Owned Ideas: When are Stock Market Run-ups Too Fast to be Rational?

John Laitner and Dmitriy Stolyarov
University of Michigan

PRELIMINARY AND INCOMPLETE, DO NOT CITE

April 9, 2004

Abstract

During the late 1990s the market value of businesses grew at 15% per year. It is hard to reconcile such fast growth with the observed rates of investment in physical capital and R&D. We therefore propose a model in which new ideas are privately owned, but discovering them does not require resources. In our model it is possible that market value rises very rapidly without drastic changes in factor prices or resource allocation. We examine possible scenarios for the rapid market run-up of the 1990s, including optimistic assessment of future productivity growth, a shift in the innovation process and a change in patent policy. At this point, plausible changes can explain no more than a third of the observed 1995-99 market run-up. This exercise suggests a method of testing for stock market bubbles.

JEL codes O16, O41, O34

1 Introduction

From 1994 to 1999 the market value of businesses has more than doubled in real terms, climbing at an average rate of more than 15% per year.[1] According to the neoclassical growth model, the market value of businesses equals the size of the capital stock. Therefore, the rate of market increase must equal the rate of net investment. Yet Figure 1 shows that throughout the 1990s the NIPA rate of net business investment was just about 2.5 percent of market value.[2] In an endogenous growth model (e.g. Grossman and Helpman, 1991, Romer, 1990) or a model with intangible capital (e.g. Laitner and Stolyarov, 2003) stock prices also incorporate investment in private knowledge. NIPA may omit knowledge investment by counting it as an intermediate good (Howitt, 1996), and then Figure 1 could be consistent with a very rapid increase in knowledge accumulation after 1994. But, one would presume that knowledge investment is correlated with private R&D spending, yet Figure 1 (dashed

[1] Source: Flow of Funds.

[2] Business investment is total investment less owner-occupied housing. See Appendix 1 for more details.

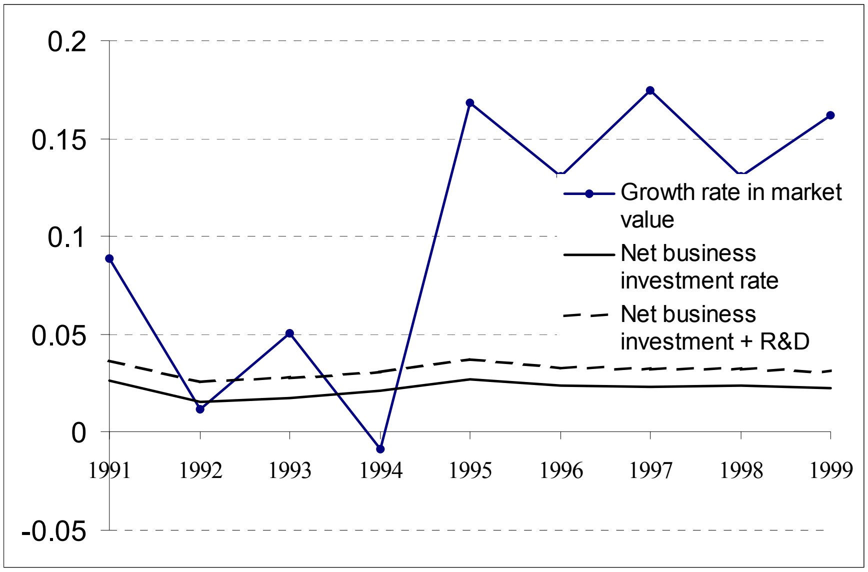

Figure 1: Growth in market value versus investment rate.

line) shows that in the 1990s the R&D investment rate stayed essentially constant.[3] Higher costs of adjustment can drive up the marginal cost of new investment and increase the price of existing capital, but these would show up as an increase in gross investment rate. Yet the share of gross investment in GDP during the 1990s was between 0.15 and 0.17 - not much different from its long-run average of 0.155. In the end, it is hard to reconcile the stock market run-up of 1995-99 with the evidence on investment rates.

This paper therefore investigates a model in which the rate of change in the market value is not limited by the rate of investment in either physical capital or knowledge. The model may help understand what the stock market participants were thinking during the late 1990s, and we can use calibrations to assess whether the observed market run-up was rational given plausible changes in the model's parameters and agents' expectations.

Table 1 maps our model's place in the existing growth literature. The well-known Solow (1956, 1960) growth models assume that technological progress is exogenous and new ideas are not privately owned; hence the benefits from new technology quickly filter through the economy and spill to labor. Models with intangible investment (Laitner and Stolyarov, 2003) or intentional R&D (Grossman and Helpman, 1991) make two changes: new knowledge requires investment and it is privately owned. The present paper considers the intermediate case: we assume that new knowledge comes from luck and inspiration but is, at least for a time, appropriated by fortunate agents or firms. For expositional convenience, we refer to any proprietary knowledge as a "patent." The last century provides numerous examples of inventors and entrepreneurs who perceived needs and opportunities, often at young ages, and developed new products and/or ideas from which they derived fortunes seemingly totally incommensurate with their inputs of time and money. If this is the case, market values need not follow investment rates

[3] Assuming that depreciation rate on R&D investment is zero, this investment rate can be measured with the ratio of R&D spending to the market value of businesses.

	Origin of Technological Progress	
	Luck	R&D or knowledge investment
Knowledge is public	Solow (1956, 1960)	No incentives for investment
Knowledge is private	this paper	Romer (1990), Grossman and Helpman (1991), Laitner and Stolyarov (2003)

Table 1: Classification of growth models

Although this model might seem to eliminate any bound on the rate of change in market value, it does provide constraints: the model produces a series of implications for observable variables that discipline the choice of parameters and restrict the simulated growth rates. We develop a general framework with exogenous, disembodied technological progress that occurs via independent Poisson processes in numerous sectors of the economy. All agents are ex-ante identical, but randomly selected fortunate individuals are endowed with new ideas every period. A fortunate agent appropriates the idea with a patent, sells it to a business (whose shares are traded on the stock market), and uses the proceeds to finance his consumption and saving.

We then perform comparative statics with respect to the parameters of the model that can cause an increase in the market value. (1) If the arrival rate of new ideas slowed down, the stock market would rise abruptly, because each patent would have a longer life. But the same change would also slow down productivity growth which seems at odds with the evidence from the late 1990s. (2) If the magnitude of improvement from each new idea increased, the market could also rise abruptly, although it is difficult to provide plausible calibrations for the magnitude and speed of observed changes.

This leads us to consider the nature of new ideas in more detail. Our basic model assumes successive innovation, in which the new idea simply replaces the one that preceded it. However, many new products depend on multiple complementary patents. Historical examples include the sewing machine, the airplane and the automobile, and most recently, the DVD (Lerner, Tirole and Strojwas, 2003). Moreover, innovations often use the same fundamental idea and improve on each other. For example, Visicalc, Lotus 1-2-3 and Microsoft Excel are all sequential improvements on one another and use the same fundamental idea of an electronic spreadsheet.[4]

We show that the additional market value that new ideas generate depends on the nature of the innovation process. Fundamental ideas become more valuable when their owners can extract surplus from future innovators. Then existing ideas will rise in value if a wave of new inventions is expected to improve on them rather than replace them. Therefore, another scenario that can generate a rapid rise in the stock market is an optimistic perception of the value of fundamental ideas. We model this as an increase in the expected number of innovations spawned by each fundamental idea. This increase can come from an increased concentration of the innovative activity in industries where ideas are highly complementary,

[4]Part of this fundamental idea is actually patented. Any spreadsheet program uses the "natural order recalculation method", U.S. Patent No. 4,398,249, granted in 1983. This patent is owned by Refac, a litigation company. In 1989 Refac sued six major spreadsheet publishers, including Lotus, Microsoft, and Ashton-Tate, for patent infringement.

such as semiconductors and computer hardware and software (Bessen and Maskin, 2000). Our numerical analysis shows that the model can match the 1995-1999 market run if the probability that the next innovation is based on the same idea permanently rises from 0 to $\frac{3}{4}$. This shock seems implausibly large, however, because fundamental ideas then stay valuable for too long (45 years). If we allow fundamental ideas to stay valuable for a more plausible 20 years, the shock would explain just a third of the observed market run-up.

The increase in the value of fundamental ideas is also consistent with a change in the patent policy that affords protection to previously unpatentable areas or allows broader patents. Both of these changes in patent policy in fact happened during the 1980s and 1990s. In particular, court decisions and administrative changes in the 1980s made computer programs and semiconductor chip designs patentable subject matter (Hunt, 1999). Computer implemented business methods (such as Amazon.com one-click shopping and Priceline.com "name your own price" auction) followed in the 1990s after seminal court decisions. At the same time, obtaining patents and defending patent rights in general became easier (Hunt 1999). Our model can therefore provide an insight into the macroeconomic effects of a change in the patent policy. We find that the effect of broader patents on physical investment is ambiguous and depends on the extent of the policy change. On the one hand, broader patents imply that patents depreciate slowly, and this stimulates physical investment. On the other hand, broader patents have higher value, and this crowds out physical investment. For small changes, investment rises, and for large changes crowding out takes over.

2 Model with successive innovation

The production side of the economy is a generalization of the quality ladder model in Grossman and Helpman (1991).[5] Final output is an aggregate of a large number of intermediate inputs

$$Y_t = \left(\sum_j x_{jt}^{\eta}\right)^{\frac{1}{\eta}}, \eta < 1 \tag{1}$$

where x_j is the quantity of input j and p_{jt} is its price. Final goods sector consists of competitive firms that maximize

$$\left(\sum_j x_{jt}^{\eta}\right)^{\frac{1}{\eta}} - \sum_j p_{jt} x_{jt}$$

which implies the following demand curve for input j

$$\left(\frac{Y_t}{x_{jt}}\right)^{1-\eta} = p_{jt} \tag{2}$$

Each intermediate good j can be produced from capital and labor using different technologies indexed by $n \leq N_{jt}$, where N_{jt} is the total number of technological innovations in industry j so far. In contrast to the endogenous growth literature, we assume that technological innovations arise from inspiration (i.e. they do not require resources) and arrive in an exogenous

[5] Barro and Sala-i-Martin (1999, Ch 7) have an alternative formulation that leads to the same aggregate production function for output.

and random fashion. Each technology is owned forever by the agent who discovered it, and the property rights are protected by patents.[6]. It is in the owner's interest to license his technology to only one producer.[7] Also, we make the following assumption about the nature of the innovation process (and later relax it in Section 3)

Successive innovation assumption Each technology n is a stand-alone alternative production method and does not rely on know-how from previously discovered technologies. To be clear, a producer that wants to produce intermediate good x_j with technology n needs to license only one patent, from the owner of technology n in that industry.

In any industry j, N_{jt} is a Poisson random variable with arrival rate λ. Although λ is equal across industries, N_{jt} are assumed to be independent random variables. Newer technologies have higher TFP levels. In particular, assume that in any industry technology n has TFP level z^n, where $z > 1$ is a constant step in the quality ladder that does not depend on j or time. If k_{nj} units of physical capital and l_{nj} units of labor are used with technology n, the resulting output in industry j is

$$x_{jt} = \sum_{n \leq N_{jt}} z^n k_{njt}^{\alpha} l_{njt}^{1-\alpha}$$

The corresponding marginal cost function for technology n (in any industry) is

$$c_{nt} = \frac{1}{z^n} \left(\frac{R_t}{\alpha} \right)^{\alpha} \left(\frac{W_t}{1-\alpha} \right)^{1-\alpha} = \frac{c_t}{z^n},$$

where R_t is the rental fee on capital and W_t is the wage. Let $Z_{jt} = z^{N_{jt}}$ denote the TFP level that corresponds to the most advanced technology in industry j.

At any point in time, several producers in an industry can make intermediate goods using technologies with different marginal costs. We assume that producers engage in Bertrand price competition. Bertrand oligopoly equilibrium has all output produced by the industry leader with the most advanced technology Z_{jt}. The leader charges either the unconstrained monopoly price $p_{jt} = \frac{1}{\eta} \frac{c_t}{Z_{jt}}$ if it exists and is below the best rival's marginal cost; otherwise he charges the limit price equal to the best rival's marginal cost, $p_{jt} = z \frac{c_t}{Z_{jt}}$. That is,

$$p_{jt} = \frac{1}{\max\left(\eta, \frac{1}{z}\right)} \frac{c_t}{Z_{jt}} = m \frac{c_t}{Z_{jt}},$$

where m is a constant markup.

Accordingly, industry output is

$$x_{jt} = Z_{jt} K_{jt}^{\alpha} L_{jt}^{1-\alpha}, \tag{3}$$

where $K_{jt} \equiv k_{N_{jt},j}$, $L_{jt} \equiv l_{N_{jt},j}$.

Poisson processes in different industries induce a distribution of technology levels by sector. Assume that at time $t = 0$ all industries have $N = 0$ and the same initial technology[8]. Let $f(N,t)$ denote the fraction of industries that have exactly N innovations at time t. It

[6] In what follows, it will be clear that probability of someone owning multiple technologies is zero.

[7] Actually, the owner would always prefer to license the patent to the current industry leader in order to ensure no competition from previous technologies.

[8] Assume that technology $z^0 = 1$ is proprietary, but there is a universally known technology $1/z$ that is freely available. Then initially every industry has a leader and a potential competitor.

turns out that aggregate output Y depends on the technologies in different sectors only through the quality index

$$Z_t = \left(\sum_j (Z_{jt})^{\frac{\eta}{1-\eta}} \right)^{\frac{1-\eta}{\eta}} = \left(\sum_{N=0}^{\infty} f(N,t)\, z^{N \frac{\eta}{1-\eta}} \right)^{\frac{1-\eta}{\eta}}. \tag{4}$$

Although output and profits in each industry grow stochastically, Z_t and aggregate output both have a deterministic growth rate. Let K_t be the capital stock at date t, L_t be the labor supply and Π_t be the aggregate profits of intermediate goods producers. The following proposition derives the aggregate production function and shows that the quality index grows at a constant rate that depends on λ, z and η.

Proposition 1

The aggregate production function is

$$Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha}. \tag{5}$$

$$Z_t = \exp\left(\gamma \frac{1-\eta}{\eta} t\right) \text{ where}$$

$$\gamma = \lambda \left(z^{\frac{\eta}{\eta-1}} - 1 \right). \tag{6}$$

Division of national income between physical capital, labor and monopoly profits

$$R_t K_t = \frac{\alpha}{m} Y_t,\ W_t L_t = \frac{1-\alpha}{m} Y_t,\ \Pi_t = \frac{1-m}{m} Y_t \tag{7}$$

Proof: See Appendix.

Corollary Profit of the leader in industry j at time t equals

$$\pi_{jt} = \Pi_t Z_t^{-\frac{\eta}{1-\eta}} \cdot \left(z^{N_{jt}} \right)^{\frac{\eta}{1-\eta}} \equiv \pi_t (N_{jt}) \tag{8}$$

Let w_{jt} denote the value of a patent in industry j at time t. It equals the expected present value of profits of the current industry leader until the moment when another innovation arrives and the leader is priced out of the market. Accordingly, for an industry whose current technology is $Z_j = z^N$, we can write

$$w_{jt} = w_t(N) = \int_t^{\infty} d\tau \lambda e^{-\lambda(\tau - t)} \int_t^{\tau} e^{-\int_t^s r_\nu d\nu} \cdot \pi_s(N), \tag{9}$$

or, differentiating with respect to t,

$$\frac{d}{dt} w_t(N) = (\lambda + r_t)\, w_t(N) - \pi_t(N). \tag{10}$$

The the aggregate value of patents in the economy equals

$$V_t = \sum_j w_{jt} = \sum_{N=0}^{\infty} f(N,t)\, w_t(N). \tag{11}$$

The following proposition derives the law of motion for the aggregate value of patents.

Proposition 2

$$\dot{V}_t = \gamma V_t + (r_t + \lambda) V_t - \Pi_t \tag{12}$$

Proof: See Appendix.

The interpretation of this expression is as follows. The first term is the *net* growth in patent wealth, where γ is defined in (6).[9] The second term is the (gross of depreciation) rate of return on current patent wealth, and the third term is payouts to owners of intellectual property.

For what follows, it is convenient to define Δ to be the depreciation rate on patent wealth. In (12), depreciation rate on a patent equals the flow probability that it is replaced by another innovation, i.e. $\Delta = \lambda$.

Equilibrium. Assume that households save a constant fraction σ of their total (gross of depreciation) income flow. In addition to labor income and capital income, whose sum is $Y_t - \Pi_t$, households receive returns on existing patent wealth in the amount of $(r_t + \Delta) V_t$ and flow of newly created patent wealth, γV_t. Depreciation payments on V add another ΔV_t. Consequently, the economy's saving function is

$$S_t = \sigma (Y_t - \Pi_t + (r_t + \Delta) V_t + \gamma V_t + \Delta V_t) = \sigma \left(Y_t + \dot{V}_t + \Delta V_t \right)$$

This saving is used to finance the growth in the economy's total net worth, $K_t + V_t$. Therefore, the market clearing condition for the economy is[10]

$$\dot{K}_t + \delta K_t + \dot{V}_t + \Delta V_t = \sigma (Y_t - \Pi_t + (r_t + \Delta) V_t + \gamma V_t + \Delta V_t) \tag{13}$$

This equilibrium condition can, of course, be written as $Y_t = C_t + \dot{K}_t + \delta K_t$, and this is equivalent to (13), since

$$C_t = (1-\sigma) Y_t + (1-\sigma) \left(\dot{V}_t + \Delta V_t \right) = Y_t - \left(\sigma Y_t - (1-\sigma) \left(\dot{V}_t + \Delta V_t \right) \right) = Y_t - \left(\dot{K}_t + \delta K_t \right).$$

Equations (12) and (13) together with (5) and the expression for the interest rate

$$r_t = \frac{\alpha}{m} \frac{Y_t}{K_t} - \delta$$

describe the time path for this economy.

Let

$$k_t = \frac{K_t}{Z_t^{\frac{1}{1-\alpha}} L_t}, \; v_t = \frac{V_t}{Z_t^{\frac{1}{1-\alpha}} L_t}, \; y_t = k_t^{\alpha}$$

denote the detrended variables. Also, let

$$g \equiv \frac{1}{1-\alpha} \frac{\dot{Z}}{Z} = \frac{1}{1-\alpha} \frac{1-\eta}{\eta} \gamma \tag{14}$$

[9] Note that if $\eta < 0$, γ is negative, because then demand for intermediate goods is inelastic and revenue falls with output. Then leader's profits shrink as cost falls and output expands.

[10] One can think of an alternative saving rule where the economy saves a constant fraction of its *net* national product:

$$\dot{K} + \dot{V} = \sigma' \left(Y - dK + \dot{V} \right).$$

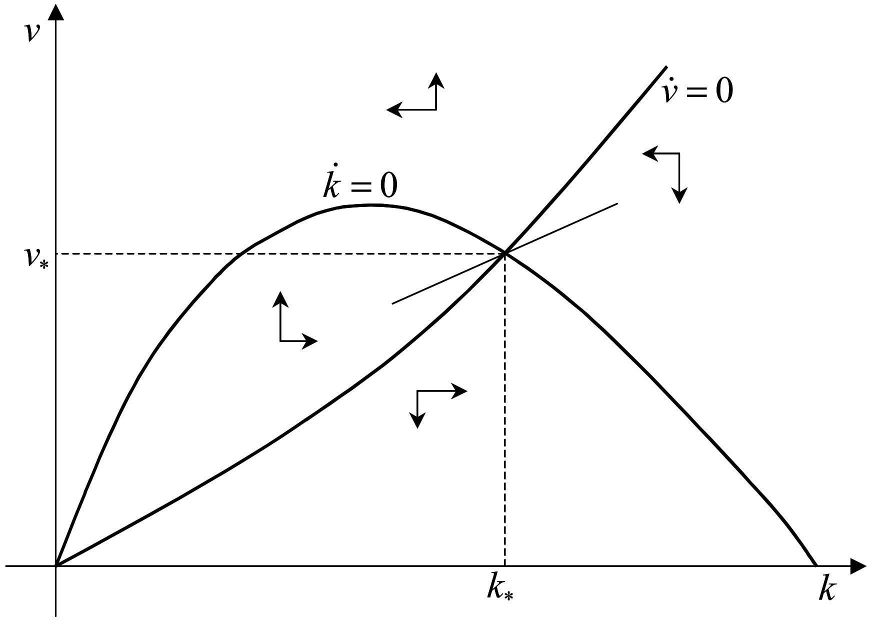

Figure 2: The phase diagram.

and

$$s = \sigma + (1-\sigma)\,\frac{m-1}{m}.$$

Then the detrended analogs of (12) and (13) are

$$\frac{\dot{V}_t}{V_t} = \frac{\dot{v}}{v} + n + g = (\gamma + r_t + \Delta) - \frac{m-1}{m}\frac{y}{v}$$

$$\frac{\dot{K}_t}{K_t} = \frac{\dot{k}}{k} + n + g = sk^{\alpha-1} - \delta - (1-\sigma)\,(\gamma + r_t + 2\Delta)\,\frac{v}{k}$$

$$\dot{v} = (\gamma + r_t + \Delta - (n+g))\,v - \frac{m-1}{m}k^{\alpha}, \tag{15}$$

$$\dot{k} = sk^{\alpha} - (\delta + n + g)\,k - (1-\sigma)\,(\gamma + r_t + 2\Delta)\,v \tag{16}$$

The phase diagram The $\dot{v} = 0$ and $\dot{k} = 0$ loci are given by

$$\dot{v} = 0\colon\ v = \frac{\frac{m-1}{m}k^{\alpha}}{(\gamma + r + \Delta - (n+g))} \tag{17}$$

$$\dot{k} = 0\colon\ v = \frac{sk^{\alpha} - (\delta + n + g)\,k}{(1-\sigma)\,(\gamma + r + 2\Delta)} \tag{18}$$

Figure 2 illustrates the phase diagram. It is a saddle with stable arm running southwest to northeast. The proof in the Appendix establishes that the steady state is unique.

2.1 Comparative Statics

We are interested in the changes in the model's parameters that lead to an increase in the value of patents.

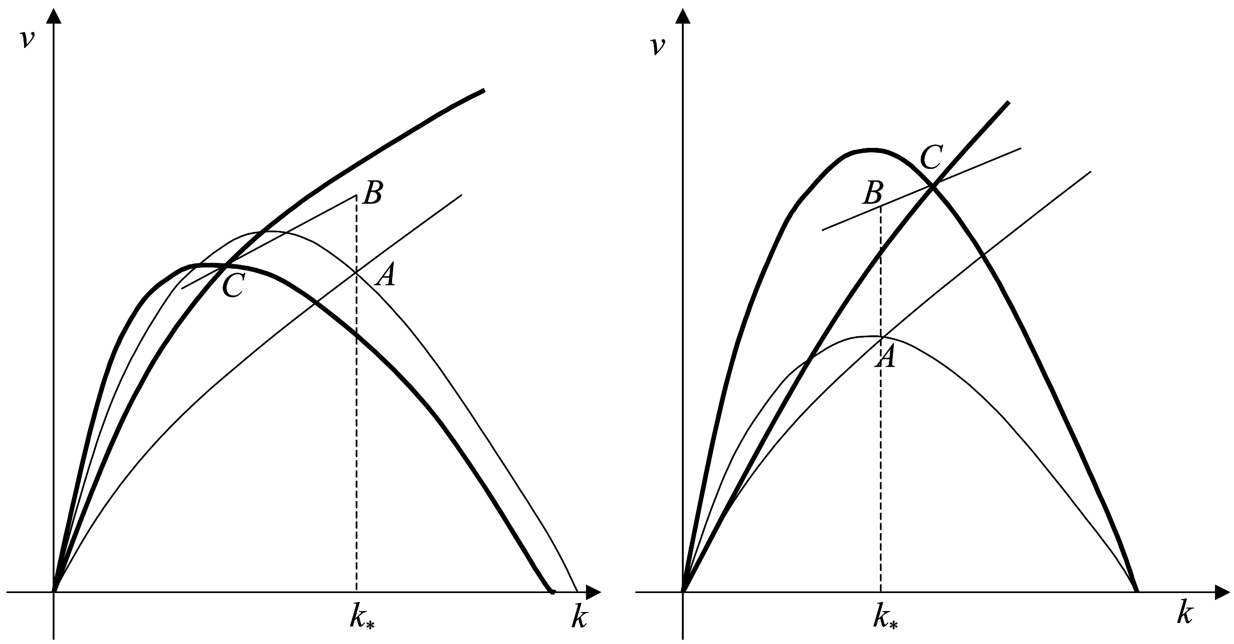

Figure 3: A permanent increase in z (left panel). A permanent decrease in Δ (right panel).

Proposition 3 Market to output ratio $\frac{v_* + k_*}{k_*^{\alpha}}$ and steady state capital intensity k^* both decrease in Δ:

$$\frac{\partial}{\partial \Delta}\left(\frac{v_* + k_*}{k_*^{\alpha}}\right) < 0$$

$$\frac{\partial}{\partial \Delta}\left(\frac{v^*}{k^*}\right) < 0$$

Let $\eta < \frac{1}{z}$, so that $m = z$. Then steady state capital intensity decreases in z:

$$\frac{\partial k_*}{\partial z} < 0,$$

$$\frac{\partial}{\partial z}\left(\frac{v^*}{k^*}\right) \lesseqgtr 0$$

Proof: See Appendix.

A permanent increase in z. Figure 3 illustrates Proposition 3. Suppose that initially the economy is in the steady state (point A on the left panel of Figure 3) and technology step z increases to z'. Assume that the increase is a complete surprise and is perceived as permanent. Let $\eta < \frac{1}{z}$ so that our limit pricing story holds. Then the increase in z has two effects: it increases the markup and also increases trend growth. On impact v jumps upward to point B overshooting its new stationary value. Subsequently, v and k will gradually fall, converging to a new stationary point (point C). Since on impact no firm has actually enjoyed the new technology step $z' > z$, the explanation of the jump in v is as follows: an existing patent faces the same probability of termination at each instant as it did previously; however, after the shock hits, a given industry faces a demand that grows more rapidly. Namely, as patents expire in other industries, the upward steps in technology are bigger than before — creating spillovers for all industries, in the form of stronger outward shifts in their demand curves than before. Existing patents experience a jump in value for this reason. Generally faster trend growth gradually leads to a lower k^*, as it would in Solow [1956]. That causes the interest rate to rise — which reduces the present value of all patents. The long-run effect of the change in z on the stationary value of v is ambiguous. The new steady state has a higher interest rate, which implies a lower v, but it also has a higher markup,

Discussion. If agents realize the economy has crossed into an era of more substantial technological change, the stock market should rise. The rise should be rapid, and the magnitude of the rise should be largest at the beginning.

There may be one insight here to the late 1990s: if z rises, existing patents can benefit from demand spillovers — despite the fact that new high–z inventions are owned by inventors who capture their direct profits. Thus, the general public could benefit from a wave of new inventions by positioning itself into the stockmarket.

In thinking about the late 1990s, however, the explanation might not fit all that well. Agents might indeed have thought that technological change was about to pick up. But, the model implies that the stocks of existing companies should have risen rapidly. It is true that existing companies at the time — such as CISCO and Sun Microsystems, and the S&P 500 in general — did experience enormous share appreciation. Nevertheless, new companies — such as Amazon.com — seemed to attract the most attention at the time, and the model does not capture this.

A decrease in Δ The right panel of Figure 3 depicts the effect of a permanent decrease in the depreciation rate of patents Δ. On impact v rises to point B, because the existing patents are now expected to live longer. As the economy converges to its new steady state at point C, interest rate falls and the value of patents keeps rising.

Since $\Delta = \lambda$ in (12), the only factor that could have decreased Δ is a slowdown in the rate of innovation - something that seems to contradict the evidence that productivity growth accelerated after 1995. This motivates us to consider a generalization of the innovation process where the rate of depreciation on patents can be different from the rate of innovation. The next section considers this extension.

3 Fundamental innovations

The model of section 2 assumed that new technologies simply replace the old. However, it seems possible to argue that some ideas are more "fundamental" than others in the sense that subsequent innovations must, or will, utilize the same fundamental idea. An example is Microsoft's computer operating system, which many generations of personal computers and computer software relied upon.

We model fundamental innovations as follows. Assume that, as before, technological progress is governed by the same Poisson process and each innovation is a fixed step up the productivity ladder. But now the innovation can be of two types: it either improves the current fundamental idea (with probability θ) or replaces it with a new fundamental idea (with probability $1-\theta$). The model of section 2 assumes that every innovation is fundamental, i.e. that $\theta = 0$. The larger is the θ, the less is the fraction of newly discovered ideas that are fundamental. Since only fundamental ideas destroy existing patents, the expected life of each patent rises in θ (When $\theta = 1$, each patent lives forever).

If the innovation is fundamental, its operation requires licensing one patent for that innovation, just as in the model of section 2. In contrast, when the innovation is, say, the

n-th improvement on some original fundamental idea, then the producer needs to license not one, but n patents, from the owners of all the previous improvements. Then the fundamental patent and the ones that follow operate together as a "patent pool."[11] For simplicity, throughout this section we assume that $\eta > \frac{1}{z}$, so when the fundamental idea changes, limit pricing against competing patent pools (that use older fundamental ideas) is never an issue. Then, regardless of its size, the patent pool operates with a constant (unconstrained monopoly) markup $m = \frac{1}{\eta}$.

If we want to determine the division of the pool's profit among the patent holders, we need additional assumptions. We require that the division of profits be a core allocation, i.e. there is no coalition of producers that can form an alternative patent pool that includes the fundamental idea and some subsequent improvements. We will consider two polar cases. (a) One is where each innovator gets the payoff equal to his contribution to the pool's profit. The other case (b) is when the owner of the fundamental patent usurps the pool's entire profit. The distribution of profits between innovators affects patent value, because it determines the rate of return on patents. Fundamental patents have two effects: on patent replacement rate and on growth rate of profits to the patent owner.

Case (a) *Innovators paid according to contribution* The aggregate resource flow to the owners of the pool that includes the frontier technology N is $\pi(N)$. Let technology N be based on some earlier fundamental idea $N_f \leq N$. Then the pool that uses technology N consists of $N - N_f + 1$ patents. Patents in the pool are indexed by $n = N_f, ..., N$, where patent N_f is the fundamental idea. Let $u(n, N)$ be the flow payoff to the owner of patent n when the most advanced technology is N. We specify the payoffs to all the owners of patents in the pool as follows.

$$\begin{aligned} u(N_f, N) &= \pi(N_f) \\ u(n, N) &= \pi(n) - \pi(n-1), \; n = N_f, ..., N. \end{aligned} \tag{19}$$

In words, each innovator gets the payoff equal to the additional profit flow that his innovation generates for as long as the pool is not replaced. With this distribution, flow payoffs to patent owners do not change as new patents are added to the pool, and the most recent innovator captures all the additional payoff that his patent generates.

Case (b) *Fundamental patent gets all.* Since the fundamental innovation constitutes a blocking patent for all the subsequent technologies in the patent pool, its owner can extract all the surplus that future innovations generate. Each time a new patent is added to the existing pool, the owner of fundamental patent gets additional profit:

$$\begin{aligned} u(N_f, N) &= \pi(N) \\ u(n, N) &= 0, \; n = N_f + 1, ..., N. \end{aligned} \tag{20}$$

The following Proposition derives the analogs of (12) for different distribution of payoffs.

Proposition 4:

$$\dot{V}_t = \gamma V_t + (r_t + \Delta(\theta)) V_t - \Pi_t,$$

where

$$\Delta(\theta) = \begin{cases} \lambda(1-\theta) & \text{in case (a)} \\ \lambda(1-\theta) - \gamma\theta & \text{in case (b)} \end{cases} \tag{21}$$

[11] In an industry which currently has $N \geq 1$ innovations, operating the best technology requires a pool of $n \leq N$ patents, where n is a random variable distributed according to $\theta^{n-1}(1-\theta)/(1-\theta^N)$.

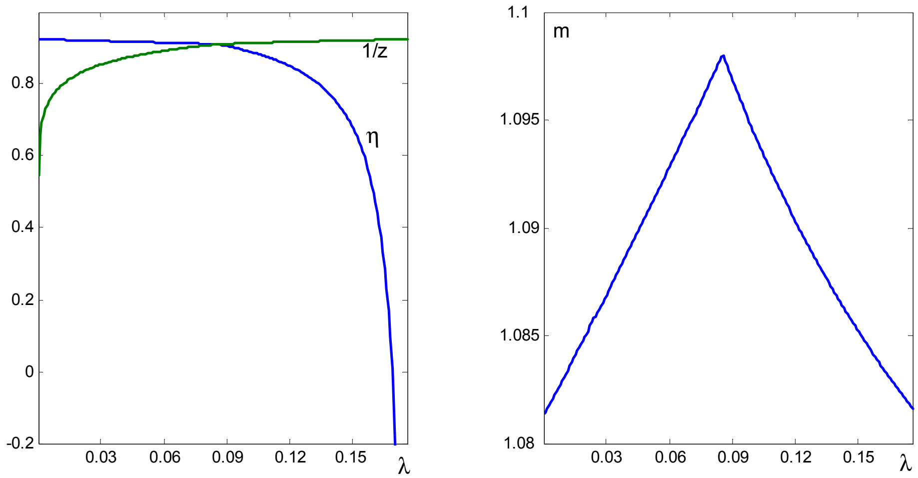

Figure 4: Calibrated η, $1/z$ and m for different values of λ.

Proof: See Appendix.

The intuition the expression (21) is as follows. When the innovators are paid according to their contribution (case (a)),.every idea in the current patent pool generates a profit flow until the pool is replaced by a new fundamental idea. In this case, the depreciation rate on patents equals the flow probability of replacement, $\lambda(1-\theta)$.

By contrast, in case (b), fundamental patents are the only ones that generate positive profits. The arrival of a non-fundamental improvement makes the fundamental patent appreciate, because its owner captures additional profits from the innovation that is added to the pool. The rate of depreciation is now the difference between the flow probability that a fundamental idea is replaced, $\lambda(1-\theta)$, and the expected growth rate in the payout to the owner of the fundamental idea, $\theta\gamma$.

The equilibrium of the model with fundamental innovations is still described by (15)-(16), although the rate of depreciation on patent wealth Δ now depends on the probability θ that a fundamental idea survives through the next innovation and the distribution of payoffs among the owners of the patent pool. Note that for a given value of θ, the lowest rate of depreciation on patents corresponds to the situation where all the surplus from innovations goes to the owner of the fundamental idea.

4 Results

In this section , we evaluate numerically the effect on an increase in θ and z and attempt to quantitatively account for the levels of the stock market in the 1990s.

4.1 Calibration

We calibrate the parameters of the model to fit three long-run statistics for the US economy: growth rate of GDP per worker $g_y = 0.02$, investment to output ratio $i = 0.137$ and market value to GDP ratio $\mu = 1.62$.

We set conventional parameters $\alpha = 0.33$, $\delta = 0.07$ and $n = 0.01$ and experiment with different values of λ. It is left to pick k^* z, η and σ. For a fixed λ, we have four calibration equations. The first equation matches the economy's steady state investment to GDP ratio to the US long-run average $i = 0.137$.

$$i = (\delta + n + g_y)\, k_*^{1-\alpha} \tag{22}$$

Equation (22) pins down k_*. Given k_*, z and η can be uniquely determined from

$$\lambda \left(z^{\frac{\eta}{1-\eta}} - 1 \right) \frac{1-\eta}{\eta} \frac{1}{1-\alpha} = g_y, \tag{23}$$

$$\frac{v_*}{y_*} = \frac{\frac{m(\eta,z)-1}{m(\eta,z)}}{(\gamma + r_* + \Delta - (n+g))} + k_*^{1-\alpha} = \mu, \tag{24}$$

where

$$\gamma = g_y (1-\alpha) \frac{\eta}{1-\eta},\ r_* = \frac{\alpha}{m(\eta,z)} k_*^{\alpha-1} - \delta$$

Equation (23) uses (14) and sets the economy's steady state growth rate of output per worker equal to g_y. Equation (24) uses (17) to match the economy's market value to output ration with the US long-run average. The remaining unknown, σ, can be found from

$$0 = \left(\sigma + (1-\sigma) \frac{m(\eta,z)-1}{m(\eta,z)} \right) y_* - (\delta + n + g)\, k_* - (1-\sigma)(\gamma + r_* + 2\Delta)\, v_*. \tag{25}$$

Equation (25) picks the value of σ that makes k_* satisfy (18).

Figure 4 shows the calibration results for different values of λ. For $\lambda \le \lambda_0 = 0.09$ there exists a solution with $\eta > \frac{1}{z}$ and $m = \frac{1}{\eta}$, and for $\lambda \in [\lambda_0, \bar{\lambda}]$ there exists a solution with $m = z$ (left panel of Figure 4). For $\lambda > \bar{\lambda} = 0.18$ equation (23) has no solutions. Calibrated markups are in a tight $[1.08, 1.1]$ range for any feasible value of λ. Consequently, values of σ are also in a tight range $[0.142, 0.180]$.

4.2 A permanent increase in growth rate

A permanent increase in z leads to a permanently higher trend growth. This shock raises market value for two reasons: markup becomes bigger and demand grows faster. We choose the higher value of z that raises trend growth to 4 percent a year. The corresponding markup is 1.172. Market to output ratio peaks on impact at 1.89 and subsequently falls to a permanently lower value. Investment to GDP ratio rises, because trend growth is permanently higher. The results on Figure 5 do not vary much with λ, because the initial value of z varies little with λ.

4.3 A change in the innovation process

We assume that at time $t = 0$ agents suddenly realize that all existing ideas are fundamental. That is, the probability that an idea survives through the next innovation permanently rises from 0 to θ. Let the owner of the fundamental idea usurp the entire profit of the patent pool, that is, let the depreciation rate on patents be $\Delta = \lambda(1-\theta) - \gamma\theta$. The impact of the increase in θ is higher for higher values of λ. The analysis in Section 3 applies only when $\eta > \frac{1}{z}$. We therefore pick the highest possible $\lambda = \lambda_0$ that is consistent with this restriction.

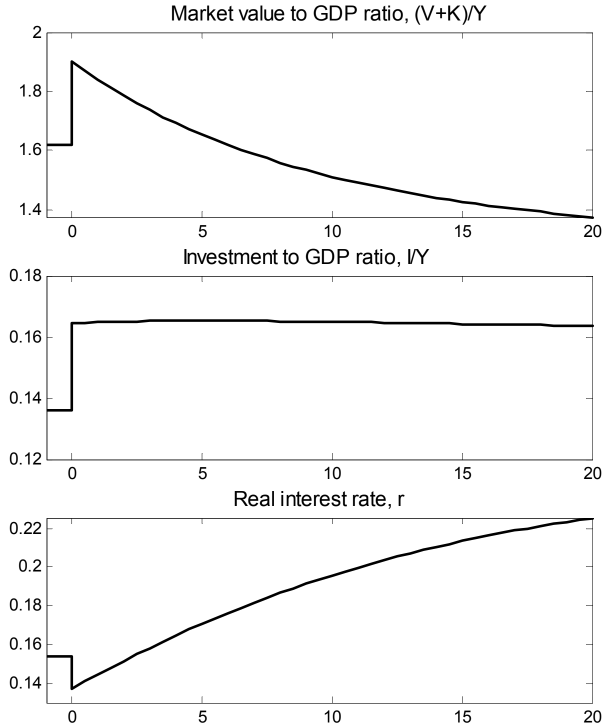

Figure 5: A permanent increase in growth rate from 0.02 to 0.04.

We also think that the plausible upper bound for the expected life of the fundamental idea is no more than $T = 20$ years. This implies that

$$\theta = 1 - \frac{1}{\lambda T} \approx 0.42.$$

Figure 6 (bold lines) shows the simulation results. Market value to GDP ratio rises to 1.81 on impact and reaches 1.9 within 5 years. In principle, it is possible to match the market value to GDP ratio that rose from 1.6 to over 2.5 late 1990s. The thin lines on Figure 6 correspond to $\theta = 0.75$. Then market value to GDP ratio rises to 2.25 on impact and reaches 2.5 within 5 years. The shock of this magnitude does not look plausible, because the implied average lifetime of a fundamental idea is 47 years.

5 Fundamental patents and rising markups

The previous section explores the case when a rise in the value of fundamental patents comes from two sources: extending their life span and allowing the owner of the fundamental patent to capture the profit from future innovators. We have shut down the third source of rising patent value, because we have assumed that markups never change as patent pool size grows. However, when $\eta < \frac{1}{z}$, a pool with n patents would limit price against the competing pool (based on the previous fundamental idea), and thus would choose the markup

$$m = \frac{1}{\max\left(\eta, \frac{1}{z^n}\right)}.$$

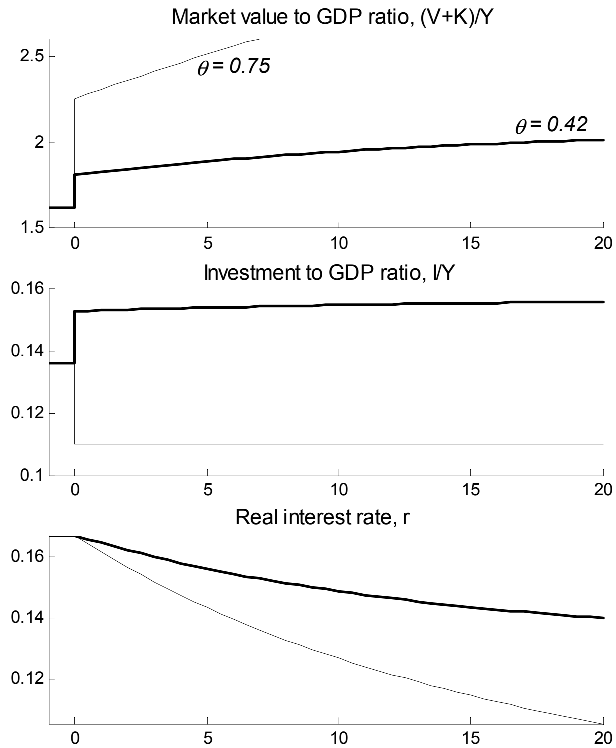

Figure 6: Impact of a change in the innovation process, $\theta = 0.42$ and $\theta = 0.75$.

Hence markups may be rising as new patents are added to the pool, and the pool's profit can temporarily grow faster than demand. The next subsection will relax the constant markup restriction and derive transitional dynamics

Also, relaxing the constant markup restriction allows us to calculate the combined effect of the shock in z and θ.

5.1 Transitional dynamics

Suppose that initially the original model of Section 2 is in the steady state equilibrium. At time $t = 0$ the agents suddenly realize that mass $\omega \in [0, 1]$ of existing active patents has become fundamental patents. As in Section 3, let θ be the probability that a patent is fundamental for the next innovation. However, assume that after a fundamental patent is replaced in a particular industry, no other fundamental patents ever emerge in that industry. Then, among industries with N innovations, $\omega\theta^N$ still have fundamental patents and $(1 - \omega) + \omega \left(1 - \theta^N\right) = 1 - \omega\theta^N$ do not. Over time, fundamental patents gradually disappear, and the economy asymptotically reaches its initial equilibrium.

The aggregate market value of patents now consists of fundamental and non-fundamental ideas, and the aggregate equation of motion for V no longer exists. Technical Appendix 3 shows that when $\eta < 0$ the economy aggregates to a system of three differential equations, one of which is the law of motion for capital stock, and the other two separately track the aggregate values of fundamental ideas and non-fundamental ideas.

Figure 7 presents our simulation results. We take λ from the interval where the corresponding η is negative. This makes the interval of admissible λ very narrow, because η changes very fast with λ (see Figure 4). The value of $\theta = 0.72$ is pinned down by the restric-

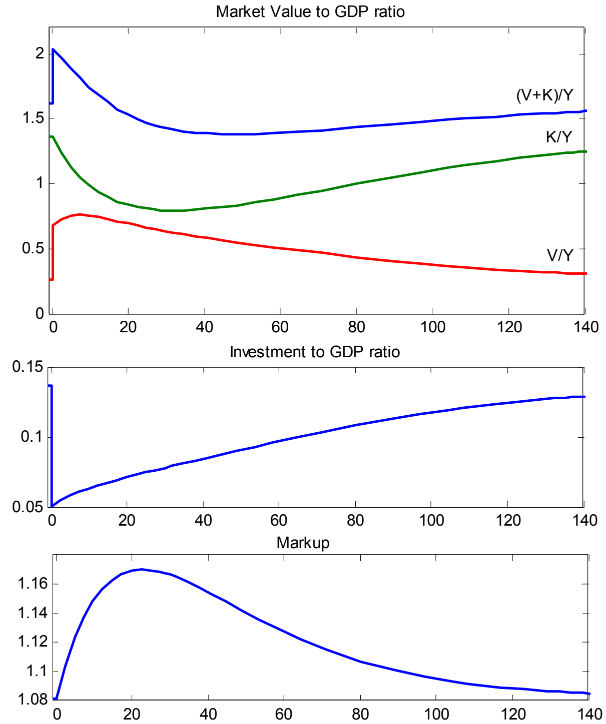

Figure 7: Simulation results with fundamental patents and rising markups.

tion that the expected life of a fundamental patent is no more than 20 years. We also set $\omega = 1$, so the magnitude of the shock is the same as that on Figure 6. The market to output ratio initially rises above 2 (see the top line, top panel of Figure 6), then keeps falling due to a fall in K (see the top line, top panel of Figure 6). Markups are building over time, which is why V keeps rising for a few years after the shock (see the bottom line, top panel of Figure 6). At the same time, output growth slows down, and this makes V fall while markups are still rising. Investment responds very strongly and negatively, because the interest rate does not change on impact and V rises sharply. Eventually, investment rises back to its initial steady state level. (middle panel of Figure 6).

References

[1] Barro, Robert J., Sala-i-Martin, Xavier, "Economic Growth"Cambridge, Mass.: MIT Press, 1999.

[2] Bessen, James and Eric Maskin, "Sequential Innovation, Patents and Imitation", MIT Economics Department Working Paper No. 00-01(2000).

[3] Grossman, Gene M., Elhanan Helpman, "Quality Ladders in the Theory of Growth", *Review of Economic Studies*, Vol. 58, No. 1. (Jan., 1991), pp. 43-61.

[4] Howitt, Peter, "On Some Problems in Measuring Knowledge-Based Growth", in Howitt, Peter, ed., The implications of knowledge-based growth for micro-economic policies, (Calgary: University of Calgary Press, 1996) 9-29

[5] Hunt, Robert M. 1999, "Patent Reform: A Mixed Blessing for the U.S. Economy?" *Federal Reserve Bank of Philadelphia Business Review* (November/December), pp. 15-29.

[6] Laitner, John P., Stolyarov, Dmitriy, "Technological Change and the Stock Market", *American Economic Review*, vol. 93, no. 4 (Sep 2003): 1240-67.

[7] Lerner, Josh, Jean Tirole and Marcin Strojwas, "Cooperative Marketing Agreements Between Competitors: Evidence from Patent Pools", NBER Working Paper No. w9680 (May 2003).

[8] Romer, Paul M., "Endogenous Technological Change", *Journal of Political Economy*, Vol. 98, No. 5, Part 2: The Problem of Development: A Conference of the Institute for the Study of Free Enterprise Systems. (Oct., 1990), pp. S71-S102.

6 Appendix 1: Data Construction

We assume that all tenant-occupied housing is counted in the market value of business. Therefore we need to construct a measure of private investment that excludes owner-occupied housing, but includes tenant-occupied housing. Gross private domestic investment (T 1.1.5 L6) less residential investment (T 1.1.5 L11) multiplied by the share of owner-occupied housing in residential investment equals business investment. To get net investment, subtract capital consumption of domestic business (T 7.5 L3).

The share of owner-occupied housing in residential investment can be imputed from the share of owner-occupied housing services in total housing services (T 7.4.5. L4 divided by T 7.4.5. L1).

Figure 8 depicts the share of investment in GDP for 1952-2002. The distance between the top line and the middle line on Figure 8 is depreciation of private capital (including owner-occupied housing). The distance between the middle line and the bottom line is net investment in owner-occupied housing.

7 Appendix 2: Proofs

Proof of Proposition 1: Omitting the time subscript for more compact notation and using the expression for the industry production function (3), the demands for capital and labor

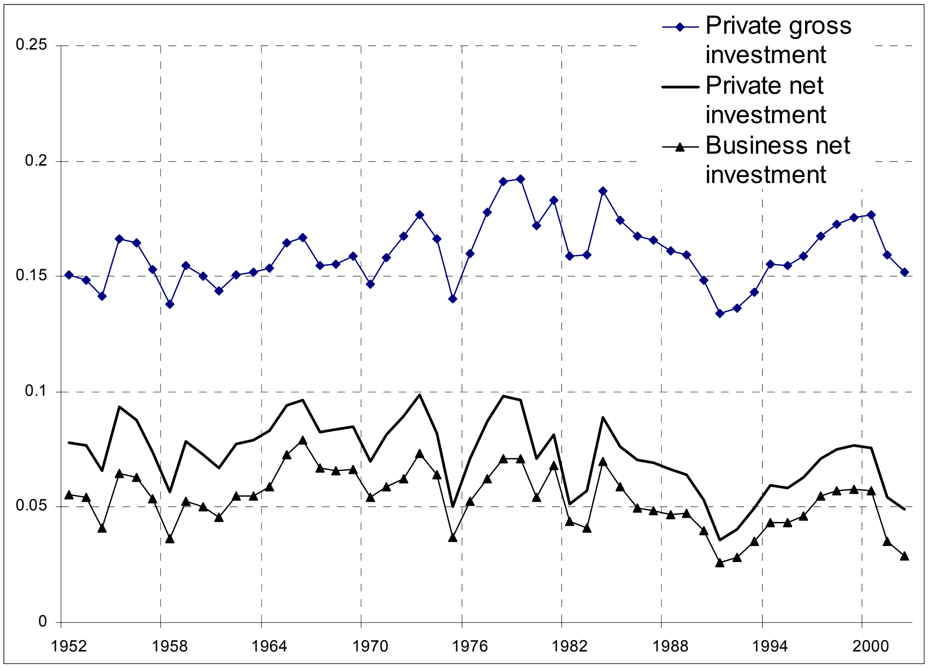

Figure 8: Different measures of investment as a share of GDP.

are given by

$$R = \alpha \frac{x_j}{K_j}, \; W = (1-\alpha)\frac{x_j}{L_j} \tag{26}$$

Then it is immediate that every industry has the same capital-labor ratio

$$k = \frac{K_j}{L_j} = \frac{K}{L}.$$

Let $l_j = L_j/L$ denote the fraction of the total labor force employed in industry j. Then, using the expression for aggregate output (1) and also(3),

$$Y = \left(\sum_j x_j^\eta\right)^{\frac{1}{\eta}} = \left(\sum_j (Z_j k^\alpha l_j L)^\eta\right)^{\frac{1}{\eta}} = K^\alpha L^{1-\alpha}\left(\sum_j (Z_j l_j)^\eta\right)^{\frac{1}{\eta}}$$

From (3),

$$\frac{x_j}{x_i} = \frac{Z_j k^\alpha l_j L}{Z_i k^\alpha l_i L}.$$

Then using the above expression and (2),

$$\frac{l_j}{l_i} = \frac{\frac{x_j}{x_i}}{\frac{Z_j}{Z_i}} = \frac{\left(\frac{p_i}{p_j}\right)^{\frac{1}{1-\eta}}}{\frac{Z_j}{Z_i}} = \left(\frac{Z_j}{Z_i}\right)^{\frac{\eta}{1-\eta}}.$$

Solving these linear equations for l_j together with $\sum_j l_j = 1$ we have

$$l_j = \frac{Z_j^{\frac{\eta}{1-\eta}}}{\sum_k Z_k^{\frac{\eta}{1-\eta}}}.$$

Substituting this into the expression for output,

$$Y = K^\alpha L^{1-\alpha} \left(\sum_j (Z_j l_j)^\eta \right)^{\frac{1}{\eta}} = K^\alpha L^{1-\alpha} \frac{1}{\sum_k Z_k^{\frac{\eta}{1-\eta}}} \left(\sum_j Z_j^{\frac{\eta}{1-\eta}} \right)^{\frac{1}{\eta}} =$$

$$= K^\alpha L^{1-\alpha} \left(\sum_j Z_j^{\frac{\eta}{1-\eta}} \right)^{\frac{1-\eta}{\eta}} = Z K^\alpha L^{1-\alpha}.$$

It is left to derive the expression for Z_t. Our assumption of an independent Poisson process with identical arrival rate λ in each of the many industries implies, by law of large numbers, that

$$f(N,t) = \frac{(\lambda t)^N e^{-\lambda t}}{N!}.$$

Then (4) becomes

$$Z_t = \left(\sum_{N=0}^{\infty} f(N,t) z^{N \frac{\eta}{1-\eta}} \right)^{\frac{1-\eta}{\eta}} = \left(e^{-\lambda t} \sum_{N=0}^{\infty} \frac{\left(\lambda t z^{\frac{\eta}{1-\eta}} \right)^N}{N!} \right)^{\frac{1-\eta}{\eta}} =$$

$$= \left(\exp\left(\lambda t z^{\frac{\eta}{1-\eta}} - \lambda t \right) \right)^{\frac{1-\eta}{\eta}} = \exp\left(\lambda \left(z^{\frac{\eta}{\eta-1}} - 1 \right) \frac{1-\eta}{\eta} t \right).$$

It is left to prove (7). From (3),

$$\frac{x_j}{l_j} = Z_j k^\alpha, \text{ and} \sum_j \frac{x_j}{Z_j} = K^\alpha L^{1-\alpha}$$

From the zero profit condition in the final goods sector,

$$Z K^\alpha L^{1-\alpha} = Y = \sum_j p_j x_j = mc \sum_j \frac{x_j}{Z_j} = mc K^\alpha L^{1-\alpha},$$

which implies that in equilibrium $mc = Z$.

From (26), the market clearing conditions for labor and capital read

$$RK = \alpha X,\ WL = (1-\alpha) X,$$

where

$$X = \sum_j x_j.$$

This implies that in equilibrium

$$c = \left(\frac{R}{\alpha} \right)^\alpha \left(\frac{W}{1-\alpha} \right)^{1-\alpha} = \frac{X}{K^\alpha L^{1-\alpha}},$$

so that

$$X = \frac{Y}{m}.$$

Therefore,

$$WL = \frac{1-\alpha}{m} Y, \; RK = \frac{\alpha}{m} Y$$

and

$$\Pi = Y - WL - RK = \frac{m-1}{m} Y.$$

■

Proof of Corollary to Proposition 1

Omitting the time subscript for more compact notation, we can write leader's current profit as

$$\pi_j = \left(p_j - \frac{c}{Z_j} \right) x_j = (m-1)\, c \frac{x_j}{Z_j}$$

Using demand function (2) and using the fact that $cm = Z$,

$$x_j = Y \cdot (cm)^{-\frac{1}{1-\eta}} Z_j^{\frac{1}{1-\eta}} = Y \cdot \left(\frac{Z_j}{Z} \right)^{\frac{1}{1-\eta}},$$

$$\pi_j = (m-1)\, c \frac{x_j}{Z_j} = \frac{m-1}{m} Y \cdot (cm)^{-\frac{\eta}{1-\eta}} Z_j^{\frac{\eta}{1-\eta}} = \frac{m-1}{m} Y \cdot \left(\frac{Z_j}{Z} \right)^{\frac{\eta}{1-\eta}}.$$

■

Proof of Proposition 2: Similarly to (11), aggregate profits of intermediate goods producers can be written as

$$\Pi_t = \sum_{N=0}^{\infty} f(N,t)\, \pi_t(N) \tag{27}$$

Also note that combining (8) and (9), we can write for any N

$$w_t(N) = z^{\frac{\eta}{1-\eta}} w_t(N-1)$$

which implies that

$$w_t(N) = w_t(0) \cdot z^{N \frac{\eta}{1-\eta}},$$

$$\pi_t(N) = \pi_t(0) \cdot z^{N \frac{\eta}{1-\eta}}$$

Accordingly, (27) and (11)can be written as

$$\Pi_t = \pi_t(0) \cdot \sum_{N=0}^{\infty} f(N,t)\, z^{N \frac{\eta}{1-\eta}} = \pi_t(0) \cdot e^{\gamma t},$$

$$V_t = w_t(0) \cdot e^{\gamma t}.$$

Differentiating the last expression and using (10), we get

$$\dot{V} = \gamma V_t + \frac{dw_t(0)}{dt} \cdot e^{\gamma t} = \gamma V_t + (\lambda + r_t)\, w_t(0) \cdot e^{\gamma t} - \pi_t(0) \cdot e^{\gamma t} =$$

$$= \gamma V_t + (\lambda + r_t)\, V_t - \Pi_t.$$

■

Proof of Proposition 3 (sketch)

$$\text{If } \frac{1}{\eta} - 1 < 1 - \alpha, \text{ then } \frac{\partial}{\partial\lambda}\left(\frac{v}{k}\right) < 0$$

If the economy is below the golden rule[12] (i.e. if $r^* > n + g$), then

$$\frac{dk^*}{d\theta} > 0, \; \frac{dk^*}{d\lambda} < 0$$

Proof: Prove that $\dot{k} = 0$ line shifts by more than $\dot{v} = 0$ line. Then their intersection has to shift to the right. Let $v_1(k, \vec{u})$ and $v_2(k, \vec{u})$ denote the $\dot{v} = 0$ and $\dot{k} = 0$ lines respectively, where $\vec{u}$ is the parameter vector that includes λ and θ. Then

$$\frac{\partial v_1(k_*, \vec{u})}{\partial\theta} = \frac{\lambda v_*}{\gamma + r_* + \Delta - (n+g)}$$

$$\frac{\partial v_2(k_*, \vec{u})}{\partial\theta} = \frac{2\lambda v_*}{\gamma + r_* + 2\Delta} = \frac{\lambda v_*}{\frac{\gamma + r_*}{2} + \Delta}$$

$$\frac{\partial v_1(k_*, \theta)}{\partial\theta} < \frac{\partial v_2(k_*, \theta)}{\partial\theta} \Leftrightarrow$$

$$\Leftrightarrow \gamma + r_* - 2(n+g) > 0$$

Similarly,

$$\frac{\partial v_1(k_*, \vec{u})}{\partial\lambda} = -\frac{v_*\left(1 - \theta + \frac{\gamma - g}{\lambda}\right)}{\gamma + r_* + \Delta - (n+g)}$$

$$\frac{\partial v_2(k_*, \vec{u})}{\partial\lambda} = -\frac{v_*\left(2(1-\theta) + \frac{\gamma}{\lambda}\right)}{\gamma + r_* + 2\Delta}$$

$\dot{k} = 0$ line shifts (down) by more than $\dot{v} = 0$ line iff

$$\frac{2(1-\theta) + \frac{\gamma}{\lambda}}{\gamma + r_* + 2\Delta} > \frac{1 - \theta + \frac{\gamma - g}{\lambda}}{\gamma + r_* + \Delta - (n+g)}$$

$$\frac{2\Delta + \gamma}{\gamma + r_* + 2\Delta} > \frac{\Delta + \gamma - g}{\gamma + r_* + \Delta - (n+g)}$$

$$\frac{\gamma + r_* + \Delta - (n+g)}{\Delta + \gamma - g} > \frac{\gamma + r_* + 2\Delta}{2\Delta + \gamma}$$

$$1 + \frac{r^* - n}{\Delta + \gamma - g} > 1 + \frac{r^*}{2\Delta + \gamma}$$

$$\frac{r^* - n}{\Delta + \gamma - g} > \frac{r^*}{2\Delta + \gamma}$$

This is always satisfied when n is sufficiently small.

[12] This is a very strong sufficient condition. For relevant parameter values the result will hold even if $r^* < n + g$.

$$\frac{\partial}{\partial\theta}\left(\frac{v}{k}\right)=\frac{\partial}{\partial\theta}\left(\frac{(m-1)\frac{R}{\alpha}}{\gamma+r+\Delta-(\delta+n+g)}\right)=$$

$$=\frac{1}{R}\left(\frac{v}{k}\right)\frac{dR}{d\theta}-\left(\frac{v}{k}\right)\frac{1}{\gamma+R+\Delta-(\delta+n+g)}\left(-\lambda+\frac{dR}{d\theta}\right)=$$

$$=\left(\frac{v}{k}\right)\left[\frac{\lambda}{\gamma+R+\Delta-(\delta+n+g)}-\frac{dR}{d\theta}\left(\frac{1}{\gamma+R+\Delta-(\delta+n+g)}-\frac{1}{R}\right)\right]=$$

$$=\left(\frac{v}{k}\right)\left[\frac{\lambda}{\gamma+R+\Delta-(\delta+n+g)}-\frac{dR}{d\theta}\left(\frac{(\delta+n+g)-\gamma-\Delta}{\gamma+R+\Delta-(\delta+n+g)}\right)\right]$$

Claim 2 In steady state

$$\frac{I}{Y}<\sigma$$

$$\frac{v}{y}=\frac{\frac{\dot{v}}{y}+\frac{m-1}{m}}{(\gamma+r+\Delta-(n+g))}$$

Substituting into (**??**),

$$\frac{I}{Y}=s-(1-\sigma)(\gamma+r+2\Delta)\frac{v}{y}=$$

$$=s-(1-\sigma)\left(\frac{\gamma+r+2\Delta}{\gamma+r+\Delta-(n+g)}\right)\frac{m-1}{m}-(1-\sigma)\frac{\frac{\dot{v}}{y}}{\gamma+r+\Delta-(n+g)}=$$

$$=\sigma-(1-\sigma)\frac{m-1}{m}\left(\frac{\gamma+r+2\Delta}{\gamma+r+\Delta-(n+g)}-1\right)-(1-\sigma)\frac{\frac{\dot{v}}{y}}{\gamma+r+\Delta-(n+g)}=$$

$$=\sigma-(1-\sigma)\frac{m-1}{m}\left(\frac{\Delta+n+g}{\gamma+r+\Delta-(n+g)}\right)-(1-\sigma)\frac{\frac{\dot{v}}{y}}{\gamma+r+\Delta-(n+g)}$$

$$\sigma-\frac{I}{Y}=\frac{(1-\sigma)}{\gamma+r+\Delta-(n+g)}\left(\frac{m-1}{m}(\Delta+n+g)+\frac{\dot{v}}{y}\right)=$$

$$=(1-\sigma)\frac{\frac{v}{y}}{\frac{\dot{v}}{y}+\frac{m-1}{m}}\left(\frac{m-1}{m}(\Delta+n+g)+\frac{\dot{v}}{y}\right)=$$

$$=(1-\sigma)\frac{1}{\frac{\dot{v}}{v}+\frac{m-1}{m}\frac{y}{v}}\left(\frac{m-1}{m}(\Delta+n+g)+\frac{\dot{v}}{y}\right)$$

In steady state,

$$\varphi(\vec{u})=\sigma-\frac{I}{Y}=\frac{m-1}{m}\frac{(1-\sigma)(\Delta+n+g)}{\gamma+r+\Delta-(n+g)}$$

$$\frac{\partial}{\partial\theta}\left[\frac{\Delta+n+g}{\gamma+r+\Delta-(n+g)}\right]=\frac{-\lambda\varphi}{\Delta+n+g}+\frac{\varphi}{\gamma+r+\Delta-(n+g)}\left(\frac{\partial r}{\partial\theta}+\lambda\right)>$$

$$>\lambda\varphi\left(\frac{1}{\gamma+r+\Delta-(n+g)}-\frac{1}{\Delta+n+g}\right)=$$

Claim 3 Any shock that makes the steady state interest rate rise also raises consumption.

Proof: The response of consumption as a fraction of GDP to a change in θ is ambiguous.

$$\frac{C}{Y} = (1-s) + (1-\sigma)(\gamma + r + 2\Delta)\frac{v}{y} =$$

As θ rises, steady state r falls and depreciation rate on patents falls, so the rate of return on patents falls as well. On the other hand, $\frac{v}{y}$ rises

$$\frac{v}{y} = \frac{v/k \uparrow}{y/k \downarrow}$$

In steady state, the $\dot{k} = 0$ line can be expressed as

$$\frac{v}{y}(1-\sigma)(\gamma + r + 2\Delta) = s - (\delta + n + g)\frac{k}{y}$$

Any shock that will make the interest rate fall will raise $\frac{k}{y}$ and the right hand side will fall. The left hand side should fall as well. Then consumption falls.

Proof of Proposition 4:

Case (a) Let $w_t(n)$ be the present value of receiving the flow payoff $\pi_t(n)$ for as long the current pool stays valuable, i.e. until the first arrival of a technology that replaces the pool.

$$w_t(n) = \int_t^\infty d\tau \lambda e^{-\lambda(\tau - t)} \left(\int_t^\tau ds e^{-\bar{r}(s,t)} \cdot \pi_s(n) + \theta e^{-\bar{r}(\tau,t)} w_\tau(n) \right) \tag{28}$$

Differentiating,

$$\frac{dw_t(n)}{dt} = -\lambda\theta w_t(n) + \lambda w_t(n) + \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \frac{d}{dt}\left(\int_t^\tau ds e^{-\bar{r}(s,t)} \cdot \pi_s(n) + \theta e^{-\bar{r}(\tau,t)} w_\tau \right)$$

$$= -\lambda\theta w_t + \lambda w_t - \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \pi_t(N) + r_t \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \left(\int_t^\tau ds e^{-\bar{r}(s,t)} \cdot \pi_s(n) + \theta e^{-\bar{r}(\tau,t)} w_\tau \right) =$$

$$= -\lambda\theta w_t(n) + (\lambda + r_t) w_t(n) - \pi_t(n)$$

Let $n \le N_j$ be a technology in a patent pool that includes N_j and let $U_t(n, N_{jt})$ denote the expected present value of payoffs to the owner of patent n in a pool that includes patent N_{jt}. Then, computing this expected value from (19) and (28), we get

$$U_t(N_f, N_{jt}) = w_t(N_f)$$
$$U_t(n, N_{jt}) = w_t(n) - w_t(n-1),\ n = N_f + 1, ..., N_{jt}.$$

Then, the current value of the patent pool is

$$\sum_{n \le N_{jt}} U_t(n, N) = w_t(N_{jt}).$$

The aggregate value of all the patent pools in the economy is

$$V_t = \sum_{N=0}^\infty f(N,t) w_t(N) = w_t(0) \cdot e^{\gamma t}$$

Corollary to Proposition 2

$$\dot{V}_t = \gamma V_t + (r_t + \lambda(1-\theta)) V_t - \Pi_t \tag{29}$$

This expression is the same as (12), except that now the flow rate of depreciation of patent value is $\Delta = \lambda(1-\theta)$, which is the flow probability of arrival of a new fundamental idea.

Case (b)

The owner of fundamental patent gets the entire profit from the pool. Let $\bar{w}_t(N)$ be the expected present value of profits from a patent pool in an industry that currently has N innovations:

$$\bar{w}_t(N) = \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \left(\int_t^\tau dse^{-\bar{r}(s,t)} \cdot \pi_s(N) + \theta e^{-\bar{r}(\tau,t)} \bar{w}_\tau(N+1) \right) =$$

$$= \int_t^\infty d\tau \lambda e^{-\lambda(\tau-t)} \left(\int_t^\tau dse^{-\bar{r}(s,t)} \cdot \pi_s(N) + \theta z^{\frac{\eta}{1-\eta}} e^{-\bar{r}(\tau,t)} \bar{w}_\tau(N) \right), \tag{30}$$

Differentiating this expression yields

$$\frac{d\bar{w}_t(n)}{dt} = \left(-\lambda\theta z^{\frac{\eta}{1-\eta}} + \lambda + r \right) \bar{w}_t(N) - \pi_t(N) =$$

$$= (\lambda(1-\theta) - \theta\gamma + r) \bar{w}_t(N) - \pi_t(N)$$

The aggregate value of patents equals the total value of all existing patent pools

$$V_t = \sum_{N=0}^{\infty} f(N,t) \bar{w}_t(N) = \bar{w}_t(0) \cdot e^{\gamma t}$$

According to Proposition 2, the differential form of this expression is

$$\dot{V}_t = \gamma V_t + (r_t + \lambda(1-\theta) - \theta\gamma) V_t - \Pi_t \tag{31}$$

■

8 Appendix 3: Aggregation of the economy with time-varying markups

For what follows, we assume that $\eta < 0$ and denote $\varepsilon = |\eta|$.

Step 1 *Aggregate production function*

$$Y = Z(t) \varphi(t) K^\alpha L^{1-\alpha}$$

Here $Z(t)$ is defined in (4). We will now derive $\varphi(t)$ in closed form.

Assume that there is a universally known technology $z^{-1}$with the associated cost function cz. Then an industry with a fundamental patent and any number of innovations sets the price $\bar{p} = cz$ and produces

$$\bar{x} = Y(cz)^{-\frac{1}{1-\eta}}$$

Let $\hat{l}$ denote the fraction of labor force employed in an industry with $N = 0$ innovations. Note that output in this industry is equal to $\bar{x}$, so that

$$\bar{x} = z^0 k^\alpha \hat{l} L.$$

Let $\bar{l}(N)$ denote the fraction of the labor force employed in an industry with a fundamental patent and N innovations ($\bar{l}(0) = \hat{l}$). Since all industries with a fundamental patent produce the same output,

$$\bar{l}(N) = \frac{\hat{l}}{z^N}.$$

Let $l_0(N)$ denote the fraction of the labor force employed in an industry with N innovations, but without the fundamental patent. Proof of Proposition 1 shows that

$$l_0(N) = \frac{\hat{l}}{z^{N\frac{\varepsilon}{\varepsilon+1}}}$$

Labor market clearing implies that

$$1 = \hat{l}\left(\sum_{N=0}^{\infty} \omega\theta^N f(N,t)\frac{1}{z^N} + \sum_{N=0}^{\infty}\left(1-\omega\theta^N\right) f(N,t)\frac{1}{z^{N\frac{\varepsilon}{\varepsilon+1}}}\right) = \hat{l}\cdot\Sigma_L,$$

where

$$\Sigma_L = \omega\exp\left(\lambda t\frac{\theta}{z} - \lambda t\right) + \exp\left(\lambda t\frac{1}{z^{\frac{\varepsilon}{\varepsilon+1}}} - \lambda t\right) - \omega\exp\left(\lambda t\frac{\theta}{z^{\frac{\varepsilon}{\varepsilon+1}}} - \lambda t\right)$$

Observe that

$$\begin{aligned}\gamma &= \frac{\lambda}{z^{\frac{\varepsilon}{\varepsilon+1}}} - \lambda, \\ |\gamma| &= \lambda - \frac{\lambda}{z^{\frac{\varepsilon}{\varepsilon+1}}} < \lambda\end{aligned}$$

Let

$$\rho = \frac{\lambda}{z^{\frac{\varepsilon}{\varepsilon+1}}} = \lambda - |\gamma|,$$

$$\kappa = \frac{\lambda}{z^{\frac{\varepsilon}{\varepsilon+1}}} - \frac{\lambda}{z} = \rho - \frac{\lambda}{z} > 0$$

so that

$$\frac{\lambda}{z} = \rho - \kappa$$

Then

$$\begin{aligned}\Sigma_L &= \omega\exp(-\lambda t + \theta(\rho-\kappa)t) + \exp(-|\gamma|t) - \omega\exp(-\lambda t + \theta\rho t) = \\ &= \exp(-|\gamma|t)\cdot(\omega\exp(-\rho t + \theta(\rho-\kappa)t) + 1 - \omega\exp(-\rho t + \theta\rho t)) = \\ &= \exp(-|\gamma|t)\cdot[\omega\exp\{-\rho(1-\theta)t - \kappa\theta t\} + 1 - \omega\exp\{-\rho(1-\theta)t\}] \qquad (32)\end{aligned}$$

Aggregate output: in an industry with N innovations and no fundamental patent output equals

$$x_0(N) = z^N k^\alpha l_0(N) L = z^{N\frac{1}{\varepsilon+1}}\bar{x}$$

Aggregate output equals

$$Y = \left(\sum_{N=0}^{\infty} \omega \theta^N f(N,t) \bar{x}^{\eta} + \sum_{N=0}^{\infty} \left(1 - \omega\theta^N\right) f(N,t) z^{N\frac{\eta}{1-\eta}} \bar{x}^{\eta} \right)^{\frac{1}{\eta}} =$$

$$= \bar{x} \left(\sum_{N=0}^{\infty} \omega\theta^N f(N,t) + \sum_{N=0}^{\infty} \left(1 - \omega\theta^N\right) f(N,t) \frac{1}{z^{N\frac{\varepsilon}{\varepsilon+1}}} \right)^{-\frac{1}{\varepsilon}} = \bar{x} \cdot \Sigma_Y^{-\frac{1}{\varepsilon}},$$

where

$$\Sigma_Y = \omega \exp(\lambda t\theta - \lambda t) + \exp(-|\gamma| t) - \omega \exp(-\lambda t + \theta(\lambda - |\gamma|) t)$$

The economy converges back to the same steady state[13] only if

$$|\gamma| > (1-\theta)\lambda \Leftrightarrow$$

$$\theta < \bar{\theta} = \frac{\lambda - |\gamma|}{\lambda} = 1 - \left(1 - \frac{1}{z^{\frac{\varepsilon}{\varepsilon+1}}}\right) = \frac{1}{z^{\frac{\varepsilon}{\varepsilon+1}}}$$

After some algebra,

$$\lambda\theta + |\gamma| - \lambda = |\gamma| - (1-\theta)(\rho + |\gamma|) = \theta|\gamma| - (1-\theta)\rho$$

$$\Sigma_Y = \exp(-|\gamma| t) \cdot (\omega \exp\{\theta|\gamma| t - (1-\theta)\rho t\} + 1 - \omega \exp\{-\rho(1-\theta) t\}). \tag{33}$$

Using (32), (33), we can write

$$\Sigma_L \cdot \Sigma_Y^{1/\varepsilon} = \exp\left(-|\gamma| t \left(\frac{1}{\varepsilon} + 1\right)\right) \cdot \left[\omega \exp\{-\rho(1-\theta) t - \kappa\theta t\} + 1 - \omega \exp\{-\rho(1-\theta) t\}\right] \times$$

$$\times \left[\omega \exp\{\theta|\gamma| t - (1-\theta)\rho t\} + 1 - \omega \exp\{-\rho(1-\theta) t\}\right]^{\frac{1}{\varepsilon}} = \frac{1}{Z(t)\varphi(t)}$$

It is easy to check that

$$\lim_{t\to 0} \varphi(t) = \lim_{t\to\infty} \varphi(t) = 1$$

Finally, we have for the aggregate production function

$$Y = Z(t)\varphi(t) K^{\alpha} L^{1-\alpha}.$$

■

Step 2 *Goods market equilibrium*
Using the zero profit condition in the final goods sector

$$Y = \sum_{N=0}^{\infty} \omega\theta^N f(N,t) \bar{p} \cdot \bar{x} + \sum_{N=0}^{\infty} \left(1 - \omega\theta^N\right) f(N,t) p_0(N) x_0(N) =$$

$$= z \cdot c\bar{x} \cdot \left(\sum_{N=0}^{\infty} \omega\theta^N f(N,t) + \sum_{N=0}^{\infty} \left(1 - \omega\theta^N\right) \frac{1}{z^{N\frac{\varepsilon}{\varepsilon+1}}} \right) = z \cdot c\bar{x} \cdot \Sigma_Y$$

[13] Otherwise, the economy goes to a different balanced growth path with a permanently lower growth rate and a larger share of monoploy profits.

yields

$$c\bar{x} = \frac{Y}{z \cdot \Sigma_Y}. \tag{34}$$

Step 3 *Aggregate monopoly profits and the interest rate.*
In an industry with N innovations and a fundamental idea,

$$\bar{\pi}_t(N) = \left(z - \frac{1}{z^N}\right) c\bar{x},$$

and in an industry with N innovations and no fundamental patent profit equals

$$\pi_t^0(N) = \frac{c}{z^N}(z-1) z^{N\frac{1}{1-\eta}} \bar{x} = c(z-1) \frac{1}{z^{N\frac{\varepsilon}{\varepsilon+1}}} \bar{x}$$

The expression for the aggregate profit is

$$\Pi_t = \sum_{N=0}^{\infty} \omega\theta^N f(N,t) c\left(z - \frac{1}{z^N}\right) \bar{x} + \sum_{N=0}^{\infty} \left(1 - \omega\theta^N\right) f(N,t) c(z-1) \frac{1}{z^{N\frac{\varepsilon}{\varepsilon+1}}} \bar{x} =$$

$$= c\bar{x}\left(z \cdot \Sigma_Y - \sum_{N=0}^{\infty} \omega\theta^N f(N,t) \frac{1}{z^N} - \sum_{N=0}^{\infty} \left(1 - \omega\theta^N\right) f(N,t) \frac{1}{z^{N\frac{\varepsilon}{\varepsilon+1}}}\right) =$$

$$= c\bar{x}\left(z \cdot \Sigma_Y - \Sigma_L\right)$$

From (34),

$$\frac{\Pi}{Y} = \frac{z\Sigma_Y - \Sigma_L}{z\Sigma_Y} = 1 - \frac{1}{z}\frac{\Sigma_L}{\Sigma_Y}$$

$$\lim_{t\to 0}\left(\frac{\Pi}{Y}\right) = 1 - \frac{1}{z}$$

$$\lim_{t\to \infty}\left(\frac{\Pi}{Y}\right) = 1 - \frac{1}{z}$$

Share of capital: Share of monopoly profits now varies over time. Let

$$m(t) = \frac{z\Sigma_Y}{\Sigma_L}$$

Then,

$$RK = \alpha(Y - \Pi) = \frac{\alpha}{m(t)},$$

$$WL = (1-\alpha)(Y - \Pi) = \frac{1-\alpha}{m(t)},$$

Step 4 *Aggregate value of fundamental patents*
The time t profit in an industry with a fundamental idea and N innovations equals

$$\bar{\pi}_t(N) = \left(z - \frac{1}{z^N}\right) c\bar{x} = \frac{1}{z-1}\left(z - \frac{1}{z^N}\right) \cdot \bar{\pi}_t(0)$$

The value of a fundamental patent in this industry equals the present value of profits for as long as the patent pool is not replaced, so it obeys (28):

$$\bar{w}_t(N) = \frac{1}{z-1}\left(z - \frac{1}{z^N}\right) \cdot \bar{w}_t(0),$$

$$\frac{d\bar{w}_t(0)}{dt} = -\lambda\theta\bar{w}_t(1) + (\lambda + r)\bar{w}_t(0) - \bar{\pi}(0) =$$

$$= -\lambda\theta\frac{z+1}{z}\bar{w}_t(0) + (\lambda + r)\bar{w}_t(0) - \bar{\pi}(0).$$

The aggregate value of fundamental patents equals

$$\bar{V}_t = \sum_{N=0}^{\infty} \omega\theta^N f(N,t)\bar{w}_t(N) = \bar{w}_t(0) \cdot \sum_{N=0}^{\infty} \omega\theta^N f(N,t)\frac{1}{z-1}\left(z - \frac{1}{z^N}\right) =$$

$$= \bar{w}_t(0) \cdot \frac{\omega}{z-1}\left(ze^{-\lambda(1-\theta)t} - e^{-\lambda t + \frac{\lambda}{z}\theta t}\right)$$

We do not have a differential equation for $\bar{V}$, but it can be expressed through $\bar{w}_t(0)$ which depends only on aggregate variables:

$$\frac{d\bar{w}_t(0)}{dt} = -\lambda\theta\frac{z+1}{z}\bar{w}_t(0) + (\lambda + r_t)\bar{w}_t(0) - \frac{(z-1)Y}{z \cdot \Sigma_Y}$$

Step 5 *Aggregate value of non-fundamental patents.* Similarly to step 4,

$$V_t^0 = \sum_{N=0}^{\infty} \left(1 - \omega\theta^N\right) f(N,t) w_t^0(N) =$$

$$= w_t^0(0) \sum_{N=0}^{\infty} \left(1 - \omega\theta^N\right) \cdot f(N,t)\frac{1}{z^{N\frac{\varepsilon}{1+\varepsilon}}} = [\text{from } (32)] =$$

$$= w_t^0(0) \cdot e^{-|\gamma|t}\left(1 - \omega e^{-\rho(1-\theta)t}\right).$$

The expression for $w_t^0(0)$ obeys (10) and also depends only on aggregate variables

$$\frac{d}{dt}w_t^0(0) = (\lambda + r_t) w_t^0(0) - \frac{(z-1)Y}{z \cdot \Sigma_Y}$$

Step 6: *The saving function.*

We need to compute depreciation rates on fundamental and non-fundamental patents. For the fundamental patents, depreciation rate is not going to be constant over time, because the expected growth rate of profits from the pool is not constant over time. One way to proceed is to leave the saving function unchanged. Overall we should have a system of three first-order non-autonomous differential equations for K, $w^0(0)$ and $\bar{w}(0)$ satisfying the following conditions:

$$\dot{K} + \delta K + \dot{V} + \Delta V = \sigma\left(Y + \dot{V} + \Delta V\right)$$

$$Y = Z(t)\varphi(t) K^\alpha L^{1-\alpha}$$

$$V_t = V_t^0 + \bar{V}_t$$

$$V_t^0 = w_t^0(0) \cdot e^{-|\gamma|t}\left(1 - \omega e^{-\rho(1-\theta)t}\right)$$

$$\bar{V}_t = \bar{w}_t(0) \cdot \frac{\omega}{z-1}\left(ze^{-\lambda(1-\theta)t} - e^{-\lambda t + \frac{\lambda}{z}\theta t}\right)$$

$$\frac{d}{dt}w_t^0(0) = (\lambda + r_t) w_t^0(0) - \frac{(z-1)Y}{z \cdot \Sigma_Y}$$

$$\frac{d\bar{w}_t(0)}{dt} = -\lambda\theta\frac{z+1}{z}\bar{w}_t(0) + (\lambda + r_t)\,\bar{w}_t(0) - \frac{(z-1)\,Y}{z\cdot\Sigma_Y}$$

The ideas stay fundamental only temporarily, so in the long run, $m = z$ and $\Delta = \lambda$

$$\dot{V} = (-|\gamma| + r + \Delta)\,V - \frac{m-1}{m}Y$$

$$\dot{K} = \sigma Y - \delta K - (1-\sigma)\left(\dot{V} + \Delta V\right) = sY - \delta K - (1-\sigma)\,(\gamma + r + 2\Delta)\,V \qquad (35)$$

Using detrended variables,

$$\dot{k} = sk^{\alpha} - (\delta + n + g)\,k - (1-\sigma)\,(-|\gamma| + r + 2\Delta)\,v$$

$$\dot{v} = (\gamma + r + \Delta - n - g)\,v - \frac{m-1}{m}k^{\alpha}$$

The $\dot{v} = 0$ and $\dot{k} = 0$ loci are given by

$$\dot{v} = 0\!:\; v = \frac{\frac{m-1}{m}k^{\alpha}}{(\gamma + r + \Delta - (n+g))}$$

$$\dot{k} = 0\!:\; v = \frac{sk^{\alpha} - (\delta + n + g)\,k}{(1-\sigma)\,(\gamma + r + 2\Delta)}$$

Step 7: *Setting up the boundary value problem.* Detrend the variables so that they converge to stationary values in the long run. In particular, let

$$U_0(t) = w_t^0(0)\cdot e^{-|\gamma|t},$$

$$u_0(t) = \frac{U_0(t)}{Z_t^{\frac{1}{1-\alpha}}L_t} = \frac{w_t^0(0)\cdot e^{-|\gamma|t}}{Z_t^{\frac{1}{1-\alpha}}L_t}.$$

$$\frac{\dot{u}_0}{u_0} = \frac{\dot{U}_0}{U_0} - (n+g) = \frac{\dot{w}^0}{w^0} - (|\gamma| + n + g) = (\lambda + r_t - (|\gamma| + n + g)) - \frac{(z-1)}{z\cdot\Sigma_Y}\frac{Y}{w^0} =$$

$$= (\lambda + r_t - (|\gamma| + n + g)) - \frac{(z-1)}{z\cdot\Sigma_Y}\frac{Y}{u_0\cdot Z_t^{\frac{1}{1-\alpha}}L_t\cdot e^{|\gamma|t}}.$$

This yields our first differential equation:

$$\dot{u}_0 = (\lambda + r_t - (|\gamma| + n + g))\cdot u_0 - \frac{(z-1)}{z\cdot\Sigma_Y\cdot e^{|\gamma|t}}\cdot y. \qquad (36)$$

The aggregate value of non-fundamental patents can be computed from from u_0:

$$v_0 = u_0\left(1 - \omega e^{-\rho(1-\theta)t}\right).$$

Since fundamental patents gradually disappear in the long run,

$$\lim_{t\to\infty} u_0(t) = \lim_{t\to\infty} v_0(t) = v_*.$$

Next, for fundamental ideas, define

$$\bar{u}(t) = \frac{\bar{w}_t(0)\cdot e^{-|\gamma|t}}{Z_t^{\frac{1}{1-\alpha}}L_t}$$

with

$$\frac{d\bar{u}}{dt} = \left(-\lambda\theta\frac{z+1}{z} + \lambda + r_t - (|\gamma| + n + g)\right)\bar{u} - \frac{(z-1)}{z \cdot \Sigma_Y \cdot e^{|\gamma|t}} \cdot y \tag{37}$$

and

$$\bar{v} = \frac{\bar{V}}{Z_t^{\frac{1}{1-\alpha}} L_t} = \bar{u}\frac{\omega}{z-1}\left(ze^{|\gamma|t-\lambda(1-\theta)t} - e^{|\gamma|t-\lambda t+\frac{\lambda}{z}\theta t}\right).$$

In the long run, $\Sigma_Y \cdot e^{|\gamma|t} \to 1$, $\bar{u} \to \bar{u}_*$, $\bar{v} \to 0$ and $u_0 \to v_*$. Therefore, from (37), the boundary condition for $\bar{u}$ is

$$\lim_{t\to\infty} \bar{u}(t) = \bar{u}_* = \frac{(z-1)/z \cdot y_*}{-\lambda\theta\frac{z+1}{z} + \lambda + r_* - (|\gamma| + n + g)} > 0.$$

Let

$$v = \bar{v} + v_0$$

be the (detrended) current aggregate value of all ideas.

It is left to derive the law of motion for the capital stock. Combining

$$\frac{\dot{V}}{V} = \frac{\dot{v}}{v} + n + g$$

and

$$\frac{\dot{K}}{K} = \frac{\dot{k}}{k} + n + g = \sigma\frac{Y}{K} - \delta - (1-\sigma)\left(\frac{\dot{V}}{K} + \Delta\frac{V}{K}\right) =$$

$$= \sigma\frac{y}{k} - \delta - (1-\sigma)\left(\frac{\dot{v}}{v} + n + g + \Delta\right)\frac{v}{k}$$

gives the last differential equation[14]

$$\dot{k} = \sigma y - (n+g+\delta)k - (1-\sigma)(\dot{v} + n + g + \Delta)v. \tag{38}$$

We solve the boundary value problem for three differential equations, (36), (37) and (38), with three boundary conditions

$$\begin{aligned} k(0) &= k_*, \\ u_0(\infty) &= v_*, \\ \bar{u}(\infty) &= \bar{u}_*. \end{aligned}$$

[14]In the long run,

$$\dot{v} + (n+g)v \to \dot{u}_0 + (n+g)u_0 = (\lambda + r - |\gamma|) - \frac{z-1}{z}y$$

and then, the above expression turns into (compare to (18))

$$\dot{k} = sy - (n+g+\delta)k - (1-\sigma)(\lambda + r - |\gamma|)v.$$