informationally equivalent to the observation of a public signal $z = Z(p)$ on θ with normal error and precision $\alpha_p = Q(\alpha_x, \alpha_\varepsilon)$.

The agent's posterior conditional on his private information x and the observed price p is

$$\theta \mid x,p\ \sim \mathcal{N}\left(\ \delta x + (1-\delta) z\ ,\ \alpha^{-1}\ \right),$$

where $\delta = \alpha_x/(\alpha_x + \alpha_p)$ and $\alpha = \alpha_x + \alpha_p$. Like in the Morris-Shin benchmark, the determinacy of equilibrium turns out to depend on the ratio

$$\frac{\alpha_p}{\sqrt{\alpha_x}},$$

that is, the ratio of the precision of the public information generated by the price to the square root of the precision of the exogenous private information. If $\alpha_p/\sqrt{\alpha_x} < \sqrt{2\pi}$, then the equilibrium is unique. If instead $\alpha_p/\sqrt{\alpha_x} > \sqrt{2\pi}$, then there are multiple equilibria. Unlike Morris-Shin, however, the precision of public information α_p, and therefore the ratio $\alpha_p/\sqrt{\alpha_x}$, are endogenous and affected by σ_x.

In what follows, we consider four alternative specifications of the asset market (stage 1). In each case, we solve for the equilibrium price function and the associated mappings Z and Q. We then examine the critical ratio $\alpha_p/\sqrt{\alpha_x}$ to study the determinacy of equilibrium as a function of the exogenous information structure, $(\alpha_x, \alpha_\varepsilon)$, or equivalently $(\sigma_x, \sigma_\varepsilon)$.

4.1 Risk Aversion – Exogenous Dividend

We start with an example that maps directly to the CARA-normal framework introduced by Grossman and Stiglitz (1976, 1980) [see also Hellwig (1980)]. The agent can invest his wealth either in a risky asset or a risk-less asset. We normalize the gross return of the risk-less asset: it costs the agent 1 in the first stage and delivers 1 in the second stage. The risky asset costs the agent p in the first stage and delivers $f = f(\theta) = \theta$ in the second. Here the return of the asset depends directly on the exogenous fundamental.

The agent enjoys utility only from second-stage consumption and his preferences over second-stage consumption exhibit constant absolute risk aversion (CARA). The indirect utility from his portfolio choice is thus given by

$$V(k, f, p) = u\left(w - pk + fk\right), \qquad u(c) = -\frac{1}{\gamma}\exp(-\gamma c), \tag{18}$$

where k is the amount invested in the risky asset, $w - pk$ is invested in the risk-less asset, $c = w - pk + fk$ is second-period consumption, and $\gamma > 0$ is the coefficient of absolute risk aversion.

Proposition 6 *Suppose V is given by (18) and $f = \theta$. The equilibrium price function $P(\theta, \varepsilon)$ is always uniquely determined. There are multiple equilibrium thresholds $x^*(p)$ and $\theta^*(p)$ if and only if $\sigma_\varepsilon^2 \sigma_x^3 < \gamma^2 (2\pi)^{-1/2}$.*

Proof. Because of the CARA-normal specification, and using $f = \theta$, we have

$$\mathbb{E}[\ V(k,f,p) \mid x,p\ ] = \frac{1}{\gamma}\exp\left\{-\frac{1}{\gamma}\left[\mathbb{E}[\theta|x,p]k - \frac{\gamma}{2}\mathrm{Var}[\theta|x,p]k^2\right]\right\}.$$

It follows that optimal demand satisfies

$$k(x,p) = \frac{\mathbb{E}[\ \theta \mid x,p\ ] - p}{\gamma \mathrm{Var}[\ \theta \mid x,p\ ]}.$$

We then guess and verify that

$$\mathbb{E}[\ \theta \mid x,p\ ] = \delta x + (1-\delta)p \quad \text{and} \quad \mathrm{Var}[\ \theta \mid x,p\ ] = \alpha^{-1},$$

for some $\delta \in (0,1)$ and $\alpha > 0$. The individual demand then reduces to

$$k(x,p) = \frac{\delta\alpha}{\gamma}(x-p),$$

in which case the aggregate demand for the asset is

$$K(\theta,p) = \frac{\delta\alpha}{\gamma}(\theta - p).$$

In equilibrium, $K = \int k_i di = \varepsilon$, and therefore the equilibrium price satisfies

$$p = P(\theta,\varepsilon) = \theta - \frac{\gamma}{\delta\alpha}\varepsilon. \tag{19}$$

By implication, the observation of p is equivalent to the observation of a public signal about θ with precision $(\delta\alpha/\gamma)^2\alpha_\varepsilon$. That is, in this case we have $Z(p) = p$, $v = -(\delta\alpha/\gamma)\varepsilon$, and $\alpha_p = (\delta\alpha/\gamma)^2\alpha_\varepsilon$.

We now determine δ and α. Note that x and $Z(p) = p$ are independent signals of θ with precision α_x and α_p, respectively. It follows that $\mathbb{E}[\ \theta \mid x,p\ ] = \delta x + (1-\delta)p$, where

$$\begin{aligned} \delta &= \frac{\alpha_x}{\alpha_x + \alpha_p} = \frac{\alpha_x}{\alpha} \\ \alpha &= \alpha_x + \alpha_p = \alpha_x + (\delta\alpha/\gamma)^2\alpha_\varepsilon \end{aligned}$$

Solving the above gives $\alpha = \alpha_x(1 + \alpha_x\alpha_\varepsilon/\gamma^2)$, $\delta = 1/(1+\alpha_x\alpha_\varepsilon/\gamma^2)$, and

$$\alpha_p = Q(\alpha_\varepsilon, \alpha_x) = \frac{\alpha_\varepsilon\alpha_x^2}{\gamma^2}. \tag{20}$$

Recall that in Section 3 we found that the precision of endogenous information increased proportionally with the precision of private information. Here the precision of public information increases *more* than proportionally with the precision of private information. This will only serve to reinforce our conclusions regarding the comparative static and limit exercises for σ_x.

To verify this, consider stage 2. A monotone (continuation) equilibrium is characterized by thresholds $x^*(p)$ and $\theta^*(p)$ such that an agent attacks if and only if $x \leq x^*(p)$ and the regime changes if and only if $\theta \leq \theta^*(p)$. The threshold $x^*(p)$ solves $b \Pr\left[\theta \leq \theta^*(p) | x, p\right] = c$, or equivalently

$$\Phi\left(\sqrt{\alpha}\left[\ \delta x^*(p) + (1-\delta)Z(p) - \theta^*(p)\ \right]\right) = \frac{b-c}{b}. \tag{21}$$

The threshold $\theta^*(p)$, on the other hand, solves $A(\theta^*(p), p) = \theta^*(p)$, or equivalently

$$x^*(p) = \theta^*(p) + \frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(p)). \tag{22}$$

Combining the above two conditions and using $Z(p) = p$, we have that $\theta^*(p)$ can be sustained in equilibrium if and only if it solves

$$\frac{\alpha_p}{\sqrt{\alpha_x}}(p - \theta^*(p)) + \Phi^{-1}(\theta^*(p)) = \sqrt{\frac{\alpha_x + \alpha_p}{\alpha_x}}\Phi^{-1}\left(1 - c/b\right). \tag{23}$$

Similar arguments as those used for Proposition 1 imply that there are multiple $\theta^*(p)$ if and only if $\alpha_p/\sqrt{\alpha_x} > \sqrt{2\pi}$, where α_p is given by (20). On the other hand, the price function is given by (19) and is uniquely determined. **QED**

As in the benchmark model, multiple equilibria survive as long as either noise is small enough. Indeed, the common-knowledge outcomes are once again obtained in the limit as $\sigma_x \to 0$ for any given σ_ε. Unlike the benchmark model, however, the indeterminacy arises with respect to individual strategies. Because the dividend is a function of the exogenous fundamentals, the price function is always uniquely determined.

Corollary 1 *As* $\sigma_x \to 0$, $P(\theta, \varepsilon) \to \theta$ *for all* (θ, ε), *in every equilibrium.*

4.2 Risk Aversion – Endogenous Dividend

We now modify the previous example by letting $f = f(A) = -\Phi^{-1}(A)$. That is, the dividend of the asset is now a function of the equilibrium size of the attack realized in the second stage.

Proposition 7 *Suppose* V *is given by (18) and* $f = -\Phi^{-1}(A)$. *The equilibrium thresholds* $x^*(p)$ *and* $\theta^*(p)$ *are always uniquely determined. There are multiple equilibrium price* *functions* $P(\theta,\varepsilon)$ *if and only if* $\sigma_\varepsilon^2\sigma_x^3 < \gamma^2/\sqrt{2\pi}$.

Proof. We earlier showed that, in equilibrium, $A = \Phi\left(\sqrt{\alpha_x}[x^*(p) - \theta]\right)$. It follows that $f = \sqrt{\alpha_x}[\theta - x^*(p)]$ and therefore

$$k = \frac{\sqrt{\alpha_x}\mathbb{E}[\ \theta \mid x, p\ ] - \sqrt{\alpha_x}x^*(p) - p}{\gamma\alpha_x \mathrm{Var}[\ \theta \mid x, p\ ]}.$$

Let

$$\widetilde{p} = \frac{1}{\sqrt{\alpha_x}}p + x^*(p), \tag{24}$$

and note that, for every p, the above defines a unique $\widetilde{p}$. We can then rewrite the optimal demand as

$$k = \frac{\mathbb{E}[\ \theta \mid x, p\ ] - \widetilde{p}}{\widetilde{\gamma}\mathrm{Var}[\ \theta \mid x, p\ ]},$$

where $\widetilde{\gamma} = \gamma\sqrt{\alpha_x}$. The rest is then as in the previous example, provided we replace γ with $\widetilde{\gamma}$. In particular, we have

$$\widetilde{p} = \theta - \frac{\widetilde{\gamma}}{\delta\alpha}\varepsilon, \tag{25}$$

so that $Z(p) = \widetilde{p}$, $v = -(\delta\alpha/\widetilde{\gamma})\varepsilon$, and $\alpha_p = \alpha_\varepsilon\alpha_x^2/\widetilde{\gamma}^2$. Using $\widetilde{\gamma} = \gamma\sqrt{\alpha_x}$, we conclude that the precision of the information revealed by the price is now given by

$$\alpha_p = Q(\alpha_\varepsilon, \alpha_x) = \frac{\alpha_\varepsilon\alpha_x^3}{\gamma^2}. \tag{26}$$

Once again, the precision of public information increases *more* than proportionally with the precision of private information, which only reinforces our results.

Indeed, consider stage 2. As in the previous example, the thresholds $x^*(p)$ and $\theta^*(p)$ solve (21) and (22). The difference is that now the endogenous signal is given by $Z(p) = \widetilde{p} = \frac{1}{\sqrt{\alpha_x}}p + x^*(p)$. Hence, (**??**) is now replaced by

$$\theta^*(p) = \Phi\left(\frac{\sqrt{\alpha_x}}{\sqrt{\alpha_x+\alpha_p}}\Phi^{-1}\left(1 - c/b\right) - \frac{\alpha_p}{\alpha_x+\alpha_p}p\right), \tag{27}$$

where α_p is given by (26). On the other hand,

$$x^*(p) = \theta^*(p) + \frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(p)). \tag{28}$$

It follows that the threshold $\theta^*(p)$ and $x^*(p)$ are uniquely determined. What may be indeterminate now is the price function.

Using (24), (25), and (27), we have that p must solve

$$F(p) \equiv \Phi\left(-\frac{\alpha_p}{\alpha_x+\alpha_p}p+\Lambda\right)+\frac{1}{\sqrt{\alpha_x}}\left[\frac{\alpha_x}{\alpha_x+\alpha_p}p+\Lambda\right]=z \tag{29}$$

where $\Lambda = \Phi^{-1}\left(\frac{b-c}{b}\right)\sqrt{\alpha_x}/\sqrt{\alpha_x+\alpha_p}$ and $z = \theta - (\delta\alpha/\widetilde{\gamma})\varepsilon$. This equation is analogous to equation (11) in the benchmark model. Note that $F(p)$ is continuous in p, and $F(p) \to -\infty$ as $p \to -\infty$, and $F(p) \to +\infty$ as $p \to +\infty$, which implies that a solution always exists. Moreover,

$$F'(p) = \frac{\alpha_p}{\alpha_x+\alpha_p}\left(\frac{\sqrt{\alpha_x}}{\alpha_p}-\phi\left(-\frac{\alpha_p}{\alpha_x+\alpha_p}p+\Lambda\right)\right),$$

so the solution is unique for all z if $\alpha_p/\sqrt{\alpha_x} < \sqrt{2\pi}$. If instead $\alpha_p/\sqrt{\alpha_x} > \sqrt{2\pi}$, there are thresholds $\underline{z}$ and $\overline{z}$ such that there exist multiple equilibrium prices whenever $z \in (\underline{z}, \overline{z})$. **QED**

The results here are reminiscent of the ones in the benchmark model. In equilibrium, the price plays the role of an anticipatory signal of the size of the attack. As a result, the strategies of the agents are uniquely determined, but the equilibrium price function may not be. In contrast, in the previous example the price played the role of a signal for the exogenous fundamental θ, in which indeterminacy emerged for individual strategies and not for the price function. In either case, however, the endogeneity of public information implies that its precision is increasing with the precision of private information, that multiplicity survives with small noise, and that the common-knowledge outcomes obtain in the limit as $\sigma_x \to 0$ for any given σ_ε.

Corollary 2 *There is an equilibrium in which, for all* $\theta \in (\underline{\theta}, \overline{\theta})$ *and all* ε, $P(\theta, \varepsilon) \to -\infty$ *as* $\sigma_x \to 0$. *There is also an equilibrium in which, for all* $\theta \in (\underline{\theta}, \overline{\theta})$ *and all* ε, $P(\theta, \varepsilon) \to +\infty$ *as* $\sigma_x \to 0$.

4.3 Risk Neutrality – Exogenous Dividend

We modify the asset market and the preferences as follows. There is no risk-less bond. One unit of the asset costs 1 in the first period and pays $f - p$ in the second period. The indirect utility from the portfolio choice is thus given by

$$V(k, f, p) = u_1(w-k) + u_2((f-p)k) \tag{30}$$

where k is the amount invested, u_1 is the utility from first-period consumption, u_2 is the utility from second-period consumption, and U is the payoff from attacking.

Proposition 8 *Suppose* V *is given by (30) and* $f = \theta$. *The equilibrium price function* P *is always uniquely determined. There are multiple equilibrium thresholds* $x^*(p)$ *and* $\theta^*(p)$ *if and only if* σ_x *is either sufficiently small or sufficiently high relative to* σ_ε.

Thus, the Morris-Shin limit result is preserved in this example. Like in all previous cases we have examined, the precision of public information increases with the precision of private information. Unlike the previous cases, however, this effect is not strong enough to restore multiplicity when σ_x is sufficiently small.

4.4 Risk Neutrality – Endogenous Dividend

We modify the previous example by letting the return of the asset be $f = -\Phi^{-1}(A)$, where A is the aggregate size of the attack occurring in the second stage. The analysis is now similar to the second example. In particular, we the strategies in stage 2 are unique but the price function in stage 1 can be indeterminate.

Proposition 9 *Suppose* V *is given by (30) and* $f = -\Phi^{-1}(A)$. *The equilibrium thresholds* $x^*(p)$ *and* $\theta^*(p)$ *are always uniquely determined. There are multiple equilibrium price functions* $P(\theta, \varepsilon)$ *if and only if* σ_x *and/or* σ_ε *are sufficiently small.*

Contrasting this case with the previous one, we see that the endogeneity of the asset's dividend helps the case for multiplicity. Indeed, the fact that sensitivity of the dividend itself to the fundamental increases with the precision of private information compensates for the fact that the sensitivity of individual demand to the expected dividend does not increase fast enough with private information. As a result, coordination once again becomes easier as private information becomes more precise, and the common-knowledge outcomes are obtained as $\sigma_x \to 0$.

5 Volatility

The results on equilibrium multiplicity translate directly to results on market volatility if we add sunspots. Indeed, the possibility that agents can condition their behavior on random variables unrelated to the fundamentals introduces "spurious" volatility.

In the exogenous-information context of Morris-Shin, multiplicity disappears when agents observe the fundamentals with small idiosyncratic noise. By implication, there can be no such spurious volatility for small σ_x. Indeed, as $\sigma_x \to 0$, the regime outcome becomes independent of the aggregate noise and volatility therefore vanishes.

With endogenous information aggregation, however, the impact of noise on volatility can be quite different. A reduction in either σ_x or σ_ε may perversely increase volatility by ensuring multiplicity and therefore introducing more spurious volatility. Indeed, for any θ in the critical region, as either $\sigma_x \to 0$ or $\sigma_\varepsilon \to 0$, the regime can either collapse or survive, purely as a function of the sunspot. In this sense, volatility is maximized when the noise vanishes.

Interestingly, the property that less noise may increase volatility does not rely the existence of multiple equilibria. In the next, we show that, even when the equilibrium is unique, a reduction in noise may increase the sensitivity of the regime outcome to the noise and may therefore lead to more rather than less market volatility.

5.1 Regime Volatility

Consider the model of Section 3, where agents observe a noisy signal about the aggregate size of the attack. The equilibrium thresholds (x^*, θ^*) and the signal y satisfy the following set of equations:

$$\begin{aligned} \theta^* &= \Phi\left(\frac{\alpha_z}{\alpha_x+\alpha_z}y + \sqrt{\frac{\alpha_x}{\alpha_x+\alpha_z}}\Phi^{-1}\left(\frac{b-c}{b}\right)\right), && (31)\\ x^* &= \theta^* + \frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*), && (32)\\ y &= \Phi^{-1}(A) = \sqrt{\alpha_x}(x^*-\theta), && (33) \end{aligned}$$

where $\alpha_z = \alpha_x\alpha_\varepsilon$. Solving these three equations gives the endogenous variables (x^*, θ^*, y) as functions of the exogenous variables (θ, ε). The solution for θ^*, in particular, satisfies[9]

$$\frac{1}{\alpha_\varepsilon}\Phi^{-1}(\theta^*) - \sqrt{\alpha_x}\theta^* = -\sqrt{\alpha_x}\theta + \varepsilon + \frac{\sqrt{1+\alpha_\varepsilon}}{\alpha_\varepsilon}\Phi^{-1}(1-c/b) \qquad (34)$$

When the equilibrium is unique, the left-hand side is monotonic in θ^*, so that the above defines θ^* as a unique function of the fundamentals θ, the supply shock ε, and the parameters $(\alpha_x, \alpha_\varepsilon, c, b)$.

To examine the comparative statics of the volatility of the regime outcome with respect the noise structure, it is useful to normalize the supply shock. The normalized shock is

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[9] To see this, substitute x^* from (32) into (33), substitute the resulting expression for y into (31), use $\alpha_z = \alpha_x\alpha_\varepsilon$, and rearrange.

defined by $\widetilde{\varepsilon} \equiv \sqrt{\alpha_\varepsilon}\varepsilon$, so that $\widetilde{\varepsilon} \sim \mathcal{N}(0,1)$. Condition (34) then becomes

$$\frac{1}{\sqrt{\alpha_\varepsilon}}\Phi^{-1}\left(\theta^*\right) - \sqrt{\alpha_\varepsilon \alpha_x}\theta^* = -\sqrt{\alpha_\varepsilon \alpha_x}\theta + \widetilde{\varepsilon} + \frac{\sqrt{1+\alpha_\varepsilon}}{\alpha_\varepsilon}\Phi^{-1}\left(1 - c/b\right). \tag{35}$$

We can then identify an increase in the volatility of the regime outcome with an increase in the sensitivity of the regime threshold θ^* with respect to the normalized shock $\widetilde{\varepsilon}$.

By (35), the slope of θ^* with respect to $\widetilde{\varepsilon}$ is given by the reciprocal of the following:

$$\frac{\partial \theta^*}{\partial \widetilde{\varepsilon}} = \left[\frac{1}{\sqrt{\alpha_\varepsilon}} \frac{1}{\phi\left(\Phi^{-1}\left(\theta^*\right)\right)} - \sqrt{\alpha_\varepsilon \alpha_x}\right]^{-1} > 0$$

It is then immediate that, keeping θ^* constant, an increase in either α_ε or α_x (that is, a reduction in either σ_ε or σ_x) necessarily increases the slope $\partial\theta^*/\partial\widetilde{\varepsilon}$. As a result, less noise may lead to more volatility in the regime outcome.[10]

5.2 Price Volatility

We next examine the comparative statics of the volatility of prices. To economize, we consider only the CARA-normal examples of Section 4; the analysis of the risk-neutral examples is similar. To emphasize the implications for volatility that do not derive from multiplicity, we first focus on case that the equilibrium is unique or that there are no sunspots.

When the dividend is exogenous, we have $f = \theta$,

$$p = \theta - \frac{1}{\sqrt{\alpha_p}}\widetilde{\varepsilon},$$

and $\alpha_p = \alpha_\varepsilon \alpha_x^2/\gamma^2$. When instead the dividend is endogenous, we have $f = \Phi^{-1}\left(A\right) = \sqrt{\alpha_x}(\theta - x^*)$,

$$p = \sqrt{\alpha_x}(\widetilde{p} - x^*) = \sqrt{\alpha_x}(\theta - x^*) - \sqrt{\frac{\alpha_x}{\alpha_p}}\widetilde{\varepsilon},$$

and $\alpha_p = \alpha_\varepsilon \alpha_x^3/\gamma^2$. In either case, we can write the equilibrium price as the sum of the dividend and the supply shock appropriately weighted:

$$p = f - \left(\gamma \sigma_\varepsilon \sigma_x^2\right)\widetilde{\varepsilon}$$

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[10] Of course, θ^* does not stay constant as α_ε or α_x change. In particular, θ^* satisfies single-crossing, it decreases for low values of $\widetilde{\varepsilon}$ and increases for high values of $\widetilde{\varepsilon}$. Given θ, let $\widetilde{\varepsilon}_0$ be the unique value of $\widetilde{\varepsilon}$ for which θ^* stays constant with a change in α_ε or α_x. For any $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$ such that $\widetilde{\varepsilon}_1 < \widetilde{\varepsilon}_0 < \widetilde{\varepsilon}_2$, we still have that $|\theta^*(\widetilde{\varepsilon}_2) - \theta^*(\widetilde{\varepsilon}_2)|$ increases with either α_ε or α_x. In this sense, volatility increases with a reduction in noise.

It remains true, however, that, at least for intermediate values of $\widetilde{\varepsilon}$, It is immediate then that, keeping the volatility of f constant, the volatility of the price decreases with a reduction in either σ_ε or σ_x. But what about the volatility of the dividend?

When the dividend is exogenous, the volatility of f is simply the volatility of exogenous fundamental and is thus indeed independent of either σ_ε or σ_x. Like in Grossman-Stiglitz, the impact of noise on price volatility is thus the natural one: More noise implies more volatility.

When instead the dividend is endogenous, the volatility of f depends, not only on the exogenous volatility of the fundamentals, but also on the endogenous volatility of the regime outcome, namely the volatility of x^*. The latter, as we discussed before, may well increase with a reduction in either σ_ε or σ_x. As a result, the overall impact of noise on price volatility is now ambiguous: A reduction in noise reduces the volatility of the price for any given volatility in the dividend, but it also increases the volatility of the dividend itself. In some cases, indeed, the second effect dominates, so that price volatility increases with a reduction in noise.

We conclude:

Proposition 10 *When the dividend is exogenous, the volatility of the price necessarily decreases with a reduction in either σ_ε or σ_x. But when the dividend depends on the coordination outcome, a reduction in either noise may increase price volatility.*

Our earlier results regarding the multiplicity of prices can thus be viewed as an extreme reincarnation of the above result. When the noise is sufficiently small, volatility can be high, not only because the dividend (and therefore the price) is very sensitive to the exogenous noise, but also because the dividend (and therefore the price) can depend on arbitrary sunspots.

6 Final Remarks

Building on Morris and Shin (1998) this paper introduced instruments that endogenized the sources of public information in models where coordination is important. We modeled public information by either: (i) a noisy signal of aggregate activity; or (ii) a financial asset's price that reveals information in equilibrium. An important feature of the equilibrium in all cases is that the precision of public information is endogenous and rises with the precision of private information.

We showed that in all but one of the six models considered this effect is strong enough to reverse the limiting uniqueness result obtained with exogenous public information. Thus, typically, with endogenous public information multiplicity is *ensured* when individuals observe fundamentals with small enough idiosyncratic noise. Conversely, uniqueness is ensured if idiosyncratic noise is large enough.

We view the main theme in Morris-Shin as emphasizing the importance of the details of the information structures for the multiplicity or uniqueness of equilibria. This paper contributes to this same theme by studying the importance of endogenous information aggregation.

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7 Appendix

Proof of Proposition 1. In a monotone equilibrium, for any realization of z, there is a threshold $x^*(z)$ such that an agent attacks if and only if $x \leq x^*(z)$. By implication, the aggregate size of the attack is decreasing in θ, so that there is also a threshold $\theta^*(z)$ such that the status quo is abandoned if and only if $\theta \leq \theta^*(z)$. A monotone equilibrium is thus identified with a pair $x^*(z)$ and $\theta^*(z)$. In step 1, below, we characterize the equilibrium θ^* for given x^*. In step 2, we characterize the equilibrium x^* for given θ^*. In step 3, we characterize both conditions and examine equilibrium existence and uniqueness.

Step 1. For given realizations of θ and z, the aggregate size of the attack is given by the mass of agents who receive signals $x \leq x^*(z)$. That is,

$$A(\theta, z) = \Phi\left(\sqrt{\alpha_x}(x^*(z) - \theta)\right),$$

where $\alpha_x = \sigma_x^{-2}$ is the precision of private information. Note that $A(\theta, z)$ is decreasing in θ, so that regime change occurs if and only if $\theta \leq \theta^*(z)$, where $\theta^*(z)$ is the unique solution to

$$A(\theta^*(z), z) = \theta^*(z).$$

Rearranging we obtain:

$$x^*(z) = \theta^*(z) + \frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(z)). \tag{36}$$

Step 2. Given that regime change occurs if and only if $\theta \leq \theta^*(z)$, the payoff of an agent is

$$\mathbb{E}\left[U(a, R(\theta, \varepsilon))|x, z\right] = a(b \Pr\left[\theta \leq \theta^*(z) \mid x, z\right] - c).$$

Let $\alpha_x = \sigma_x^{-2}$ and $\alpha_z = \sigma_z^{-2}$ denote, respectively, the precision of private and public information. The posterior of the agent is

$$\theta \mid x, z \sim \mathcal{N}\left(\ \delta x + (1-\delta)z\ ,\ \alpha^{-1}\ \right),$$

where $\delta \equiv \alpha_x/(\alpha_x + \alpha_z)$ is the relative precision of private information and $\alpha \equiv \alpha_x + \alpha_z$ is the overall precision of information. Hence, the posterior probability of regime change is

$$\Pr\left[\ \theta \leq \theta^*(z) \mid x, z\ \right] = 1 - \Phi\left(\sqrt{\alpha}(\delta x + (1-\delta)z - \theta^*(z))\right),$$

which is monotonic in x. It follows that the agent attacks if and only if $x \leq x^*(z)$, where

$x^*(z)$ solves the indifference condition

$$b\Pr\left[\ \theta \leq \theta^*(z) \mid x^*(z), z\ \right] = c.$$

Substituting the expression for the posterior and the definition of δ and α, we obtain:

$$\Phi\left(\sqrt{\alpha_x+\alpha_z}\left(\frac{\alpha_x}{\alpha_x+\alpha_z}x^*(z)+\frac{\alpha_z}{\alpha_x+\alpha_z}z-\theta^*(z)\right)\right)=\frac{b-c}{b}. \tag{37}$$

Step 3. Combining (36) and (37), we conclude that $\theta^*(z)$ can be sustained in equilibrium if and only if it solves

$$G\left(\theta^*(z), z\right) = g, \tag{38}$$

where $g = \sqrt{1+\alpha_z/\alpha_x}\Phi^{-1}\left(1-c/b\right)$ and

$$G(\theta, z) \equiv \frac{\alpha_z}{\sqrt{\alpha_x}}\left(z-\theta\right)+\Phi^{-1}\left(\theta\right).$$

With $\theta^*(z)$ given by (38), $x^*(z)$ is then given by (36). We are now in a position to establish existence and determinacy of the equilibrium by considering the properties of the function G. Note that, for every $z \in \mathbb{R}$, $G(\theta, z)$ is continuous in θ, with $G(\underline{\theta}, z) = -\infty$ and $G(\overline{\theta}, z) = \infty$, which implies that there necessarily exists a solution and any solution satisfies $\theta^*(z) \in (\underline{\theta}, \overline{\theta})$. This establishes existence; we now turn to uniqueness. Note that

$$\frac{\partial G(\theta, z)}{\partial \theta} = \frac{1}{\phi(\Phi^{-1}\left(\theta\right))} - \frac{\alpha_z}{\sqrt{\alpha_x}}$$

Since $\max_{w\in\mathbb{R}}\phi(w) = 1/\sqrt{2\pi}$ then if $\alpha_z/\sqrt{\alpha_x} \leq \sqrt{2\pi}$ we have that G is strictly increasing in θ, which implies a unique solution to (38). If instead $\alpha_z/\sqrt{\alpha_x} > \sqrt{2\pi}$, then G is non-monotonic in θ and there is an interval $(\underline{z}, \overline{z})$ such that (36) admits multiple solutions $\theta^*(z)$ whenever $z \in (\underline{z}, \overline{z})$ and a unique solution otherwise. We conclude that monotone equilibrium is unique if and only if $\alpha_z/\sqrt{\alpha_x} \leq \sqrt{2\pi}$. **QED**

Proof of Proposition 2. Consider the limits as $\sigma_x \to 0$ for given σ_z, or $\sigma_z \to \infty$ for given σ_x. In either case, $\alpha_z/\sqrt{\alpha_x} \to 0$ and $\sqrt{\left(\alpha_x+\alpha_z\right)/\alpha_x} \to 1$. Condition (38) then implies that $\theta^*(z) \to \widehat{\theta} = 1 - c/b$ for all z, so that the regime-change threshold is unique and independent of z. Similarly, condition (36) gives $x^*(z) \to \widehat{x}$, where $\widehat{x} = \widehat{\theta}$ if we consider the limit $\sigma_x \to 0$, and $\widehat{x} = \widehat{\theta} + \sigma_x\Phi^{-1}(\widehat{\theta})$ if we instead consider the limit $\sigma_z \to \infty$. **QED.**

Proof of Proposition 4. From conditions (9) and (10) we have that, for every y, $\theta^*(y) \to 1 - c/b = \widehat{\theta}$ and $x^*(y) \to \widehat{\theta} + \sigma_x\Phi^{-1}(\widehat{\theta}) = \widehat{x}$ as $\sigma_\varepsilon \to \infty$. Condition (6) then implies

$\theta - \sigma_x \varepsilon = x^*(y) - \sigma_x y \to \widehat{x} - \sigma_x y$ and therefore the unique signal function in the limit is $Y(\theta, \varepsilon) \to (\widehat{x} - \theta)/\sigma_x + \varepsilon$.

Proof of Proposition 5. First, note that $\underline{y} \to -\infty$ and $\overline{y} \to +\infty$ as $\sigma_x \to 0$. Next, note that both $|\sigma_\varepsilon^2 \sigma_x - \phi(\underline{y})|$ and $|\sigma_\varepsilon^2 \sigma_x - \phi(\overline{y})|$ vanish. Since $\lim_{y \to -\infty} \phi(y)y = \lim_{y \to +\infty} \phi(y)y = 0$, the latter implies $\sigma_x \underline{y} \to 0$ and $\sigma_x \overline{y} \to 0$. Hence, $\underline{z} \to \Phi(-\infty) = \underline{\theta}$ and $\overline{z} \to \Phi(+\infty) = \overline{\theta}$ as $\sigma_x \to 0$. Moreover, for every θ and ε, $\theta - \sigma_x \varepsilon \to \theta$ as $\sigma_x \to 0$. It follows that

$$\Pr\left[\ \theta - \sigma_x \varepsilon \in (\underline{z}, \overline{z}) \mid \theta \in (\underline{\theta}, \overline{\theta})\ \right] \to 1 \ \text{ as } \ \sigma_x \to 0.$$

Next, let $\underline{Y}(\theta, \varepsilon) \equiv \min \mathcal{Y}(\theta - \sigma_x \varepsilon)$ and $\overline{Y}(\theta, \varepsilon) \equiv \max \mathcal{Y}(\theta - \sigma_x \varepsilon)$ and consider (θ, ε) such that $\theta - \sigma_x \varepsilon \in (\underline{z}, \overline{z})$. Note that $\underline{Y}(\theta, \varepsilon) < \underline{y} < \overline{y} < \overline{Y}(\theta, \varepsilon)$ and therefore

$$\underline{Y}(\theta, \varepsilon) \to -\infty \ \text{ and } \ \overline{Y}(\theta, \varepsilon) \to +\infty \ \text{ as } \ \sigma_x \to 0.$$

From (9), $\theta^*(y)$ is independent of σ_x, $\theta^*(y) \to \Phi(-\infty) = \underline{\theta}$ as $y \to -\infty$, and $\theta^*(y) \to \Phi(+\infty) = \overline{\theta}$ as $y \to +\infty$. It follows that, as long as $\theta \in (\underline{\theta}, \overline{\theta})$,

$$\Pr\left[\ \theta \leq \theta^*\left(\underline{Y}(\theta, \varepsilon)\right)\ \right] \to 0 \ \text{ and } \ \Pr\left[\ \theta \leq \theta^*\left(\overline{Y}(\theta, \varepsilon)\right)\ \right] \to 1 \ \text{ as } \ \sigma_x \to 0,$$

which establishes the result. **QED**

Non-simulataneous signal. Let $\mathcal{E}_{(\mu)}$ denote the economy with the non-simultaneous signal introduced above and $\mathcal{E}$ the economy with the simultaneous signal analyzed in the previous sections.

Consider any monotone equilibrium of $\mathcal{E}_{(\mu)}$. An early agent attacks if and only if $x \leq x_1^*$, for some threshold x_1^*. The aggregate attack of early agents is thus

$$A_1(\theta) = \Phi\left(\sqrt{\alpha_x}\,[\theta - x_1^*]\right). \tag{39}$$

Hence, in equilibrium

$$y_1 = \Phi^{-1}\left(A_1(\theta)\right) + \varepsilon = \sqrt{\alpha_x}\,[x_1^* - \theta] + \varepsilon.$$

The observation of y is thus equivalent to the observation of a public signal z, which is defined by

$$z = x_1^* - \frac{1}{\sqrt{\alpha_x}} y = \theta - \sigma_x \varepsilon.$$

The strategy of a late agent is contingent on his private signal x and the public signal y.

Since y and z has the same informational content, we can equivalently express the strategy of a late agent as a function of x and z. Hence, in a monotone equilibrium, a late agent attacks if and only if $x \leq x_2^*(z)$, for some threshold function x_2^*. It follows that the aggregate attack of late agents is

$$A_2(\theta, z) = \Phi\left(\sqrt{\alpha_x}\left[\theta - x_2^*(z)\right]\right). \tag{40}$$

Combining (39) and (40), we obtain the overall size of attack:

$$A(\theta, z) = \mu\Phi\left(\sqrt{\alpha_x}\left[\theta - x_1^*\right]\right) + (1-\mu)\Phi\left(\sqrt{\alpha_x}\left[\theta - x_2^*(z)\right]\right).$$

It follows that the regime changes if and only if $\theta \leq \theta^*(z)$, where $\theta^*(z)$ solves $A(\theta^*(z), z) = \theta^*(z)$, or equivalently

$$\mu\Phi\left(\sqrt{\alpha_x}\left[\theta^*(z) - x_1^*\right]\right) + (1-\mu)\Phi\left(\sqrt{\alpha_x}\left[\theta^*(z) - x_2^*(z)\right]\right) = \theta^*(z). \tag{41}$$

Next, consider the optimal behavior of the agents. Note that the realization of z is known to the late agents but unknown to the early agents. The threshold $x_2^*(z)$ thus solves $b \Pr\left[\theta \leq \theta^*(z) | x_2^*(z), z\right] = c$, or equivalently

$$\Phi\left(\sqrt{\alpha}\left(\delta x_2^*(z) + (1-\delta) z - \theta^*(z)\right)\right) = \frac{b-c}{b}, \tag{42}$$

where $\delta = \alpha_x/(\alpha_x + \alpha_z)$ and $\alpha = \alpha_x + \alpha_z$. The threshold x_1^*, on the other hand, solves $b \Pr\left[\theta \leq \theta^*(y) | x_1^*\right] = c$, or equivalently

$$\int \Phi\left(\sqrt{\alpha_x}\left[\theta^*(z) - x_1^*\right]\right) \sqrt{\alpha_1}\phi\left(\sqrt{\alpha_1}\left[z - x\right]\right) dz = \frac{b-c}{b}, \tag{43}$$

where we used the fact that, conditional on x, z is distributed normal with precision $\alpha_1 = \alpha_x\alpha_\varepsilon/(1+\alpha_\varepsilon)$.[11]

Solving (41) for $x_2^*(z)$ gives

$$x_2^*(z) = \theta^*(z) - \sqrt{\alpha_x}\Phi^{-1}\left(\theta^*(z) + \frac{\mu}{1-\mu}\left[\theta^*(z) - \Phi\left(\sqrt{\alpha_x}\left[\theta^*(z) - x_1^*\right]\right)\right]\right).$$

Substituting the above into (42) and using $\delta = \alpha_x/(\alpha_x + \alpha_z)$ and $\alpha = \alpha_x + \alpha_z$, we obtain:

$$\Gamma(\theta^*(z), z, x_1^*, \mu) = g, \tag{44}$$

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[11] To see this, note that $z = \theta - \sigma_x\varepsilon = x - \xi - \sigma_x\varepsilon$, so that $z|x \sim \mathcal{N}\left(0, \sigma_x^2 + \sigma_x^2\sigma_\varepsilon^2\right)$.

where $g = \sqrt{1+\alpha_z/\alpha_x}\Phi^{-1}\left(1-c/b\right)$ and

$$\Gamma(\theta, z, x_1, \mu) = \frac{\alpha_z}{\sqrt{\alpha_x}}\left(z-\theta\right) + \Phi^{-1}\left(\theta + \frac{\mu}{1-\mu}\left[\theta - \Phi\left(\sqrt{\alpha_x}\left[\theta - x_1\right]\right)\right]\right).$$

For any $x_1^* \in \mathbb{R}$ and any $z \in \mathbb{R}$, we have that $\Gamma(\theta, z, x_1^*, \mu)$ is continuous in θ, with $\Gamma(\underline{\theta}, z, x_1^*, \mu) = -\infty$ and $\Gamma(\overline{\theta}, z, x_1^*, \mu) = \infty$. Hence, for any given threshold $x_1^* \in \mathbb{R}$, condition (44) determines a function $\theta^* : \mathbb{R} \to [\underline{\theta}, \overline{\theta}]$. On the other hand, for any given function $\theta^* : \mathbb{R} \to [\underline{\theta}, \overline{\theta}]$, condition (43) determines a threshold $x_1^* \in \mathbb{R}$. An equilibrium is any joint solution to (43) and (44).

Consider now the limit as $\mu \to 0$. Note that, for all $(\theta, z, x_1^*) \in \mathbb{R}^2$, $\mu \to 0$ implies

$$\Gamma(\theta, z, x_1^*, \mu) \to G(\theta, z) = \frac{\alpha_z}{\sqrt{\alpha_x}}\left(z-\theta\right) + \Phi^{-1}\left(\theta\right).$$

Note that G is independent of x_1^* and is the same as in the Morris-Shin benchmark. Consider now a function $\theta^* : \mathbb{R} \to [\underline{\theta}, \overline{\theta}]$ such that, for all z,

$$G(\theta^*(z), z) = g. \tag{45}$$

As shown earlier, θ^* is unique if and only if $\alpha_z/\sqrt{\alpha_x} \leq \sqrt{2\pi}$. If instead $\alpha_z/\sqrt{\alpha_x} > \sqrt{2\pi}$, there are multiple θ^* solving (45). Moreover, for any such θ^*, (43) admits a unique solution $x_1^* \in \mathbb{R}$. We conclude that, for μ small enough, there are multiple solutions to (43) and (44) whenever $\alpha_z/\sqrt{\alpha_x} > \sqrt{2\pi}$. But $\alpha_z = \alpha_\varepsilon \alpha_x$, so that multiple equilibria again emerge as long as $\alpha_\varepsilon\sqrt{\alpha_x} > \sqrt{2\pi}$.

The equilibria of $\mathcal{E}_{(\mu)}$ approximate the equilibria of $\mathcal{E}$ also in the following sense. Let $\mathcal{E}_{(\mu)}$ denote the "dynamic" economy of this section and $\mathcal{E}$ the "static" economy of the previous section. Let x^* and θ^* denote the equilibrium thresholds for $\mathcal{E}$, $\mathcal{Y}$ the correspondence defined in (7), and $\widetilde{Y}$ a function selected from this correspondence. For any $\omega > 0$, we can find μ small enough such that $\left|\theta^*_{(\mu)}(z) - \theta^*(\widetilde{Y}(z))\right| < \omega$ and $\left|x^*_{2(\mu)}(z) - x^*(\widetilde{Y}(z))\right| < \omega$ for all z, where $x^*_{2(\mu)}$ and $\theta^*_{(\mu)}$ are the thresholds associated with an equilibrium of $\mathcal{E}_{(\mu)}$. To see this, note that θ^* and $\widetilde{Y}$ are part of an equilibrium for $\mathcal{E}$ if and only if the composite $\theta^* \circ \widetilde{Y}$ is a solution to (45).

Finally, let us introduce a random variable y_2 defined by

$$y_2 = \Phi^{-1}\left(A_2\right) + \varepsilon,$$

where ε is the same realization as the one in (12). y_2 is a noisy indicator about the activity of late agents. If y_2 is unobservable, late agents continue to attack if and only if $x \leq x_2^*(z)$.

Hence, in equilibrium, $\Phi^{-1}(A_2) + \varepsilon = \sqrt{\alpha_x}\,[x_2^*(z) - z]$ and y_2 is a function of z alone:

$$y_2(z) = \sqrt{\alpha_x}\,(x_2^*(z) - z)\,.$$

Since y_2 conveys no more information than z and thus no more information than y_1, nothing changes if we allow late agents to condition their behavior on y_2 in addition to y_1. That is, the equilibria of the economy where late agents observe both y_1 and y_2 coincide with the equilibria of the economy where late agents observe only y_1.

Now consider again the limit as $\mu \to 0$. Since $x_{2(\mu)}^*(z) \to x^*(\widetilde{Y}(z))$, we have $y_{2(\mu)}(z) \to \sqrt{\alpha_x}\left[x^*(\widetilde{Y}(z)) - z\right]$. By definition of $\widetilde{Y}(z)$, we have $\widetilde{Y}(z) = \sqrt{\alpha_x}\left[x^*(\widetilde{Y}(z)) - z\right]$. Hence, $y_{2(\mu)}(z) \to \widetilde{Y}(z)$. That is, in the limit as $\mu \to 0$, the random variable $y_{2(\mu)}$ that is part of an equilibrium in economy $\mathcal{E}_{(\mu)}$ solves the fixed-point condition $y = \Phi^{-1}(A(\theta, y)) + \varepsilon$ for economy $\mathcal{E}$. Conversely, any $\widetilde{Y}(z)$ that is part of an equilibrium for $\mathcal{E}$ can be approximated by a random variable $y_{2(\mu)}$ of economy $\mathcal{E}_{(\mu)}$. This indeed provides a justification for the equilibrium selection we made in Section 3.1. Any equilibrium of the "static" economy in which the signal y is not a function of z alone, if it exists, can not be approximated by an equilibrium of a "dynamic" economy.

Proof of Proposition 8. Consider an agent who receives a private signal x and observes a price p. His optimal investment k solves

$$u_1'(w-k) = \mathbb{E}\left[\ (f-p)u_2'\left((f-p)k\right)\ \mid x, p\ \right]. \tag{46}$$

We assume that $u_1(c)$ is quadratic and $u_2(c)$ is linear, in which case (46) reduces to a simple linear relation, $k_i = \kappa\left\{\mathbb{E}\left[f|x,p\right] - p\right\} + \lambda$, for some constants $\kappa > 0,\ \lambda \in \mathbb{R}$. With out any loss of generality, we normalize $\lambda = 0$. Finally, we let $f = f(\theta) = \theta$. That is, the return of the asset depends only on the exogenous fundamental.

The analysis here is similar to that in the first example. The optimal individual demand for the asset is

$$k = \kappa\left\{\mathbb{E}\left[\ f \mid x, p\ \right] - p\right\} = \kappa\left\{\mathbb{E}\left[\ \theta \mid x, p\ \right] - p\right\}.$$

We conjecture

$$\mathbb{E}\left[\ \theta \mid x, p\ \right] = \delta x + (1-\delta)p$$

for some $\delta \in (0,1)$ to be determined. It follows that $k = k(x,p) = \kappa\delta(x-p)$ and therefore $K(\theta, p) = \kappa\delta\,(\theta - p)$. In equilibrium, $K = \varepsilon$. Hence, the equilibrium price is

$$p = P(\theta, \varepsilon) = \theta - \frac{1}{\kappa\delta}\varepsilon.$$

By implication, p is a public signal about θ with precision $\kappa^2\delta^2\alpha_\varepsilon$. That is, in this example $Z(p) = p$ and $v = -\frac{1}{\kappa\delta}\varepsilon$. It remains to pin down δ and the function Q.

Note that α_p is bounded above by $\kappa^2\alpha_\varepsilon$ and therefore we immediately have that uniqueness is ensured for α_x high enough. To complete the analysis, note that

$$\delta = \frac{\alpha_x}{\alpha_x + \alpha_p} = \frac{\alpha_x}{\alpha_x + \alpha_\varepsilon\delta^2\kappa^2}.$$

The above uniquely determines $\delta \in (0, 1)$ as an increasing function of α_u and a decreasing function of α_ε. To see this, let $\alpha = \alpha_x / (\alpha_\varepsilon\kappa^2)$ and rewrite the above as $\alpha = \delta^3/(1-\delta)$. Obviously, this gives a monotonic relation between α and δ, with $\delta \to 0$ as $\alpha \to 0$ and $\delta \to 1$ as $\alpha \to \infty$. Using these results, we find

$$\begin{aligned}\frac{\alpha_p}{\sqrt{\alpha_x}} &= \frac{\kappa^2\delta^2\alpha_\varepsilon}{\sqrt{\alpha_x}} = (\kappa\sqrt{\alpha_\varepsilon})\frac{\delta^2}{\sqrt{\alpha_x}} \\ &= (\kappa\sqrt{\alpha_\varepsilon})\sqrt{\delta(1-\delta)}.\end{aligned}$$

The fact that $\delta(1-\delta) \to 0$ as either $\alpha_x \to 0$ or $\alpha_x \to \infty$ then implies that, given α_ε, we have that $\alpha_p/\sqrt{\alpha_x} < \sqrt{2\pi}$ and therefore the equilibrium in unique if and only if α_x is either sufficiently small or sufficiently high. On the other hand, for given α_x, we have $\delta(1-\delta) \leq 1/4$ necessarily and therefore $\alpha_\varepsilon < 8\pi/\kappa^2$ is sufficient for uniqueness, whereas α_ε sufficiently high is sufficient for multiplicity. **QED**

Proof of Proposition 9. Let $x^*(p)$ denote the threshold agents use in stage 2 in deciding whether to attack. In equilibrium,

$$A = A(\theta, p) = \Phi^{-1}\left(\frac{x^*(p) - \theta}{\sigma_x}\right),$$

so that the asset return is $f = \sqrt{\alpha_x}[\theta - x^*(p)]$. The demand for the asset is thus

$$k = \kappa\left\{\mathbb{E}[\, f \mid x, p\,] - p\right\} = \kappa\left\{\sqrt{\alpha_x}\mathbb{E}[\,\theta \mid x, p\,] - p - \sqrt{\alpha_x}x^*(p)\right\}.$$

Let

$$\widetilde{p} = \frac{1}{\sqrt{\alpha_x}}p + x^*(p) \tag{47}$$

and note that, for every p, the above defines a unique $\widetilde{p}$. We can thus write the demand as

$$k = \widetilde{\kappa}\left\{\mathbb{E}[\,\theta \mid x, p\,] - \widetilde{p}\right\}$$

where $\widetilde{\kappa} = \kappa\sqrt{\alpha_x}$. We now conjecture

$$\mathbb{E}[\ \theta \mid x, p\ ] = \delta x + (1-\delta)\widetilde{p}.$$

It follows that $K = \widetilde{\kappa}\delta(\theta - \widetilde{p})$ and therefore

$$\widetilde{p} = \theta - \frac{1}{\widetilde{\kappa}\delta}\varepsilon. \tag{48}$$

Hence, the observation of p is equivalent to the observation of $\widetilde{p}$, which is a public signal for θ with precision $\alpha_p = \widetilde{\kappa}^2\delta^2\alpha_\varepsilon$. It follows that

$$\mathbb{E}[\ \theta \mid x, p\ ] = \mathbb{E}[\ \theta \mid x, \widetilde{p}\ ] = \delta x + (1-\delta)\widetilde{p},$$

where

$$\delta = \frac{\alpha_x}{\alpha_x + \alpha_p} = \frac{\alpha_x}{\alpha_x + \alpha_\varepsilon\delta^2\widetilde{\kappa}^2}.$$

This is the same as in the previous example, with $\widetilde{\kappa}$ replacing κ. Using $\widetilde{\kappa} = \kappa\sqrt{\alpha_x}$, we infer

$$\delta = \frac{1}{1 + \alpha_\varepsilon\delta^2\kappa^2},$$

so that δ is decreasing in α_ε but independent of α_x. This means that α_p is proportional to α_x, like in the benchmark model. Indeed, the critical ratio is now given by

$$\frac{\alpha_p}{\sqrt{\alpha_x}} = \frac{\widetilde{\kappa}^2\delta^2\alpha_\varepsilon}{\sqrt{\alpha_x}} = \left(\kappa^2\delta^2\alpha_\varepsilon\right)\sqrt{\alpha_x}, \tag{49}$$

and is increasing in both α_ε and α_x.

The rest of the analysis is similar to the second example. In particular, the thresholds $\theta^*(p)$ and $x^*(p)$ are uniquely determined and are given by

$$\begin{aligned}
\theta^*(p) &= \Phi\left(\frac{\sqrt{\alpha_x}}{\sqrt{\alpha_x + \alpha_p}}\Phi^{-1}(1 - c/b) - \frac{\alpha_p}{\alpha_x + \alpha_p}p\right), \\
x^*(p) &= \theta^*(p) + \frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(p)),
\end{aligned}$$

with α_p as in (49). Next, combining (47) and (48), we infer that the equilibrium price solves

$$p = \sqrt{\alpha_x}\theta - \frac{1}{\kappa\delta}\varepsilon - \sqrt{\alpha_x}x^*(p).$$

If $\alpha_p/\sqrt{\alpha_x} < 2\pi$, the above has a unique solution. If instead $\alpha_p/\sqrt{\alpha_x} > 2\pi$, the above has multiple solutions. **QED**