Persuasion and Limited Communication

Itai Sher*
University of Minnesota

April 11, 2008

Abstract

This paper studies persuasion as a game. A speaker must decide which arguments to present and a listener which arguments to accept. Glazer and Rubinstein (2006) showed that a listener's optimal persuasion rule is credible in the sense that commitment has no value for the listener. This paper provides an algorithm for finding an optimal persuasion rule in persuasion problems without time constraints along with the speaker's strategy in the credible implementation of the rule. The credibility result, as well as the algorithm, is extended to the case where the listener's decision is not binary under a concavity assumption. Qualitative properties of optimal rules are derived. In particular, in the absence of time constraints there exists an optimal rule that treats equivalent evidence equivalently whereas this may fail with time constraints. All of the results depend on a relation between the persuasion problem and the maximum flow problem, a well-known combinatorial optimization problem.

JEL Classification: C61, D82, D83.

Keywords: communication, optimal persuasion rules, credibility, commitment, evidence, maximum flow problem.

*I am very grateful to Eddie Dekel for many valuable comments and advice. I would also like to thank David Austen-Smith, Marciano Siniscalchi, Asher Wolinsky, Ariel Rubinstein, Jesse Bull, Elchanan Ben-Porath, Melissa Koenig, David Rahman, Peter Sher, Tomasz Strzalecki, and Jan Werner. I am also grateful to seminar participants at The University of British Columbia, University of Illinois at Urbana-Champaign, Princeton University, University of Minnesota, University of Rochester, Penn State, Lund University, and Université de Montréal for their comments. All errors are mine.

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1 Introduction

Persuasion is important in a wide array of political and economic environments. People often argue over the best course of action, or attempt to persuade others that their own personal desires are really best for everyone. But what makes an argument persuasive? One approach is to define the concept of persuasiveness endogenously so that it optimizes the listener's objective. For example, suppose that a speaker makes a request of a listener, and whether the listener would like to accept the request depends on facts known only to the speaker. The speaker presents evidence about these facts in order to attempt to persuade the listener to accept the request. Now imagine a rule which specifies which pieces of evidence will cause the listener to accept the request. Say that an argument is *persuasive* if it is accepted by the optimal rule. But could such a notion be credible? Can the listener credibly commit to such an optimal rule? In a series of papers, Glazer and Rubinstein (2001, 2004, 2006) showed that the answer is "yes". In other words, the ability to commit has no value in the persuasion problem.

This paper enriches our understanding of persuasion in three ways. First, I present an algorithm for finding an optimal persuasion rule in settings in which the presentation of evidence satisfies a "no time constraints assumption" known as *normality.* (Green and Laffont 1986, Bull and Watson 2004).[1] In other words, if the speaker has several pieces of evidence, there are no time constraints which prevent him from presenting all of it. This algorithm not only finds an optimal rule, but also the speaker's strategy in the credible implementation of the rule. By the credible implementation, I mean the equilibrium of the game without commitment which leads to the same outcome as the optimal rule. The above amounts to a constructive proof of the credibility result in a subclass of persuasion problems. The algorithm reveals that the credible implementation can be assumed to take a canonical form: in states of the world in which the listener would like to accept the request, the speaker tells the whole truth (i.e., presents all of his evidence), and in states of the world in which the listener would like to reject the request, the speaker randomizes over lies. This qualitative conclusion survives with an amendment in the presence of time constraints: in states of the world in which the speaker would like to accept the request, the speaker tells as much of the truth as he can subject to time constraints.

The second contribution is the extension of the credibility result to the case where the listener does not face a binary decision. This is important first, because many of the examples from the literature on persuasion games and strategic communication involve decisions over multiple actions (Milgrom and Roberts 1986, Shin 2003), and secondly because it leads to additional insights about what drives the credibility result. In contrast with

[1] Green and Laffont (1986) referred to a related condition known as the *nested range condition.* Along with these papers, several other papers including Lipman and Seppi (1995), Deneckere and Severinov (2001), Singh and Wittman (2001), Bull and Watson (2004) and Forges and Koessler (2005) have explored the consequences of related properties in mechanism design environments where agents possess hard evidence.

the binary case, the credibility result may fail with multiple actions. This was established by a counter-example in Glazer and Rubinstein (2006). However, this example involved a listener utility function which is "convex" at some state of the world. I show that with an appropriate concavity assumption, the credibility result can be restored. This assumption holds trivially with a binary decision. A similar assumption is relevant for the existence of optimal deterministic rules, where determinism means that the persuasion rule does not randomize in response to any message. This establishes a relationship between credibility and determinism which is invisible in the binary case. The algorithm for finding an optimal rule is also extended to this case.

The third contribution concerns a qualitative analysis of the properties of optimal rules, and in particular the role of time constraints. Glazer and Rubinstein (2006) present examples in which optimal persuasion rules treat differently two pieces of evidence which are essentially equivalent in terms of their content. For instance, if the evidence consists of the written testimony of certain experts, the optimal persuasion rule may take into account the gender of the experts even though this really has no bearing on the matter at hand. I establish that in the absence of time constraints, there always exists an optimal persuasion rule which is symmetric in the sense that it treats equivalent evidence equivalently. So it is only in the presence of time constraints that optimal persuasion rules must treat equivalent evidence differently. I also establish monotone comparative statics results concerning how the difficulty of persuading the listener and the probability that the listener selects the wrong action change as the ex ante probability that the listener would like to accept the speaker's request changes. Again, the no time constraints assumption is critical. The importance of this assumption is accounted for by the following straightforward observation; without time constraints, the listener must decide only which arguments to find persuasive, but time constraints introduce an additional ingredient: the listener must also decide which information to try to elicit.

All of the conclusions mentioned above are consequences of a single technical result; namely, I show that under normality, the persuasion problem is equivalent to the maximal closure problem. The latter is an optimization problem introduced by Picard (1976), and shown by him to be a special case of the well-known maximum flow problem. This leads to the conclusion that the Ford-Fulkerson algorithm–or for that matter any algorithm for solving the maximum flow problem–can be used to solve the persuasion problem under normality. Secondly, the maximal closure problem is known to be an instance of supermodular optimization, leading to the qualitative conclusions concerning symmetry and comparative statics. The relation to the maximum flow problem also serves as a foundation for extending the credibility result to multiple actions.

2 Examples

This section will present three canonical examples from the literature on persuasion games. These examples demonstrate interesting properties of persuasion, which will be examined in the body of the paper. In all the examples, the speaker's preferences are common knowledge, but the messages that the speaker can send are known only to the speaker and depend on the state of the world.

Example 1 This example is from Milgrom (1981) and Milgrom and Roberts (1986), and is similar to Grossman and Hart (1980) and Grossman (1981). A seller has a product whose quality $x \in X := \{0, 1, \ldots, n\}$ is privately known to the seller. $p_x > 0$ is the probability that the quality is x. When the quality is x, the seller may tell a buyer that the quality is in any set $Y \subseteq X$ with $x \in Y$. So, the seller may withhold information but may not lie. The communication constraint that $x \in Y$ allows us to interpret any statement Y as hard evidence that the state is in Y. After observing the seller's report, the buyer may purchase any quantity $q \in \mathbb{R}_+$ at price π, which is exogenous. The buyer's utility is given by $v(x, q) - \pi q$. Assume that v is continuously differentiable and strictly concave in q, that $\frac{\partial}{\partial q} v(0, 0) > 0$ and that $\frac{\partial}{\partial q} v(x, q)$ is strictly increasing in x. The seller's utility is given by $(\pi - c)q$ where c is the unit cost. Milgrom (1981) and Milgrom and Roberts (1986) show that in every sequential equilibrium of this game (i) the buyer always infers the true quality, and hence purchases the optimal quantity given the quality, and (ii) given any message Y, the speaker has *skeptical* beliefs in that he assumes that the quality is the lowest quality in Y. This is known as the *unraveling result* in the persuasion game literature. The result suggests that with sufficient availability of evidence, laws mandating the disclosure of evidence would have no effect because even without such laws, there would be full disclosure in equilibrium.

Example 2 This example is from Shin (2003), and is similar to Shin (1994a), Shin (1994b) and Dzuida (2007). A firm undertakes N projects, each of which succeeds with probability r, which is independent across projects. In period 1, each project is completed with probability θ, which is independent across projects. The firm's manager privately knows which of the completed projects have been successful but knows nothing about uncompleted projects. A state is a pair (S, F) where S (respectively, F) is the number of successful (respectively, unsuccessful) projects at period 1. The manager may report any pair (S', F') with $S' \leq S$ and $F' \leq F$ to the market. S' (respectively, F') is the number of *reported* successes (respectively, failures). So, the manager may withhold information, but may not lie. At period 2 the remaining projects are completed and the outcomes of all projects are revealed to the market. The firm then has a value of $W = B^{S^*} C^{N-S^*}$, where $0 < C < B$ and S^* is the total number of successes over projects which were completed in either period. The market is treated as a player who chooses a price π at period 1 after the manager's report, and gets utility $-(\pi - W)^2$. So the market would like to choose a price as close to the value as possible and in particular, to minimize the square loss. The manager's objective is to maximize π. Shin shows that there is a sequential equilibrium in which the manager uses a *sanitization* strategy whereby he reveals all successes and no failures. There does not exist an equilibrium with full disclosure. The difference between this example and the previous one is that here there is uncertainty how much information the speaker has, which mutes the listener's skepticism.

Example 3 This example is from Glazer and Rubinstein (2006) and is similar to examples in Fishman and Hagerty (1990), Glazer and Rubinstein (2001), and Glazer and Rubinstein (2004). A speaker attempts to persuade a listener that the majority of evidence supports the speaker's opinion. The speaker observes five facts, each of which may either support or oppose the speaker's position. This is represented as a state $x \in \{0,1\}^5$, where $x_i = 0$ means that fact i opposes the speaker's position, and $x_i = 1$ means that fact i supports the speaker's position. The speaker only has time to show the listener 2 facts. So a message can be represented as an element $m \in \{-1, 0, 1\}^5$, where $m_i = -1$ means that the ith fact has been omitted. At state x, the listener may present any message m such that $m_i = -1$ for at least three indices i and otherwise $m_i = x_i$. Again, the speaker cannot lie. However, in contrast to the previous examples, the speaker *must* withhold some of his information. The speaker makes a request of the listener and the listener's goal is to accept the speaker's request if and only if the majority of the facts support the speaker's position. Say that the listener makes an error in any state where he fails to achieve his goal. Suppose that each state $x \in X$ occurs with probability 1/32. The listener's objective is to minimize the probability of error. Suppose that the listener can commit to how he will respond to any message. Then it is optimal for the listener to partition the indices $\{1, 2, 3, 4, 5\}$ into two sets of size 2 and 3, which we will call *categories*, for example $\{1, 2\}$ and $\{3, 4, 5\}$, and to accept the speaker's request only if the speaker presents two facts supporting his opinion *in the same category*. For example, if the evidence consists of the opinions of five experts, two of which are women and three of which are men, it would be optimal to require the speaker to present supporting opinions of two experts *of the same gender* in order to win acceptance. Categories such as gender are ex ante irrelevant but the optimal rule may have to make use of some such categories. Glazer and Rubinstein (2006) interpret this as a pragmatic phenomenon, where pragmatics is the subfield of linguistics that studies the nonliteral meanings of words which arise in conversation. The authors view the persuasion model is contributing to pragmatics in noncooperative settings. Observe that since even under commitment, there is a positive probability of error, there will not be full information revelation in the game without commitment, in contrast to Example 1.

In the first two examples above the listener's decision is not binary. Below I will first study a model with a binary decision, and then extend the model to multiple actions.

3 Models

In this section I will present the persuasion problem and some related optimization problems. Sections 3.4 and 3.5 are not necessary for Section 4.1, so a reader wishing to get more quickly to the main results may postpone these sections until after 4.1.

3.1 The Persuasion Problem

Glazer and Rubinstein (2006) present the following problem. A speaker would like to persuade a listener to accept a request. There is a finite set of states X and a subset A of X such that the speaker would like to accept the request if and only if the state is in A. Define $R := X \setminus A$. For any $x \in X$, let p_x be the probability of x. Assume that for all $x, p_x > 0$. Let $p = (p_x)_{x \in X}$. For every state $x \in X$, there is a finite set $\sigma(x)$ of statements which are available at x. Let $M = \bigcup_{x \in X} \sigma(x)$.[2] We refer to (X, M, σ) at the **message structure** and to σ as the **message correspondence**. A **persuasion problem** is a tuple (X, M, σ, A, p).

The listener may select a **persuasion rule** f which is a function $f : M \to [0, 1]$. $f(m)$ is the probability that the listener accepts the speaker's request conditional on statement m. A persuasion rule is **deterministic** if for all m, $f(m) \in \{0, 1\}$. Let F be the set of all persuasion rules. Given a state x, define $\alpha(f, x) := \max_{m \in \sigma(x)} f(m)$. This is the maximal probability of acceptance that a speaker can induce given the persuasion rule f at state x. Define $\alpha(f) := (\alpha(f, x) : x \in X)$. Define $\mu_x(f) := 1 - \alpha(f, x)$ if $x \in A$, and $\mu_x(f) := \alpha(f, x)$ if $x \in R$. $\mu_x(f)$ is the probability of error induced by persuasion rule f in state x. The listener would like to solve: $\min_{f \in F} \sum_{x \in X} p_x \mu_x(f)$. The objective is to minimize the **error probability**. The formulation of the problem is such that the listener may commit to a persuasion rule, and the speaker knows of this commitment at the time that he selects this message.

Glazer and Rubinstein (2006) prove that there always exists an optimal persuasion rule which is deterministic, and provide an integer program whose solutions correspond to the state by state error probabilities induced by optimal persuasion rules. Define an L to be a pair (x, T) such that $T \subseteq R$ and $x \in A$, and $\sigma(x) \subseteq \bigcup_{y \in T} \sigma(y)$. An L, (x, T) is **minimal** if for all proper subsets T' of T, (x, T') is not an L.

[2] We also allow $M \supseteq \bigcup_{x \in X} \sigma(x)$.

Theorem 1 *(Glazer and Rubinstein (2006)) Let* $(\mu_x^*)_{x\in X}$ *be a solution to:*

$$\begin{aligned} & \min_{\{\mu_x\}_{x\in X}} \sum_{x\in X} p_x \mu_x \\ \text{s.t.} \quad & \mu_x \in \{0,1\},\ \forall x \in X \\ & \sum_{y\in\{x\}\cup T} \mu_y \geq 1 \ \text{ for every minimal } L,\ (x,T). \end{aligned} \tag{1}$$

Then there is an optimal persuasion rule that induces error probabilities $(\mu_x^*)_{x\in X}$*. Moreover, any optimal deterministic persuasion rule induces a vector of error probabilities equal to some solution* $(\mu_x^*)_{x\in X}$ *of (1).*

The linear constraints in (1) are referred to by the authors as the **L-principle**, and are perhaps more easily understood, if rewritten as: $\sum_{y\in T}\mu_y \geq 1-\mu_x$ for every L, (x,T). Suppose that the listener would like to avoid a mistake in state $x \in A$. Then the listener must accept the listener's request in state x, so there must be some message $m \in \sigma(x)$ that the listener accepts. Then $\mu_x = 0$, so $\sum_{y\in T}\mu_y \geq 1$. Why? Because at least one member y of T has message m as well, and therefore the listener would have to accept the request at state y as well, making a mistake, so that $\mu_y = 1$.

Nothing essential in the above model changes if we introduce a state dependent error cost $\ell_x > 0$. The listener's objective is then $\min_{f\in F}\sum_{x\in X}\ell_x p_x \mu_x(f)$. Theorem 1 remains valid if the objective is changed to $\min_{\{\mu_x\}_{x\in X}}\sum_{x\in X}\ell_x p_x \mu_x$.

3.2 The Game Without Commitment

In the **game without commitment**, the listener's choice of persuasion rule and the speaker's choice of strategy are simultaneous. Equivalently, the speaker first sends a message and the listener responds with either acceptance or rejection. The speaker's strategy is a function $\zeta : X \times M \to [0,1]$, where $\zeta(x,m)$ is interpreted as the probability that the speaker plays m at state x. Glazer and Rubinstein (2006) establish that if f is an optimal persuasion rule, then there exists a speaker strategy ζ such that (f,ζ) is a Bayesian Nash equilibrium of the game without commitment. Thus commitment has no value in this problem. I refer to this as the **credibility result**, and to the equilibrium (f,ζ) as the **credible implementation**. Below, I ignore sequential rationality off the equilibrium path, because consideration of sequential rationality would complicate the statement of results without adding insight. A new simple and direct proof of the credibility result based on the minimax theorem for zero-sum games is given in Appendix A. This proof is discussed in Section 5 shortly after the proof of Theorem 11.

3.3 The Maximal Closure Problem

The **maximal closure problem** was formulated by Picard (1976), who generalized Rhys (1970) and Balinski (1970). This problem turns out to be related to the persuasion problem. Consider a directed graph (V, E), where V is a set of vertices, and $E \subseteq V \times V$ is a set of directed edges. For each vertex $x \in V$, there is a positive or negative number w_x. Define:

$$\mathcal{C} := \{U \subseteq V : \forall x, y \in V, ((x \in U \text{ and } (x, y) \in E) \Rightarrow y \in U)\} \tag{2}$$

$\mathcal{C}$ is the set of **closed** subsets of V. The maximal closure problem is: $\max_{U \in \mathcal{C}} \sum_{x \in U} w_x$. Under one interpretation, V is a set of activities. Some activities produce benefits (those for which $w_x > 0$), and some require costs (those for which $w_x \leq 0$). $(x, y) \in E$ means that activity x requires activity y. One would like to find the profit maximizing collection of activities subject to the constraint that for every activity undertaken, all required activities are undertaken as well.

Let $V^+ = \{x \in V : w_x > 0\}$ and $V^- = V \setminus V^+$. A **path** P in a graph (V, E) is a sequence of vertices $(v_1, \ldots, v_n)$ such that for all $i = 1, \ldots, n-1$, $(v_i, v_{i+1}) \in E$. For $x = v_1, y = v_n$, P is an **x-y-path**. The following is proved by Hochbaum (2001), and attributed to Johnson (1968):

Theorem 2 *$U^* \in \mathcal{C}$ is a solution to the maximal closure problem if and only if, there exists $(\nu_x^*)_{x \in V}$ such that (i) $\nu_x^* = 1$ if and only if $x \in U^*$, and (ii) $(\nu_x^*)_{x \in V}$ solves:*

$$\begin{aligned} & \max_{(\nu_x)_{x \in V}} \sum_{x \in V} w_x \nu_x \\ \text{s.t.} \quad & \nu_x \in \{0, 1\}, \ \forall x \in V \\ & \nu_x \leq \nu_y \ \text{ for all } x\text{-}y\text{-paths in } (V, E) \text{ with } x \in V^+, y \in V^-. \end{aligned} \tag{3}$$

3.4 The Max-Flow Min-Cut Problem

In the next section I will describe an algorithm for solving normal persuasion problems. This algorithm involves translating the maximal closure problem into into a special case of the well known maximum flow problem. In this section, I describe the maximum flow problem, and its dual the minimum cut problem, as well as their relation to the maximal closure problem.

The **maximum flow problem** is a well known optimization problem. A **network** is a tuple (V, E, c, s, t), where (V, E) is a directed graph, $c : E \to \mathbb{R}_+ \cup \{\infty\}$ gives the capacity of each edge, $s \in V$ is the **source** while $t \in V$ is the **sink**. For any subset W of V, define $\delta^+(W) := \{(w, v) \in E : w \in W, v \in V \setminus W\}$. For any edge $v \in V$, define $\delta^+(v) := \{(u, w) \in E : u = v\}$ and $\delta^-(v) := \{(u, w) \in E : w = v\}$. A **flow** is a function $\varphi : E \to \mathbb{R}_+ \cup \{\infty\}$ satisfying $\varphi(e) \leq c(e), \forall e \in E$ (the **capacity constraints**) and $\sum_{e\in\delta^+(v)} \varphi(e) = \sum_{e\in\delta^-(v)} \varphi(e), \forall v \in V \setminus \{s,t\}$ (the **flow conservation constraints**). This says that the flow into any vertex (other than the source or sink) is equal to the flow out of that vertex. The objective is to choose a flow to maximize the net flow out of the source:

$$\text{value}(\varphi) := \sum_{e\in\delta^+(s)} \varphi(e) - \sum_{e\in\delta^-(s)} \varphi(e) \tag{4}$$

The dual of the of the maximum flow problem is the **minimum cut problem**. A **cut** in a network is a subset W of V with $s \in W, t \notin W$. The **capacity** of a cut W is: $\sum_{e\in\delta^+(W)} c(e)$. The **minimum cut problem** is the problem of choosing a cut of minimum capacity. The capacity of any cut is greater than the value of any flow. The well known **max-flow min-cut theorem** due to Ford and Fulkerson (1956) and Dantzig and Fulkerson (1956) says that the value of the maximum flow is equal to the capacity of the minimum cut.

3.5 Converting Maximal Closure into Maximum Flow

Picard (1976) showed how to translate any instance (V, E, w) of the maximal closure problem into an instance (V', E', c, s, t) of the max-flow min-cut problem. First add a source s and a sink t, setting $V' := V \cup \{s,t\}$. For each vertex $v \in V^+$, add an edge from the source to v, and for each $v \in V^-$ add an edge from v to the sink, so that: $E' := E \cup \{(s,v) : v \in V^+\} \cup \{(v,t) : v \in V^-\}$. Finally define:

$$c(e) = \begin{cases} w_v, & \text{if } e = (s,v) \text{ for some } v \in V^+; \\ \infty, & \text{if } e \in E; \\ -w_v, & \text{if } e = (v,t) \text{ for some } v \in V^-. \end{cases}$$

The capacity of a cut W is finite if and only if it is equal to $Y \cup \{s\}$ for some closed set Y in the original maximal closure graph. Following Hochbaum (2004), when the cut capacity is finite, we can then write:

$$\begin{aligned} \sum_{e\in\delta^+(Y\cup\{s\})} c(e) &= \sum_{v\in V^+\setminus Y} w_v - \sum_{v\in Y\cap V^-} w_v \qquad (5) \\ &= \sum_{v\in V} w_v - \sum_{Y\cap V^+} w_v - \sum_{Y\cap V^-} w_v = \sum_{v\in V} w_v - \sum_{v\in Y} w_v, \end{aligned}$$

For the first equality, observe that for any cut W with finite capacity, (i) if $v \in V^+ \setminus W$, then $(s,v) \in \delta^+(W)$, and (ii) if $v \in W \cap V^-$, then $(v,t) \in \delta^+(W)$. (5) establishes that the capacity of any cut (with finite capacity) is equal to a constant ($\sum_{v\in V} w_v$) minus the value of some closed set in the original maximal closure graph. Therefore finding a minimum cut is equivalent to finding a maximal closure. Observe that in the corresponding maximum flow problem there are no edges leading into the source, and therefore the objective in (4) can be simplified to value$(\varphi) = \sum_{e\in\delta^+(s)} \varphi(e)$.

4 Normal Persuasion Problems

In this section, I introduce a property on the message structure called *normality*, under which the maximal closure problem and the persuasion problem are equivalent. Normality was introduced by Bull and Watson (2007), and is closely related to the *nested range condition* of Green and Laffont (1986).[3]

Definition 1 *A message structure* (X, M, σ) *is* ***normal*** *if:*

$$\forall x \in X, \exists m_x \in \sigma(x), \forall y \in X, m_x \in \sigma(y) \Rightarrow \sigma(x) \subseteq \sigma(y).$$

m_x *is* x*'s* ***maximal message***.[4]

Intuitively, m_x summarizes all of x's information. If m_x is available at state y, then all messages available at x are also available at y. Normality can be interpreted as the **absence of time constraints** in communication: without time constraints, it is possible to present a message which summarizes all of one's evidence, simply by presenting all of one's evidence; with time constraints, this is impossible. I now categorize the examples in Section 2 with respect to normality.

Example 1 The set of states is $X = \{0, 1, \ldots, n\}$. The set of messages M is the set of all nonempty subsets of X. The message correspondence is $\sigma(x) = \{Y \subseteq X : x \in Y\}$ This is a normal message structure with $m_x = \{x\}$ for all $x \in X$.
Example 2 $X = M = \{(S, F) \in \{0, 1, \ldots, N\}^2 : S + F \leq N\}$. $\sigma(S, F) = \{(S', F') \in M : S' \leq S, F' \leq F\}$ is normal with $m_x = x$ for all $x \in X$.
Example 3 $X = \{0, 1\}^5$. $M = \{m \in \{-1, 0, 1\}^5 : |m_i = -1| \geq 3\}$. The message correspondence is $\sigma(x) = \{m \in M : m_i \neq -1 \Rightarrow m_i = x_i\}$. σ is not normal. To see this, consider $x = (1, 1, 1, 1, 1)$ and $m \in \sigma(x)$. For example, let $m = (1, 1, -1, -1, -1)$. Note that $(1, 1, -1, -1, -1) \in \sigma(1, 1, 1, 1, 0)$ but $\sigma(1, 1, 1, 1, 1) \not\subseteq \sigma(1, 1, 1, 1, 0)$ because $(-1, -1, -1, 1, 1) \notin \sigma(1, 1, 1, 1, 0)$. So m cannot be m_x. A similar argument holds for any $m \in \sigma(x)$.

Observe that in the intuitive explanation of Examples 1 and 2 in Section 2, there were no time constraints, and these message structures are normal. The explanation of Example 3 involved time constraints, and indeed it is not normal. In any persuasion problem in which

[3] Several other papers on mechanism design with evidence and related topics note the importance of these notions or similar notions, including Lipman and Seppi (1995), Deneckere and Severinov (2001), Singh and Wittman (2001), and Forges and Koessler (2005).

[4] The maximal message may not be unique. However, if there are two messages m and m' such that both satisfy the definition of x's maximal message, then it must be the case that for all $y \in X$, $m \in \sigma(y) \Leftrightarrow m' \in \sigma(y)$; in other words, the messages are equivalent.

at each state, the speaker receives n pieces of information–where n may even be random–but at least sometimes, has time only to present some but not all of the information, the persuasion problem is not normal.[5] The argument is similar to that presented with regard to Example 3 above.

There is a very rich family of message correspondences which satisfy normality. To see this observe that for any quasi-order (i.e., reflexive and transitive relation) $\preccurlyeq$, and any set X, the message correspondence $\sigma(x) := \{y \in X : y \preccurlyeq x\}$ is normal with $m_x = x$.[6] The interpretation of $x \preccurlyeq y$ is that y can mimic x. There are many variations of the message structure in Example 2 which would still be normal. For example, suppose that rather than presenting the number of favorable arguments and the number of unfavorable arguments, each favorable argument had specific characteristics. The speaker could present evidence that for example, "project 1 was successful and project 3 was successful, while project 5 was a failure", rather than presenting evidence that two project were successful and one was a failure. If the speaker observed a subset of the project outcomes and could present any subset of what he observed, the message structure would still be normal. One could also allow for the possibility that the speaker could observe more than just success or failure, but degrees of success or other characteristics of the outcome. If the speaker has the option of presenting all of his evidence or some parts of it, then the message structure is still normal. In any such example, normality allows for the possibility that like in Example 2, the listener does not know how much evidence the speaker has collected.

4.1 Equivalence of Persuasion and Maximal Closure Problems

The following lemma relates normality to the L's of Glazer and Rubinstein (2006).

Lemma 1 1. *In a normal persuasion problem, every minimal L, (x, T) is such that $T = \{y\}$ for some $y \in R$.*

2. *Fix (X, M, σ, p). If for all $A \subseteq X$, every minimal L, (x, T) is such that $T = \{y\}$ for some $y \in R$, then the message structure is normal.*

Proof of 1. Consider an L, (x, T). Then $\sigma(x) \subseteq \bigcup_{y \in T} \sigma(y)$. So there exists $y \in T$ such that $m_x \in \sigma(y)$. Then by normality $\sigma(x) \subseteq \sigma(y)$. So $(x, \{y\})$ is an L.
Proof of 2. If σ is not normal, there exists $x \in X$ such that for all $m \in \sigma(x)$, there exists $y_m \in X$ such that $m \in \sigma(y_m)$ but $\sigma(m) \not\subseteq \sigma(y_m)$. Let $A = \{x\}$. Then $(x, \{y_m : m \in \sigma(x)\})$ is an L, but for every $m \in \sigma(x)$, $(x, \{y_m\})$ is not an L. $\square$

This lemma allows us to transform any normal persuasion problem into a maximal closure problem. Consider a minimal L, (x^*, T) where $x^* \in A, T \subseteq R$ in the persuasion

[5] The pieces of evidence are assumed to be truly distinct in the sense that there does not exist at every state one piece of evidence from which it is possible to infer the existence and identity of all the others.

[6] A similar observation is made by Green and Laffont (1986).

problem, and the corresponding L-constraint $\sum_{y\in T}\mu_y \geq 1-\mu_{x^*}$. The lemma shows that under normality, T is a singleton $\{y^*\}$. So the constraint simplifies to $\mu_{y^*}\geq 1-\mu_{x^*}$. For all $x\in X$ define:

$$w_x = \begin{cases} p_x, & \text{if } x\in A; \\ -p_x, & \text{if } x\in R. \end{cases} \qquad \nu_x = \begin{cases} 1-\mu_x, & \text{if } x\in A; \\ \mu_x, & \text{if } x\in R. \end{cases} \tag{6}$$

Consider a graph (V,E) with vertices $V := X$. We have $V^+ = A, V^- = R$, and the L-constraint simplifies to $\nu_{x^*}\leq \nu_{y^*}$ with $x^*\in V^+, y^*\in V^-$. Define

$$E := \{(x,y) : x\in A, y\in R, \sigma(x)\subseteq\sigma(y)\} \tag{7}$$

and observe that $(x^*,\{y^*\})$ is an L if and only if $(x^*,y^*)\in E$. Because (V,E) is bipartite with all edges pointing from V^+ to V^-, $(x^*,y^*)\in E$ if and only if there is an x^*-y^*-path in (V,E). Next observe that

$$\sum_{x\in X} w_x\nu_x = \sum_{x\in A} p_x\nu_x - \sum_{x\in R} p_x\nu_x = \sum_{x\in A} p_x(1-\mu_x) - \sum_{x\in R} p_x\mu_x = \sum_{x\in A} p_x - \sum_{x\in X} p_x\mu_x$$

So the objective of the maximal closure problem (which is maximized) is equal to a constant $\sum_{x\in A} p_x$ minus the objective of the persuasion problem (which is minimized). In this way, the persuasion program (1) is transformed into the maximal closure program (3). It is easy to see that one can go in the opposite direction as well, from (3) to (1).[7] To summarize:

Theorem 3 *When restricting attention to normal message structures, the persuasion problem is equivalent to the maximal closure problem.*

I now revisit Examples 1 and 2, which are normal, altering them so that the listener faces a binary decision. In Example 1, $X = \{0,1,\ldots,n\}$, where a higher value of x means that the seller's good is of a higher quality. Suppose the buyer may buy either 1 unit of the object or 0 and $v(x)$ is his value for a good of quality x. Let h be the smallest number such that $v(x) > \pi$. Then $R = \{1,\ldots,h-1\}, A = \{h,\ldots,n\}$ and the cost of making the wrong decision ℓ_x is decreasing on R and increasing on A. Constructing the maximal closure graph (V,E), $V = X$ and $E = \emptyset$, because the message $\{x\}$ is only available in state x. It follows that every subset of V is closed. So, assuming $v(h-1) < \pi$, A is the unique solution to the corresponding maximal closure problem. So the optimal persuasion rule f accepts the speaker's request to buy the item precisely in A and has an expected error $\sum_{x\in X}\ell_x p_x\mu_x(f) = 0$. The credibility result implies that the expected error is 0 in some equilibrium of the game without commitment. This conclusion is a partial analog to

[7] When going in the other direction, assume wlog that $w_x \neq 0$ and $\sum_{x\in X}|w_x| = 1$. The message structure induced by the maximal closure problem is: $\sigma(x) := \{x\}$ if $x\in A$, and $\sigma(x) := \{x\}\cup\{y\in A :$ there is a y-x path in (V,E).$\}$ if $x\in R$. σ is normal with $m_x = x$.

the unraveling result.[8] The conclusion is robust to some alternative specifications of the message structure under which the unraveling result is known to hold. If at x, the seller could only prove that (i) the quality belonged to any *interval* $I = \{k, k+1, \ldots, k+r\}$ with $x \in I$, or alternatively that (ii) the quality is at least y for any $y \leq x$, there still exists an equilibrium with an expected error of 0. Under (ii), $E = \{(x, y) \in V \times V : x \leq y\}$ and not every subset of X is closed in (V, E), but A is still closed.

In Example 2, suppose the market may choose between two prices π_0 and π_1 with $\pi_0 < \pi_1$. Accepting the speaker's request amounts to choosing the higher price. Then $A = \{x \in X : \mathbb{E}[(W - \pi_1)^2|x] < \mathbb{E}[(W - \pi_0)^2|x]\}$, where $\mathbb{E}$ is the expectation operator. Again $V = X$ and: $E := \{((S, F), (S', F')) \in A \times R : S \leq S', F \leq F'\}$. Since a state is defined at the interim stage in this example, a persuasion rule which achieves an expected error of 0, would be a persuasion rule which induced the market to always choose the best price conditional on the manager's information at the time of disclosure. There are many specifications of the model for which $(1, 0) \in A$ but $(1, N-1) \in R$. However, any closed set in (V, E) containing $(1, 0)$ also contains $(1, N-1)$. It follows that there does not exist an optimal persuasion rule which achieves an expected error of 0, and hence it is not possible to achieve an expected error of 0 in any equilibrium of the game without commitment.

I conclude this section with some observations about when it is possible to achieve an expected error of 0 in normal persuasion problems. Say that a persuasion problem has a **revealing equilibrium** if and only if there exists an equilibrium of the game without commitment which induces an expected error of 0. By the credibility result, this holds if and only if there is exists a persuasion rule which induces an expected error of 0. Say that a persuasion problem is **separating** if for all $x \in A$ and $y \in R$, there exists a message which is available at x but not at y, and that it is **strongly separating** if at every state $x \in A$, there is a message available at x which is not available at any state in R. Strong separation implies separation, and it turns out that in normal problems, the two are equivalent.

Observation 1 *1. A persuasion problem has a revealing equilibrium if and only if it is strongly separating.*[9]

2. In normal persuasion problems, the following are equivalent:

(a) There is a revealing equilibrium.

(b) A is a closed set in the corresponding maximal closure graph.

(c) The problem is separating.

[8] It is only partial because some equilibria–even sequential equilibria–have positive expected error.

[9] This result is related to Proposition 11 of Lipman and Seppi (1995) and Theorem 1 or Singh and Wittman (2001).

Proof. 1 is immediate.[10] The equivalence of 2a and 2b follows from Theorem 3 and the credibility result. 2c means that for all $x \in A, y \in R$, $\sigma(x) \not\subseteq \sigma(y)$. By (7) this is equivalent to: for all $x \in A, y \in R$, $(x, y) \notin E$, which in turn is equivalent to 2b. □

For this observation, recall that we have assumed above that the probability of every state is positive and that the listener is never indifferent between the two actions. This theorem may be viewed as providing necessary and sufficient conditions for a partial analog of the unraveling result in the binary action case. Example 3 is separating but not strongly separating, and the optimal persuasion rule has a positive error probability. It follows separation is not sufficient for revealing equilibrium in non-normal problems. On an intuitive level, if the listener knows how much information the speaker has, then without time constraints, Observation 1 implies that the problem has a revealing equilibrium since the listener can recognize a complete complete report, and a complete report always separates any state from any other state. However, in the presence of time constraints, even if the listener knows how much information the speaker has, he may not be able to elicit a complete report and hence full revelation may be infeasible. So uncertainty about the amount of information the speaker has is a necessary condition for the listener to incur a positive probability of error at the optimal rule without time constraints but not with time constraints.

4.2 An Algorithm for Persuasion

In this section, I will provide an algorithm for solving normal persuasion problems. This algorithm is based on the equivalence of the maximal closure problem and the normal persuasion problem. In order to apply the algorithm, I will first convert the maximal closure problem into an instance of the well known maximum flow problem.

Start with a normal persuasion problem, and construct the corresponding maximal closure problem as in Section 4.1, and then the corresponding maximum flow problem as in Section 3.4. In the resulting network, there are three types of edges (i) for every $x \in A$, there is an edge pointing from the source s to x with capacity p_x, (ii) for every $x \in A$ and $y \in R$ with $m_x \in \sigma(y)$ (or equivalently, $\sigma(x) \subseteq \sigma(y)$), there is an edge pointing from x to y with infinite capacity, and (iii) for every $y \in R$, there is an edge pointing from y to the sink t with capacity p_y.

I now describe the Ford-Fulkerson algorithm which was put forth in Ford and Fulkerson (1957) for solving maximum flow problems. For any edge $e = (v, w)$, define $\overleftarrow{e} := (w, v)$ to be the edge pointing in the opposite direction. For any network $N = (V, E, c, s, t)$ and any feasible flow φ in that network define the **residual network** to be $N^\varphi = (V, E^\varphi, c^\varphi, s, t)$,

[10] Observe that a persuasion problem is strongly separating precisely it it does not have any L's.

where:

$$\begin{aligned} E^\varphi &:= \underbrace{\{e : e \in E, \varphi(v,w) < c(v,w)\}}_{E_1^\varphi} \dot\cup \underbrace{\{e : \overleftarrow{e} \in E, \varphi(\overleftarrow{e}) > 0\}}_{E_2^\varphi} \\ c^\varphi(e) &:= \begin{cases} c(e) - \varphi(e), & \text{if } e \in E_1^\varphi \ ; \\ \varphi(\overleftarrow{e}), & \text{if } e \in E_2^\varphi. \end{cases} \end{aligned}$$

Above, $\dot\cup$ refers to the disjoint union, so that if e is contained in both E_1^φ and E_2^φ, then we suppose that there are really two distinct copies of e (i.e., two parallel edges), one in each set. This ensures that the capacity c^φ is well defined. For any path $P = (v_1, \ldots, v_n)$, and edge $e \in E$, $e \in P$ means that $e = (v_i, v_{i+1})$ for some $i = 1, \ldots, n-1$. Recall that an **s-t-path** in a network (V, E, c, s, t) is a path in (V, E) from s to t. For any feasible flow φ define: $V^\varphi := \{v \in V :$ there is a path from s to v in $(V, E^\varphi).\}$[11] The following algorithm finds a maximum flow in a network.

1. Set $\varphi(e) := 0$ for all $e \in E$.

2. Find an s-t-path P in N^φ. If none exists, then stop.

3. Let $\gamma := \min\{c^\varphi(e) : e \in P\}$. Set $\varphi(e) := \varphi(e) + \gamma$ for all $e \in P$, and leave $\varphi(e)$ unchanged otherwise. Go to step 2.

The algorithm was designed for networks in which all edge capacities are finite, whereas we have assumed that some edge capacities are infinite; however the problem would not be changed if all infinite capacities were replaced by sufficiently large finite capacities, and therefore, the algorithm is valid for the problem studied here.

On general graphs, the Ford-Fulkerson algorithm may require an exponential number of steps, and when edge capacities are *irrational*, it may not terminate at all; however, Edmonds and Karp (1972) solved these problems by showing that if at step 2, a shortest s-t-path is always chosen, then φ must be augmented at most $|V| \cdot |E|/2$ times. This is only one algorithm which can be used to solve the persuasion problem. Showing that the persuasion problem is a max-flow problem allows us to use a variety of algorithms which are known for this problem.[12]

Given the equivalence between the persuasion problem and the maximal closure problem under normality, the next theorem shows that the Ford-Fulkerson algorithm can be used to find: (1) an optimal persuasion rule, (2) a speaker strategy in a credible implementation of the optimal rule, (3) and the probability of error induced by the optimal rule.

[11] Observe that (s) is path starting and ending at s. So $s \in V^\varphi$.

[12] See Chapter 8 of Korte and Vygen (2006) for further details about facts in this paragraph.

Theorem 4 *Fix a normal persuasion problem satisfying $m_x = m_y \Rightarrow x = y$ for all $x, y \in A$.*[13] *Let φ be a maximum flow in the corresponding maximum flow problem. Then f defined by:*

$$f(m) := \begin{cases} 1, & \text{if } m = m_x \text{ for some } x \in A \cap V^{\varphi}; \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

is an optimal persuasion rule. A speaker strategy in the credible implementation is given by:

$$\zeta(x, m) := \begin{cases} 1, & \text{if } x \in A, m = m_x. \\ \frac{\varphi(y,x)}{\varphi(x,t)}, & \text{if } x \in R, m = m_y \in \sigma(x) \text{ with } y \in A; \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

unless $y \in R$ and $\varphi(y, t) = 0$, in which case the speaker strategy at state x can be chosen arbitrarily. The error probability is given by value(φ).

Proof. In Appendix. □

This theorem amounts to a constructive proof of the credibility result for normal problems. One important qualitative conclusion is that in every normal persuasion problem there is an optimal rule and a credible implementation of that rule such that at all states in A, the speaker tells the whole truth, and in states in R, the speaker randomizes (possibly degenerately) over lies (unless he cannot lie). By the capacity constraints, $\varphi(s, x) \leq p_x$ for all $x \in A$ and $\varphi(y, t) \leq p_y$ for all $y \in R$. Therefore $\varphi(s, x)$ can be thought of as part of the probability mass of state x and $\varphi(y, t)$ can be thought of similarly. Consider $x \in A$. A maximum flow exactly matches up all the mass in $\varphi(s, x)$ with $\sum_{y \in R:(x,y) \in E} \varphi(x, y)$, where $\varphi(x, y)$ is a part of $\varphi(y, t)$, and $(x, y) \in E$ means that y can mimic x. Whenever m_x is rejected, $p_x = \varphi(s, x)$. This means that rejecting x amounts to making a mistake with probability $p_x = \varphi(s, x)$, because the listener would have liked to accept the speaker at x. Whenever it is optimal to accept m_x, then for any y that can mimic x, $p_y = \varphi(y, t)$, and holding fixed the speaker's strategy in the credible implementation, accepting m_x leads to an error with probability $\varphi(x, y)$ for each $y \in R$ that can mimic x, given that at y (which occurs with probability $p_y = \varphi(y, t)$), the speaker pretends the state is x with probability $\frac{\varphi(x,y)}{\varphi(y,t)}$, and the listener would like to reject the speaker at state y. But by the flow conservation constraints, $\varphi(s, x) = \sum_{y \in R:(x,y) \in E} \varphi(x, y)$, so accepting m_x (when it is optimal to do so) again leads to an error probability of $\varphi(s, x)$. But since value$(\varphi) = \sum_{x \in A} \varphi(s, x)$, the total error probability is simply the value of the flow φ.

The analysis of the persuasion problem as a maximum flow problem can add to our understanding of the full revelation result in Observation 1. In particular, it can be used to derive a **no revelation result**. Suppose that A is not closed in the maximal closure

[13] This last assumption is innocuous because if it fails, we could always combine x and y into a single state z with $p_z = p_x + p_y$ and $\sigma(z) = \sigma(x) = \sigma(y)$ (noting that the last equality follows from normality and $m_x = m_y$), arriving at an equivalent persuasion problem.

graph corresponding a normal persuasion problem. In saying this, we are negating the condition which was found to be necessary and sufficient for full revelation in Observation 1, assuming that all states have positive probability. Now let A^* be the largest subset of A such that at no state in R can the speaker pretend that the state is in A^* by sending m_x for $x \in A^*$. Similarly, let R^* be the largest subset of R such that at no state in R^* can the speaker pretend that the state is in A. A^* is the largest closed subset of A and R^* is the complement of the closure of A in the corresponding maximal closure graph. Assuming that every state occurs with positive probability, it is easy to see that regardless of the probability distribution over states, any optimal persuasion rule will accept the speaker's request in A^* and reject the speaker's request in R^*.

Theorem 5 *Fix all aspects of a normal persuasion problem except p. If A is not a closed set in the corresponding maximal closure graph, then there exists a probability distribution p on X such that:*

1. $p_x = 0 \Leftrightarrow x \in A^* \cup R^*$
2. *Any optimal persuasion rule induces a probability of error of* $1/2$.
3. *In some credible implementation of any optimal rule, the listener is indifferent between accepting and rejecting every message that is sent with positive probability.*

In contrast to the rest of the paper, here I allow for zero probability states. The theorem can be interpreted as follows. Suppose that nature chooses the probability of states so as to maximize the probability of error at the optimal persuasion rule. If A is closed in the maximal closure graph, then nature cannot induce any error; otherwise nature can induce an error of $1/2$, which is the highest possible at the optimal rule, since either rejecting all messages produces an error probability of at most $1/2$, or accepting all messages does. Allowing for zero probability states without assuming property 1, the result is immediate as by Observation 1, if A is not closed, the problem is not separating: simply assign probability $1/2$ each to some $x \in A$ and $y \in R$ with $\sigma(x) \subseteq \sigma(y)$. In fact, $X \setminus (A^* \cup R^*)$ is the largest support for any probability distribution for which properties 2 and 3 can hold, since on $(A^* \cup R^*)$ it is always feasible to avoid error. Observe that Theorem establishes no revelation in two distinct senses given by properties 2 and 3.

A proof sketch for Theorem 5 is as follows. Consider any maximum flow in the network corresponding to the normal persuasion problem with an arbitrary full support probability distribution. One can show that it is always possible to find a maximum flow φ with the property that for all $x \in A$, $\varphi(s,x) = 0 \Leftrightarrow x \in A^*$ and for all $x \in R$, $\varphi(x,t) = 0 \Leftrightarrow x \in R^*$. Now construct a probability distribution on X such that p_x is proportional to $\varphi(s,x)$ if $x \in A$ and to $\varphi(y,t)$ for all $y \in R$. By construction, the corresponding network is such that any maximum flow φ' satisfies $\varphi'(s,x) = p_x$ for all $x \in A$ and $\varphi'(y,t) = p_y$ for all

$y \in R$. This, along with the interpretation of the maximum flow in the paragraph following Theorem 4, implies Theorem 5.

4.3 Qualitative Properties of Persuasion

Topkis (1998) establishes that the maximal closure problem is an instance of a supermodular optimization problem.[14] Theorem 3 then implies that under normality, the persuasion problem is also an instance of supermodular optimization. This has several important qualitative consequences, which will be explored in this section. These consequences involve symmetry, which will be discussed in Subsection 4.3.1 and comparative statics, which will be discussed in Subsection 4.3.2. Appendix B further formalizes the relation of supermodularity to normality. Definitions of supermodularity and related notions can be found in Topkis (1998).

4.3.1 Symmetry

Example 3 shows that messages which have equivalent evidentiary content may be treated differently by optimal persuasion rules. In that example, evidence that two experts of the same gender support the speaker's position is persuasive, but the opinions of two experts of different genders is not. Yet gender is irrelevant. A similar example resembles Fishman and Hagerty (1990): a speaker attempts to persuade a listener that the speaker is talented in some task. The listener knows that the speaker will take five attempts at the task regardless of his performance. On each attempt, the speaker will be either successful or unsuccessful. Conditional on his talent, his performance on the tasks is i.i.d.. The speaker will only report the outcome of one task, but cannot lie. Intuitively, it still may be more persuasive for the speaker to say "I was successful on the first attempt", than "I was successful on the fourth attempt," even though in a setting in which the speaker randomly chooses which outcome to report neither should be more persuasive. This may be because it is common knowledge that the listener expects the speaker to report the first task on which he was successful. If the speaker reports the first successful task because it is more persuasive, the listener would acquire more information than if the listener were to respond to any claim of the form "I was successful on attempt n" in the same way and the speaker randomly chose a success to report (whenever he was successful on some attempt). Both of the examples above contain a constraint on the amount of evidence which the speaker reports. This turns out to be essential to the asymmetric treatment of identical evidence. I will now formalize this observation.

[14] Topkis (1998) has also shown that the maximum flow and minimum cut problems are instances of supermodular optimization.

Definition 2 *A pair of bijections (π, ξ) where $\pi : X \to X$ and $\xi : M \to M$ is a* ***symmetry*** *if for all x: (i) $\sigma(\pi(x)) = \{\xi(m) : m \in \sigma(x)\}$, (ii) $x \in A \Leftrightarrow \pi(x) \in A$, and (iii) $p_{\pi(x)} = p_x$. A persuasion rule f is* ***symmetric*** *if for every symmetry (π, ξ), $f = f \circ \xi$.*

A symmetry is a pair of functions, one from states to states, and the other from messages to messages, which preserve all relevant properties of the model. A persuasion rule is symmetric if it treats any pair of messages which cannot be distinguished without labels–or in other words, in terms of their intrinsic properties–in the same way. The two examples described above involving (i) gender (Example 3) and (ii) reporting only one task showed how optimal persuasion rules may be asymmetric.

Theorem 6 *Every normal persuasion problem admits a symmetric optimal persuasion rule.*

Proof. See Appendix. □

Theorem 6 follows from the supermodularity of the persuasion problem under normality. Observe that Theorem 6 allows for the possibility that some optimal persuasion rule is asymmetric, but under normality this phenomenon is inessential in the sense that whenever there is an asymmetric optimal rule, there is also a symmetric optimal rule. This establishes that time constraints, leading to a failure of normality, are critical for asymmetric treatment of equivalent evidence in an essential manner.

I will now present a class of non-normal persuasion problems, which I will refer to as the class of **typical time constrained** persuasion problems, in which all optimal rules are asymmetric whenever the problem is not trivial. This class includes the two examples discussed in the beginning of this section as special cases. In particular, suppose that $X = \{0, 1\}^n$. For any $J \subseteq \{1, \ldots, n\}$, and $x = (x_1, \ldots, x_n) \in X = \{0, 1\}^n$, define $x_J = \{(x_j, j) : j \in J\}$, and suppose that $\sigma(x) := \{x_J : |J| \leq h\}$. This means that at any state, the speaker can reveal (at most) h components of the vector he observes. If the speaker reveals a component, he reveals both its value (in $\{0, 1\}$) and its index (in $\{1, \ldots, n\}$). Assume that $1 \leq h < n$, so at x, the speaker can show at least 1 but not all of the components. Assume there exists some function $g : X \to [0, 1]$ such that for all x, $p_x = g(\sum_i x_i)$, so the probability of a string depends only on the number of 1's and 0's in the string. For example, his would be true if the components of x are n i.i.d. draws from a binary random variable, or if first some random variable θ is realized, and then x is the realization of n draws which are i.i.d. conditional on θ. Assume moreover that for some ℓ with $h < \ell \leq n$, $A = \{x \in X : \sum_i x_i \geq \ell\}$. In other words, there is some critical number ℓ such that if the speaker has at least ℓ components equal to 1, then the listener would like to accept the speaker's request, and otherwise, the listener would like to reject it. ℓ is sufficiently large, that it is always impossible for the speaker to prove that he has ℓ components equal to 1.

Theorem 7 *In typical time constrained persuasion problems, if it is not optimal to reject every message, then every optimal persuasion rule is asymmetric.*

Proof. In Appendix.

Observe that it is critical for the results in this section that the listener knows which action the speaker prefers. Imagine, in contrast a situation of "cheap talk" in which every message is available in every state of the world. Suppose there are two actions, a and b, and that the speaker and listener's interests are perfectly aligned in every state. Also, the optimal action depends on the state which is privately known to the speaker. Any optimal persuasion rule would employ at least two messages, which would be treated differently. For example, if a were optimal, the speaker would report one message, say m^a, and if b were optimal, the speaker would report another message, say m^b, and the listener would respond to m^a with a and to m^b with b. The message structure is trivially normal, and every optimal persuasion rule is asymmetric, because all messages are ex ante the same, so a symmetric persuasion rule would have to treat them in the same way. A little reflection shows that asymmetric treatment of equivalent messages is essential to all linguistic communication. For example, one word is used to refer to a house in English, and a different word is used in French. These two words are ex ante equivalent; before a language develops, it is does not matter which one will be used to refer to a house. The systematically different treatment of arbitrary signs is what allows people to communicate. Why then is the symmetry result in Theorem 6 relevant? The answer is that we are dealing with a situation in which the speaker's preferences are common knowledge. If there were only cheap talk, meaningful communication would be impossible, because at every state the speaker would use the most persuasive message, which would not allow the listener to infer any information about the state. So what allows for communication in this case is the presence of evidence whose availability depends on the state. The question then becomes, in such a setting is there any reason for the listener to treat equivalent pieces of evidence differently, and the above theorems show that the answer is: with time constraints, yes, but without time constraints, no.

4.3.2 Comparative Statics

As the maximal closure problem is an instance of maximizing a supermodular function on a lattice, monotone comparative statics are known for this problem (Topkis 1998). These comparative statics have an interesting interpretation when the maximal closure problem is interpreted as a problem of persuasion, in terms of the difficulty of persuasion rules. A persuasion rule f is **more difficult** than persuasion rule f' if for all $x \in X$, $\alpha(f, x) \leq \alpha(f', x)$. A more difficult persuasion rule is one such that in each state the speaker is less persuasive, and hence worse off. A persuasion rule f is a **most difficult optimal rule** if f is optimal and for all optimal f', f is (weakly) more difficult than f'. There may be more than one most difficult optimal rule, but in this case all such rules are equivalent in the sense that they induce the same probability of acceptance at every state. **Least difficult**

optimal rules are defined similarly. An important consequence of the supermodularity of normal persuasion problem under is that under normality, there always exist most difficult and least difficult optimal rules (see Theorem 18). Fix all aspects of a persuasion problem except p. Then for two probability distributions $p = (p_x)_{x \in X}$ and $p' = (p'_x)_{x \in X}$ on X write:

$$p \leq^* p' \Leftrightarrow \begin{cases} \forall x \in A, & p_x \leq p'_x, \\ \forall x \in R, & p'_x \leq p_x. \end{cases} \tag{10}$$

Theorem 8 *Consider any (A, p) and (A', p') satisfying either:*

1. *$p = p'$ and $A \subseteq A'$, or*

2. *$p \leq^* p'$ and $A = A'$.*

(i) If the message structure is normal, then both the most and least difficult optimal persuasion rules are are (weakly) more difficult given (A, p) than given (A', p').
(ii) Even without normality, it is impossible for there to be an optimal rule f given (A, p) which is strictly less difficult than all optimal rules given (A', p').
(iii) However, whenever the message structure is not normal there exist (A, p) and (A', p') satisfying 2 such that for all optimal persuasion rules f given (A, p) and f' given (A', p'), f and f' are unordered in terms of difficulty.

Proof. In Appendix.

Remark 1 *An identical comparative result holds when the cost of error ℓ_x depends on the state, and p is held constant so that $p = p'$ but $\ell \leq^* \ell'$, where the definition of $\leq^*$ for the state dependent cost of error $\ell = (\ell_x)_{x \in X}$ is analogous to (10).*

Theorem 8 says that given a normal message structure if either the probability mass is shifted from states in R to in A states or A grows at the expense of R, then optimal persuasion rules accept the speaker's message in more states, where more may mean either "superset" or "a greater number". It is tempting to conclude that the probability of acceptance must also go up. However, this conjecture would be false in the case that probability mass is shifted from states in R to states in A, as the following example shows:

Example 4 Suppose that $X = \{1, 2, 3, 4\}$, $A = \{1, 2\}$, $M = \{o, e\}$, $\sigma(1) = \sigma(3) = \{o\}, \sigma(2) = \sigma(4) = \{e\}$. (X, M, σ) is a normal message structure. Assume that:

$$\begin{array}{llll} p_1 = 1/3 & p_2 = 1/9 & p_3 = 2/9 & p_4 = 1/3 \\ p'_1 = 1/3 & p'_2 = 2/9 & p'_3 = 1/9 & p'_4 = 1/3 \end{array}$$

Then in moving from $(p_x)_{x \in X}$ to $(p'_x)_{x \in X}$ probability mass is shifted from a state in R to a state in A. However in both cases the unique optimal persuasion rule accepts o and rejects e, and the probability of acceptance moves down from 5/9 to 4/9.

The previous example shows that if probability mass is shifted from states in R to states in A, the probability of the listener accepts the speaker's request may go down. So condition 2 in Theorem 8 does not imply that the probability the listener will accept the speaker's request goes up; however condition 1 does:

Theorem 9 *Fix a normal* (X, M, σ) *and* p. *Let* $A \subseteq A'$. *Then the probability that the listener accepts the speaker's request is higher given* A' *than given* A *(evaluated at either the least or most difficult optimal rule.)*

Proof. By Theorem 18, the most difficult optimal rule is deterministic. Let Y (resp., Y') be the set of states at which the listener accepts the speaker's request at the most difficult optimal rule given A (resp., A'). Then by Theorem 8, $Y \subseteq Y'$. The observation that the probability of states has been held fixed completes the proof. The proof for the least difficult optimal rule is similar. $\square$

Next, consider the effect of moving probability mass from states in R to states in A on on the probability of error. This effect is ambiguous since when the probability of A is 0 or 1, the probability of error is zero. Fixing all aspects of the persuasion problem except p, let $\mu(p)$ be the error probability of error at the optimal rule.

Theorem 10 *Fix a normal message structure* (X, M, σ) *and a set* A. *Choose* $\overline{x} \in A, \underline{x} \in R$. *Suppose that* $p \leq^* p'$, *and let* $\epsilon \in [0, \min\{p_{\underline{x}}, p'_{\underline{x}}\}]$. *Then:*

$$\mu(p_{\overline{x}} + \epsilon, p_{\underline{x}} - \epsilon, p_{-\{\underline{x},\overline{x}\}}) - \mu(p) \geq \mu(p'_{\overline{x}} + \epsilon, p'_{\underline{x}} - \epsilon, p'_{-\{\underline{x},\overline{x}\}}) - \mu(p') \tag{11}$$

This theorem is similar to Theorem 3.7.3 in Topkis (1998), and so the proof is omitted. The theorem states that the shifting a fixed probability mass from a state in R to a state in A reduces the probability of error by more (or increases it by less), after some probability mass has already been shifted from states in R to states in A. Counter-examples to the theorem can be constructed without normality.

5 Non-normal Persuasion Problems

In this section, I will use the analysis of normal persuasion problems to draw conclusions about non-normal persuasion problems. Appendix B provides several characterizations of normality which shed further light on the relation between normal and non-normal persuasion problems. Say that a persuasion problem (X, M, σ, A, p) is **normal on** A if for all $x \in A$, there exists $m_x \in \sigma(x)$ such that for all $y \in X, m_x \in \sigma(y) \Rightarrow \sigma(x) \subseteq \sigma(y)$.

Observation 2 *For any persuasion problem which is normal on* A, *the construction of the corresponding maximal closure problem exactly as in Section 4.1 is valid. Therefore in*

Theorem 4, which provides the algorithm for persuasion is still true if the assumption that the persuasion problem is normal is weakened to the assumption that it is normal on A.

Let Z be the set of deterministic speaker strategies. So $\zeta \in Z$ if and only if $\zeta(x,m) \in \{0,1\}$ for all x and m. To simplify notation write $\zeta(x) = m$ if $\zeta(x,m) = 1$. For any $\zeta \in Z$, let ζ_A be the restriction of ζ to A. A similar notation may also be used for non-deterministic strategies. Let $Z_A := \{\zeta_A : \zeta \in Z\}$. For any $\zeta_A \in Z_A$, define $\sigma_{\zeta_A}(x) := \{\zeta_A(x)\}$ if $x \in A$ and $\sigma_\zeta(x) = \sigma(x)$ if $x \in R$. Since in A, the speaker has only one message according to σ_{ζ_A} it is immediate that (X, M, σ_{ζ_A}) is normal on A.

Theorem 11 *Let f^* be an optimal persuasion rule in (X, M, σ, A, p), and let $\zeta_A^* \in Z_A$ be a best reply to f^* on A. Then (i) f^* is an optimal persuasion rule in $(X, M, \sigma_{\zeta_A^*}, A, p)$. Moreover, (ii) an optimal persuasion rule f' in (X, M, σ, A, p) and a credible implementation (f', ζ') (with $\zeta'_A = \zeta_A^*$) can be found by solving the maximum flow problem corresponding to $(X, M, \sigma_{\zeta_A^*}, A, p)$ as in Theorem 4.*

Proof. Since in $\sigma_{\zeta_A^*}$, messages have only been removed in A (as compared to σ), on A, any persuasion rule f accepts with a weakly lower probability given $\sigma_{\zeta_A^*}$ than given σ, and on R, f accepts with the same probability given either σ or $\sigma_{\zeta_A^*}$. So (*) for every persuasion rule, the error probability is weakly greater given $\sigma_{\zeta_A^*}$ than given σ_A. However, since ζ_A^* is a best reply to f^* on A, f^* induces the same acceptance probability at every state given σ and $\sigma_{\zeta_A^*}$. (i) now follows and moreover the error probability at the optimal rule is the same given σ and $\sigma_{\zeta_A^*}$. (ii) now follows from Observation 2, (*), and Theorem 4.[15] $\square$

Theorem 11 can be interpreted as follows: suppose that one were to correctly guess a best reply ζ_A to the optimal persuasion rule on A. Then one could find an optimal persuasion rule, the probability of error, and a credible implementation by solving the maximal flow problem corresponding to $(X, M, \sigma_{\zeta_A}, A, p)$. Thus, in general, solving the persuasion problem can be decomposed into two parts:

1. Assigning a reporting strategy ζ_A to the speaker on A (The set of states on which the speaker's and listener's preferences coincide).

2. Using an appropriate maximum flow problem to determine which messages of the form $\{\zeta_A(x) : x \in A\}$ to accept.

[15] The one detail which may not be obvious in this last argument concerns the question of how we know that ζ_A^* is a best reply on A to f' since under σ, the speaker has more strategies available on A. Let φ be the maximum flow in the network corresponding to $\sigma_{\zeta_A^*}$. Then $\sum_{x \in A} \varphi(s,x)$ is the error probability induced by both f^* and f' (by (i)). Assume for contradiction that for $x \in A$, $f'(\zeta_A^*(x)) = 0$ but for some $y \in A$, $f'(\zeta_A^*(y)) = 1$ and $\zeta_A^*(y) \in \sigma(x)$. Let Z be the set of $x \in A$ satisfying the condition in the previous sentence. Note that for all $x \in Z$, $\varphi(s,x) > 0$, since otherwise x would be reachable from the source in the residual graph induced by φ. Then it is easy to see that f' must actually induce the error probability $\sum_{y \in A \setminus Z} \varphi(s,y)$ because for $x \in Z$, $\varphi(s,x)$ represents the error due to rejecting x under $\sigma_{\zeta_A^*}$. However, this contradicts the fact that f' induces error probability $\sum_{x \in A} \varphi(s,x)$.

In normal problems, the first step is trivial: simply instruct the speaker to report the whole truth m_x on A. In non-normal problems, the first step is far from trivial. In fact, it may be impossible to achieve the first step without first solving the second step for every possible choice of ζ_A, which may require a very large number of choices. Solving every persuasion problem of the form $(X, M, \sigma_{\zeta_A}, A, p)$ and then taking the solution with minimal error would be one valid, although perhaps inefficient, way of solving the persuasion problem.

An interesting consequence of part (i) of Theorem 11 is a simple and direct proof of the credibility result. In particular, (i) says that any optimal persuasion rule f^* is still optimal when the speaker is forced to play the best reply ζ_A^* on A. But then the game essentially becomes a zero-sum game between the speaker on R and the listener. Credibility then follows from the fact that commitment has no value in a zero-sum game. A formal version of this argument is given in Appendix A. Notice that (i) itself follows from a very simple argument that does not depend on any other result in this paper. Notice also that the original persuasion game is usually actually not a zero-sum game. This is intuitively obvious from the fact that the interests of the speaker and listener coincide on A, and formally it can be shown by the fact that there generally exist multiple equilibria of the game which give the players different utilities.

A further consequence of Theorem 11, is that, as in the normal case, without normality, the credible implementation can be assumed to take a canonical form. Say that a message $m \in \sigma(x)$ is **irreducible for** x if $\sigma^{-1}(m) = \{x \in X : m \in \sigma(x)\}$ is a minimal (according to $\subseteq$) in $\{\sigma^{-1}(m') : m' \in \sigma(x)\}$. An irreducible message provides as much information as possible. For example, in the class of *typical time constrained problems* presented in Section 4.3.1, in which a speaker observes a vector with n components, but has time only to show at most h of those components, the irreducible messages are precisely those in which the speaker reveals exactly h components. In non-normal problems there may be multiple non-equivalent irreducible messages at a state. In the above example, the speaker can present different collections of exactly h components. In contrast, in normal problems, a message is irreducible at x if and only if it can serve as the maximal message at x.

Corollary 1 *For every persuasion problem, there exists an optimal deterministic persuasion rule f such that (i) $f(m) = 1$ only if m is irreducible for some $x \in A$, and (ii) f has a credible implementation (f, ζ) such that for all $x \in A$, $\zeta(x, m) = 1$ for some m which is irreducible for x.*

Proof. Let f be an optimal persuasion rule. For every $x \in A$, there exists $h_x \in \sigma(x)$ such that $f(h_x) = \alpha(f, x)$. For each $x \in A$, it is possible to choose irreducible $h'_x \in \sigma(x)$ such that $\sigma^{-1}(h'_x) \subseteq \sigma^{-1}(h_x)$. Consider the persuasion rule f' defined by $f'(h'_x) = \alpha(f, x)$ for all $x \in A$ and $f'(m) = 0$ for all other messages m. For all $x \in A$, $\alpha(f, x) \leq \alpha(f', x)$ and for all $y \in R$, $\sigma^{-1}(h'_x) \subseteq \sigma^{-1}(h_x), \forall x \in A$ implies $\alpha(f', y) \leq \alpha(f, y)$. So optimality of f implies optimality of f'. Moreover, $\zeta_A(x) = h'_x$ is a best reply to f' on A. The result now follows from Theorem 11. $\square$

The corollary can be interpreted as saying that there always exists a credible implementation in which in A, the speaker reports as much truthful information as possible subject to time constraints, and in R, the speaker randomizes over lies. This extends a qualitative finding from Section 4.2 to the non-normal case. Next, I extend some results from Section 4.3 to the non-normal case.

Definition 3 *A collection of G of persuasion rules has the **common best reply property (CBRP)** if there exists a collection H of persuasion rules such that (i) $\{\alpha(f) : f \in G\} = \{\alpha(f) : f \in H\}$ and (ii) there exists $\zeta_A \in Z_A$ which is a best reply to every persuasion rule in H on A.*

Observation 3 *In a normal persuasion problem the set of all persuasion rules has the common best reply property.*

In particular, the common best reply is $\zeta_A(x) = m_x$ on A. Essentially the same observation was also made by Bull and Watson (2007) in the context of a more general class of mechanism design problems with evidence. Notice the parallel with the revelation principle in standard mechanism design; any implementable social choice function can be implemented in a direct mechanism with truthful revelation. In contrast, as we have seen above, in the non-normal case, selecting a reporting strategy is a non-trivial part of the problem of finding an optimal persuasion rule. Further discussion of the revelation principle and its interpretation in this context is found in the conclusion. It turns out that the CBRP, which holds trivially in the normal case, is at the heart of the qualitative results found in Section 4.3 concerning symmetry and comparative statics.

Theorem 12

1. *In any persuasion problem in which the set of optimal persuasion rules has the CBRP, there exist a most difficult and least difficult optimal persuasion rule, both of which are symmetric.*

2. *Fix a message structure (X, M, σ). Statement (i) of Theorem 8 holds without normality if the union of optimal persuasion rules given (A, p) and (A', p') has the CBRP. Theorem 9 holds without normality if the union of optimal persuasion rules given A and A' has the CBRP. Theorem 10 holds without normality as long as the union of optimal persuasion rules given $p, p', (p_{\overline{x}} + \epsilon, p_{\underline{x}} - \epsilon, p_{-\{\underline{x},\overline{x}\}})$, and $(p'_{\overline{x}} + \epsilon, p'_{\underline{x}} - \epsilon, p'_{-\{\underline{x},\overline{x}\}})$ has the CBRP.*

Proof. Let G be the set of optimal persuasion rules, and let H be as in Definition 3 with respect to ζ_A. Theorem 18, Observation 2, and normality of σ_{ζ_A} on A imply the existence of a most difficult and optimal rule f given σ_{ζ_A}. Theorem 11 and its proof imply that f is optimal given σ and f induces the same acceptance probability in every state given

σ and σ_{ζ_A}. If f is not a most difficult optimal rule given σ, CBRP implies there exists optimal f' more difficult than f such ζ_A is a best reply to f'. Then f' is optimal given σ_{ζ_A} and again induces the same acceptance probability at every state given σ and σ_{ζ_A}, so f' is more difficult than f given σ_{ζ_A}, a contradiction. So f is a most difficult optimal rule given σ. Now apply the construction in the proof of Theorem 6 starting with f to arrive at an optimal f^* such that $\alpha(f) = \alpha(f^*)$ and f^* is symmetric. For 2, an argument similar to the one in the previous paragraph, shows that performing the relevant comparative statics with σ_ζ leads to the same conclusion with σ. Normality on A of σ_{ζ_A} and Observation 2 complete the result. $\square$

Above we saw that a non-normal persuasion problem can be decomposed into two steps: (1) finding a speaker reporting strategy on A, ζ_A, and (2) Solving $(X, M, \sigma_{\zeta_A}, A, p)$. (1) corresponds to deciding which information to elicit, and (2) corresponds to deciding which arguments to accept. Theorem 12 shows that the qualitative properties found in Section 4.3 hold as long as there is a single solution to (1) which holds for all optimal persuasion rules in the sense of Definition 3. So it is only when different optimal persuasion rules require different elicitation of information–either for fixed parameter values for symmetry or as parameter values vary for comparative statics–that these qualitative results break down. Without time constraints, (1) is essentially vacuous explaining why the qualitative results always hold in that case.

6 Multiple Actions

In this section, I extend the persuasion model to one in which the speaker may choose one of many actions, and not just two.

6.1 The Persuasion Problem with Multiple Actions

I consider two possibilities for the listener's choice set. Under **discrete actions**, the listener selects an action from $J = \{1, \ldots, n\}$ with $n \geq 2$. j is **adjacent** to j' if $j = j'+1$ or $j = j'-1$. Under **continuous actions**, the listener's choice set is $J = [0, 1]$. The speaker has a utility function $u : J \to \mathbb{R}$ which does not depend on the state. Assume that $j < j' \Rightarrow u(j) < u(j')$. So the actions are ordered by the speaker's preference. The listener has a utility function $v : J \times X \to \mathbb{R}$, which does depend on the state. When actions are continuous, I assume that u and v are continuous. (X, M, σ) and p, as well as speaker strategies ζ, are as in the binary case. As in the binary case, the above captures the assumption that the listener knows what the speaker would like the listener to do but does not know what the speaker can say; what the listener would prefer to do depends on the speaker's information.

I now describe persuasion rules for the discrete actions case; analogous definitions hold with continuous actions. A persuasion rule is redefined as a function $f : M \times J \to [0, 1]$, where $f(m,j)$ is interpreted as the probability assigned to j given that message m is received. Of course, we assume that for all m and j, $f(m,j) \geq 0$, and $\sum_{i \in J} f(m,i) = 1$. F is the set of all persuasion rules. A persuasion rule f is **deterministic** if for all m and j, $f(m,j) \in \{0,1\}$. D is the set of all deterministic persuasion rules. If $f \in D$, I write $\tilde{f}(m)$ for the unique action j such that $f(m,j) = 1$. For any rule f, let $B(f)$ be the set of speaker best replies to f. A persuasion rule f^* is **optimal** if it maximizes $\max_{\zeta \in B(f)} \sum_{x \in X} \sum_{m \in M} \sum_{j \in J} \zeta(x,m) f(m,j) v(j,x) p_x$. Notice that f enters the objective in two ways, one of which is through $B(f)$. Implicitly, in determining an optimal persuasion rule, it is as if the listener could select the speaker's strategy as well, provided that it is a best reply to the selected rule. A persuasion rule is **deterministically optimal** if it is optimal among rules in D. In contrast, an **optimal deterministic persuasion rule** is optimal among rules in F and belongs to D.

The **game without commitment** is defined similarly to the binary case. However, in the case of many actions, a persuasion rule f^* is **credible** if there exists a ζ^* such that (i) (ζ^*, f^*) is a Bayesian Nash equilibrium, and (ii) ζ^* is the element of $B(f)$ that maximizes the *listener's* expected utility given that the listener uses f^*. The rationale for (ii) is that we are interested in the question of whether optimal persuasion rules are credible, and optimal persuasion rules are evaluated on the assumption that the listener may choose among speaker best replies; we want to be sure that both the persuasion rule and the speaker best reply that the listener chooses are consistent with equilibrium in the game without commitment.[16] The model presented in this section is a straightforward generalization of the model with binary actions.

6.2 Determinism

Glazer and Rubinstein (2006) established that in the case of two actions, there always exists an optimal deterministic persuasion rule. This is no longer true in the case of many actions.

Example 5 Suppose that there are two states x_1 and x_2 and three actions $J = \{1,2,3\}$, and that $\sigma(x_i) = \{m_1, m_2\}$ for $i = 1,2$. $p_x > 0$ for both states x. Suppose that the listener's utility function is given by:

v	1	2	3
x_1	1	0	1
x_2	0	1	0

I assume moreover that the speaker's utility function is such that $u(j) = j$. In any optimal rule, the response to one message is action 2 with probability 1, and the response to the other puts half the probability on 1 and the other half on 3. □

[16] In the binary case, condition (ii) adds nothing. Also, when restricting attention to deterministic rules, (ii) adds nothing.

In this subsection, I present two generalizations of the determinism result to many actions. Throughout this subsection, I will restrict attention to the discrete actions case, but at the end, I will comment on how one of the results extends to the continuous action case. The following concept is weaker than determinism:

Definition 4 *A persuasion rule f is **quasi-deterministic** if for all $m \in M$:*

$$\sum_{j \in J} u(j) f(m,j) \in \{u(j) : j \in J\} \tag{12}$$

$$|\{j \in A : f(m,j) > 0\}| \leq 2 \tag{13}$$

(12) says that every message gives the speaker an expected utility equal to that of some pure action. (13) says that given any message, at most two actions receive positive probability. Every deterministic persuasion rule satisfies both of these properties, so quasi-determinism generalizes determinism. In the model with binary actions, a persuasion rule satisfies (12) if and only if it is deterministic. This implies that in the binary case, determinism coincides with quasi-determinism. (12) and (13) imply that if for any message, two actions receive positive probability, then these two actions are nonadjacent. Since with two actions, it is trivially true that every pair of distinct actions is adjacent, this nonadjacency property is satisfied exactly (and vacuously) by the deterministic persuasion rules in the binary case. This is another way of seeing that determinism and quasi-determinism coincide in the binary case.

Define $\mathcal{R}(\zeta) := \{f \in F : \zeta \in B(f)\}$. $\mathcal{R}(\zeta)$ is the set of persuasion rules which **rationalize** ζ. $\mathcal{R}(\zeta)$ is always nonempty because it always contains all persuasion rules which respond to all messages in the same way. Being defined by a set of linear inequalities, bounded, and nonempty, $\mathcal{R}(\zeta)$ is always a polytope (i.e., the convex hull of a finite number of points), which may be thought of as a subset of $[0,1]^{M \times J}$.

Lemma 2 *The extreme points of $\mathcal{R}(\zeta)$ are quasi-deterministic.*

Proof. In appendix.

Theorem 13 *There exists an optimal quasi-deterministic persuasion rule.*

Proof. Let f^* be an optimal persuasion rule, and let ζ^* be a speaker strategy which maximizes the listener's expected utility within $B(f^*)$. Then any rule which maximizes the listener's expected utility within $\mathcal{R}(\zeta^*)$ on the assumption that the speaker will use ζ^* is optimal. There is a solution to the latter problem among the extreme points of $\mathcal{R}(\zeta^*)$ because holding ζ^* fixed, the listener's objective is linear in the rule f considered as an element of $[0,1]^{M \times J}$. The result now follows from Lemma 2. $\square$

Since quasi-determinism coincides with determinism in the binary case, this theorem implies as a corollary the Glazer and Rubinstein (2006) result that in the binary case, there always exists an optimal deterministic persuasion rule.

Using an argument similar to that in Theorem 13, it follows that:

Theorem 14 *For every equilibrium (f^*, ζ^*) of the game without commitment, there exists quasi-deterministic f^{**} such that (f^{**}, ζ^*) is an equilibrium. In the case of two actions, f^{**} is deterministic.*

The next assumption is important for both determinism and credibility.

Assumption 1 *For all $x \in X$, there exists a concave function $c_x : \mathbb{R} \to \mathbb{R}$ such that for all $j \in J$, $v(j, x) = c_x(u(j))$.*

Theorem 15 *Given Assumption 1, there exists an optimal deterministic persuasion rule.*

Proof. By Theorem 13, we can choose an optimal quasi-deterministic persuasion rule f^*, and let ζ^* be the speaker strategy which maximizes the listener's utility in $B(f^*)$ given that the listener uses f^*. By quasi-determinism, for any message m such that f^* selects a nondegenerate distribution δ in response to m, there is an action j such that j with probability 1 gives the speaker the same utility as δ. Construct a new persuasion rule f^{**} which replaces all such δ's with the corresponding j's. Then $\zeta^* \in B(f^{**})$. Moreover, by Jensen's inequality, the listener is better off. □

Since every function on a two element set is the restriction of some concave function to that set, Theorem 15 is a second generalization of Glazer and Rubinstein's result concerning deterministic optimal rules in the binary case.

In the case of continuous actions, Theorem 15 still holds. In the proof, it is unnecessary to appeal to quasi-determinism or an analogous concept. In particular, in the proof, let f^* be any optimal persuasion rule. Then continuity of the speaker's utility function guarantees that for any probability measure δ on J, there exists an action $j \in J$ such that the speaker is indifferent between j and δ. Then the remainder of the proof proceeds in the same way.

6.3 Credibility

Glazer and Rubinstein (2006) show that with many actions, the credibility result no longer holds. The following example is adapted from their paper:

Example 6 Suppose that there are two states $X = \{x_1, x_2\}$, and the probability distribution p is such that $p_{x_1} = .4$ and $p_{x_2} = .6$. $J = \{1, 2, 3\}$ and $u(j) = j$. Moreover, $\sigma(x_1) = \{m_1\}, \sigma(x_2) = \{m_1, m_2\}$. The listener's utility function is:

v	1	2	3
x_1	0	-1	1
x_2	0	1	-1

The rule f which assigns action 1 to m_1 and 2 to m_2 is the unique deterministically optimal rule. However, in this case, upon seeing m_1, the listener would know that the state is x_1, and therefore would prefer to take action 3. So f is not credible.[17] Allowing for random rules, the optimal rule f' responds to m_1 by randomizing over actions 1 and 3 with equal probability, and responds to m_2 by taking 2 with probability 1. Again, this rule is not credible. If the speaker chooses from his best replies, the one which is optimal from the listener's perspective, at x_1, the speaker will send m_1 and at x_2, the speaker will send m_2. But in this case the listener will prefer to select 3 with probability 1 upon seeing m_1. So f' is not credible. $\square$

Observe that in the above example, the listener's utility function is "convex" in actions at x_1. One can restore the credibility result under an appropriate concavity assumption.

Assumption 2 *There exists a strictly increasing function $r : J \to \mathbb{R}$ and for all $x \in X$, there exists a concave function $c_x : \mathbb{R} \to \mathbb{R}$ such that for all $j \in J$, $v(j, x) = c_x(r(j))$.*[18]

Assumption 2 is always satisfied in the case of two actions. Also, Assumption 2 is weaker than Assumption 1. With many actions, if r can be chosen to be equal to the speaker's utility function u (as in Assumption 1), then by Theorem 15, all deterministically optimal rules are optimal. It is interesting that an assumption which is sufficient for optimality of deterministic rules is similar to an assumption which is sufficient for credibility of rules which are optimal *among* deterministic rules.

Theorem 16 *Given Assumption 2, every deterministically optimal persuasion rule is credible.*

Proof. In Appendix.

Corollary 2 *Given Assumption 1, there exists an optimal persuasion rule which is deterministic and every optimal deterministic rule is credible.*

Proof. This follows from Theorems 15 and 16, and the fact that Assumption 1 implies Assumption 2. $\square$

Both Theorem 16 and Corollary 2 apply both when actions are discrete and when actions are continuous. To establish the credibility result in the binary case, one only needs to construct a speaker strategy ζ such that if the optimal rule accepts m, then if ζ uses m with positive probability, the listener would still like to accept the speaker's request conditional on seeing m, as well as an analogous fact for rejected messages. However, with

[17] Although, they do not say so explicitly, Glazer and Rubinstein's analysis of a counter-example shows that with many actions, a *deterministically* optimal persuasion rule may not be credible. This is natural, because they are making a comparison with the two action case, where deterministic optimality implies optimality. Here I extend the example to consider optimal rules, which are not deterministic.

[18] When J is continuous, I also assume that r is continuous.

many actions, one has to construct a speaker strategy ζ such that if the deterministically optimal $\tilde{f}$ is such that $\tilde{f}(m) = j$, then if ζ uses m with positive probability, *all* other actions will be less attractive for the listener than j conditional on seeing m. Consider the case of discrete actions. Given Assumption 2, it is easy to see that it is sufficient to show that actions $j-1$ and $j+1$ are less attractive to the listener than j conditional on seeing m. So it is necessary to construct a speaker strategy which makes $j-1$ and $j+1$ unattractive simultaneously for all m with $\tilde{f}(m) = j$. To further simplify matters, let us consider normal persuasion problems. The following separation theorem for separating closed from open sets in maximal closure problems implies the existence of such a strategy, and is also useful for explicitly constructing such a strategy in an algorithm to be presented below.

Lemma 3 *Let Z be a finite set and $\mathcal{C} \subseteq 2^Z$ be a family of sets closed under union and intersection and containing $\emptyset$ and Z, and consider $(\underline{w}_x)_{x\in Z}$ and $(\overline{w}_x)_{x\in Z}$ in $\mathbb{R}^Z$ such that for all $z \in Z$ and $U \in \mathcal{C}$:*

$$(i)\ \underline{w}_z \leq \overline{w}_z, \quad (ii)\ \textstyle\sum_{x\in U} \underline{w}_x \leq 0, \ \text{ and } \quad (iii)\ \textstyle\sum_{x\in Z\setminus U} \overline{w}_x \geq 0.$$

Then there exists $(w_x)_{x\in Z} \in \mathbb{R}^Z$ such that for all $z \in Z$ and $U \in \mathcal{C}$:

$$(i')\ \underline{w}_z \leq w_z \leq \overline{w}_z, \quad (ii')\ \textstyle\sum_{x\in U} w_x \leq 0, \ \text{ and } \quad (iii')\ \textstyle\sum_{x\in Z\setminus U} w_x \geq 0.$$

Proof. In Appendix. □

Note that any family $\mathcal{C}$ as in the lemma corresponds to a graph (Z, E) where the edges are taken to be $E = \{(x, y) \in Z \times Z : \forall U \in \mathcal{C}, x \in U \Rightarrow y \in U\}$. The closed sets in (Z, E) are precisely $\mathcal{C}$. Conversely, starting with any graph (Z, E), the set $\mathcal{C}$ of closed sets in (Z, E) satisfy the assumptions of the lemma. Therefore, (ii) above says that $\emptyset$ is a maximum weight closed set in the maximal closure problem corresponding to $\underline{w}$ and (iii) says that Z is a maximum weight closed set in the maximal closure problem corresponding to $\overline{w}$. The weight vector w constructed in the lemma is such that both $\emptyset$ and Z are maximum weight closed sets given w. Mapping this onto the discussion of credibility, Z is the set of states in which the speaker receives action j given $\tilde{f}$, $\underline{w}_x = \frac{v(j+1,x)-v(j,x)}{r(j+1)-r(j)} p_x$, which is proportional to the listener's benefit at state x of selecting $j+1$ rather than j and $\overline{w}_x := \frac{v(j,x)-v(j-1,x)}{r(j)-r(j-1)} p_x$, which is proportional to the benefit at state x of selecting j rather than $j-1$.[19] Assumption 2 implies that (i) is satisfied. The existence of w in the theorem is used to construct a speaker strategy which makes both deviations to $j-1$ and to $j+1$ unattractive for the listener for every m with $\tilde{f}(m) = j$. Then arguments similar to those presented in Section 5 are used to extend the argument to non-normal persuasion problems.

Next, I provide an algorithm for solving normal persuasion problems in the discrete multiple action case. For discrete multiple action normal persuasion problem $(X, M, \sigma, J, v, u, p)$,

[19] Notice that the constant of proportionality is different for deviations from j to $j+1$ than for deviations from j to $j-1$.

deterministic persuasion rule $\tilde{f}$, and action j, define $X^j_{\tilde{f}} := \{x \in X : \max_{m \in \sigma(x)} \tilde{f}(m) = j\}$, and define $E := \{(x, y) \in X \times X : \sigma(x) \subseteq \sigma(y)\}$. Let $\overline{w}^j_x := \frac{v(j,x)-v(j-1,x)}{r(j)-r(j-1)} p_x$ for $j = 2, \ldots, n$ and $\underline{w}^j_x := \frac{v(j+1,x)-v(j,x)}{r(j+1)-r(j)} p_x$ for $j = 1, \ldots, n-1$. I now define several maximal closure problems: for $j = 2, \ldots, n$, $\mathcal{W}^j := (X, E, \overline{w}^j)$.

Lemma 4 *Let $(X, M, \sigma, J, v, u, p)$ be a discrete action normal persuasion problem satisfying Assumption 2. Then $\tilde{f}$ is a deterministically optimal persuasion rule if and only if there exists $Y^2 \supseteq Y^3 \supseteq \cdots \supseteq Y^n$ such that each Y^j is optimal in $\mathcal{W}^j$ and $Y^j = \bigcup_{i=j}^n X^i_{\tilde{f}}$.*

This theorem follows in part from Theorem 5.1 of Hochbaum and Queyranne (2003) which shows that to solve the convex cost closure problem, it is sufficient to solve a sequence of related maximal closure problems.[20] The convex cost closure problem is similar to the maximal closure problem expect that the objective is to minimize a convex function rather than to maximize a linear function. Under normality and Assumption 2, finding a deterministically optimal persuasion rule is equivalent to solving the convex cost closure problem. Since the complete proof of 4 amounts to spelling out this equivalence and then appealing to the theorem mentioned above, this proof is omitted. Lemma 4 says that under normality and Assumption 2, finding a deterministically optimal persuasion rule is equivalent to solving a sequence of binary persuasion problems, where the first problem determines whether the speaker gets action at least 2, the second determines whether he gets action at least 3, and so on.

The following algorithm builds on the Hochbaum and Queyranne (2003) result as well as Lemma 3. It takes as input a multiple action discrete normal persuasion problem, and returns as output several maximum flows in related networks. It can be used not only to find an optimal persuasion rule, but also its credible implementation.

1. Use any max flow algorithm to find a sequence $Y^2 \supseteq Y^3 \supseteq \cdots \supseteq Y^n$ such that each Y^j is optimal in $\mathcal{W}^j$ for $j = 2, \ldots, n$.[21]

2. For $j = 2, \ldots, n-1$, consider maximal closure problems $(Y^j, E^j, \overline{w}^j)$ and $(Y^j, E^j, \underline{w}^j)$, where E^j is the restriction of E to Y^j. It follows from Lemma 4 that Y^j is optimal in $(Y^j, E^j, \overline{w}^j)$ and $\emptyset$ is optimal in $(Y^j, E^j, \underline{w}^j)$. Then by Lemma 3, there exists w^j with $\underline{w}^j_x \leq w^j_x \leq \overline{w}^j_x$ such that $\emptyset$ and Y^j are both optimal in (Y^j, E^j, w^j). This w^j can be constructed by finding maximum flows in several related networks as described in the proof of Lemma 3. Define $Y^1 =: X \setminus Y^2$, $w^1 = \underline{w}^1$ and $w^n = \overline{w}^n$.

3. For each $j = 1, \ldots, n$ use any max-flow algorithm to find a maximum flow φ^j in (Y^j, E^j, w^j).

[20] The lemma also bears a relation to Theorem 3.8.2 of Topkis (1998), which says that in the dynamic selection problem in which a firm must make decisions in a series of periods, myopic decisions (i.e., those which only take into account today's profits) are optimal in the problem as a while.

[21] If initially the sequence of solutions $Y^2, Y^3, \ldots, Y^n$, then the sequence $Y^2, Y^2 \cap Y^3, \ldots, Y^2 \cap Y^3 \cdots \cap Y^n$ is decreasing and by Assumption 2 and Theorem 3.7.4 of Topkis (1998) this is also a sequence of optimal solutions. Observe that the term "increasing" in that theorem means "increasing in the strong set order."

Theorem 17 *Fix a multiple action discrete normal persuasion problem satisfying $m_x = m_y \Rightarrow x = y$.*[22] *The preceding algorithm yields a deterministically optimal persuasion rule and a credible implementation of that rule. A deterministically optimal rule is given by:*

$$\tilde{f}(m) := \begin{cases} j, & \text{if } m = m_x \text{ for } x \in Y^j \text{ with } w_x^j \geq 0; \\ 1, & \text{otherwise.} \end{cases}$$

The speaker's strategy in the credible implementation is given by:

$$\zeta(x, m) := \begin{cases} 1, & \text{if } m = m_x,\ x \in Y^j \text{ and } w_x^j \geq 0; \\ \frac{\varphi^j(y,x)}{\varphi^j(x,t)}, & \text{if } m = m_y \text{ with } x, y \in Y^j,\ w_x^j < 0, w_y^j \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

If $y \in Y^1, w_x^1 < 0, \varphi(x,t) = 0$, then ζ is arbitrary.

The fact that $\tilde{f}$ is actually a deterministically optimal rule follows from Lemma 4. That the speaker strategy ζ is usable in a credible implementation follows from an argument similar to that in the proof of Theorem 16, and in particular the proof of Lemma 6. I omit the details because they are similar to arguments made elsewhere. The conclusion that when the speaker and listener's interests are aligned, then the speaker tells the whole truth and otherwise randomizes over lies survives to the multiple action case, although this set of states on which speaker and listener interests are aligned must be defined locally relative to the optimal rule. In particular, the set of states at which the speaker tells the whole truth is $\bigcup_{j=1}^{n} A^j$ where $A^j = \{x \in Y^j : w_x^j \geq 0\}$. So A^j contains (i) the set of all states at which the speaker is assigned action j by $\tilde{f}$ and if the state were known, the listener would prefer a strictly higher action, and (ii) possibly some states at which the speaker is assigned j, and if the state were known the listener would exactly prefer j, but not necessarily all states satisfying the latter condition. Clearly, within (i), the interests of the speaker and listener are aligned locally since both would prefer a higher action. Within (ii), interests are aligned when considering deviations to lower actions, which are dispreferred by both the speaker and listener, but not to higher actions, which are preferred by the speaker but not by the listener. At any state x at which according to $\tilde{f}$, the speaker receives an action which would be too high from the listener's perspective, the speaker randomizes (possible degenerately) over lies.

I conclude this section by discussing the relation of the credibility result to the unraveling result mentioned in Example 1. For each state x, define j_x to be a listener-optimal action if the listener knew the state was x. Say there there exists a **revealing equilibrium** in the game without commitment if there exists an equilibrium which induces the action j_x at each state x. The following observation applies with both continuous and discrete actions.

[22] The assumption $m_x = m_y \Rightarrow x = y$ is innocuous.

Observation 4 *If there exists a deterministic persuasion rule $\tilde{f}$ with $\max_{m \in \sigma(x)} \tilde{f}(m) = j_x$ for all $x \in X$, then there is a revealing equilibrium in the game without commitment.*

It is easy to prove this without appeal to the credibility result. However, for our purposes it is interesting to understand the relation to the credibility result. First suppose Assumption 2 holds. Observe that $\tilde{f}$ is clearly an optimal rule and it is deterministic, so it is deterministically optimal as well. So Theorem 16 implies that $\tilde{f}$ can be credibly implemented, which amounts to a revealing equilibrium. If Assumption 2 fails, then at each state x, we replace $v(j,x)$ with the smallest concave function of j which is always weakly greater than $v(j,x)$. This does not affect the optimal rule and credibility in this new problem implies credibility in the old problem because any deviation for the listener is less attractive in the old problem than the new. Observation 4 is weaker than the unraveling result in several ways. First, it does not make any primitive assumption on the evidence structure, but it is not difficult to characterize the conditions under which there exists a persuasion rule with the property in the observation.[23] For example, given normality an analog of the weak separation property given in section 4.1 is necessary and sufficient. Secondly, this observation only states that there exists a revealing equilibrium whereas the unraveling result states that all equilibria are revealing. Nevertheless, the proof suggested above shows that at least the part of the unraveling result stating existence of revealing equilibrium may be viewed as a consequence–or under Assumption 2, a special case–of the credibility result. To sum up, one way of viewing part of the unraveling result is as saying: if the listener can commit to a persuasion rule which elicits all of the speaker's relevant information, then there is an equilibrium in the game without commitment in which all relevant information is elicited; commitment has no value.

7 Conclusion and Discussion

In this section I discuss some methodological issues, as well as some issues which clarify the contribution of this paper.

7.1 Cheap Talk, Hard Evidence, and Dynamic Communication

A common distinction in the literature on persuasion games and mechanism design with evidence is that between cheap talk and hard evidence.[24] The idea is that there is a possibly unlimited stock of "cheap" messages which are available at every state and a collection of hard evidence whose availability depends on the state. The addition of cheap talk and dynamic communication may affect optimal persuasion rules. For instance, the speaker could be asked to report all of his information with a cheap talk message. Then

[23]See Proposition 11 of Lipman and Seppi (1995).

[24]The classic paper on strategic communication with cheap talk is Crawford and Sobel (1982).

depending on the cheap talk report, the listener randomly requests a piece of evidence from the speaker. If the speaker fails to provide this evidence, his request is rejected, and if he provides it, the request is accepted with some probability. This is similar to the mechanism used in Glazer and Rubinstein (2004) and the canonical mechanisms in Bull and Watson (2007). In contrast, such dynamic mechanisms are disallowed in Glazer and Rubinstein (2006) as well as in this paper. Unlimited cheap talk is not without loss of generality in any meaningful sense in the presence of communication constraints. For example the message correspondence may impose a limit on the size of the message space, such as a limit in the number of bits he may use to communicate. For instance, in presenting a talk, a speaker may be limited to an hour, and in the context of a debate, debaters may only have limited time to present their case. In fact, this is the intuitive interpretation offered here for why normality may fail. In principle, one might put constraints directly on the sequences of back and fourth communication. However, here I interpret the message correspondence as limiting all communication and disallow back and forth communication. It should be noted that given normal message structures, allowing cheap talk and dynamic communication would not change the optimal persuasion rule.[25] Therefore many of the results in this paper would be completely unaltered by this possibility. Glazer and Rubinstein (2004) established a credibility result in a model with cheap talk and dynamic communication and binary actions, so it would be interesting to explore the question of whether something like Assumption 1 or 2 would be sufficient for credibility in such a model with multiple actions when normality fails. As just explained, these assumptions are sufficient even with cheap talk and dynamic communication in the presence of normality.

7.2 The Revelation Principle

Several authors have discussed the interpretation of the revelation principle in settings where the available evidence depends on the state of the world (Green and Laffont 1986, Deneckere and Severinov 2001, Singh and Wittman 2001, Forges and Koessler 2006, Bull and Watson 2007). Different authors have interpreted both the message correspondences and the revelation principle differently in such a setting, so there does not appear to be a consensus on this issue. Here I would like to point out that the results about the canonical form which the speaker's strategy takes in equilibrium do not follow from any "revelation principle" in this literature. For example, Bull and Watson (2007) establish within their framework under normality, it is without loss of generality that any agent presents maximal message as well as a cheap talk report of his type to the mediator. However, the current paper shows that under normality there always exists a *credible implementation* of the optimal rule, such that in A (but not R) the speaker presents all of his evidence to the listener. This

[25]This would not hold in every mechanism design problem with a normal evidence structure, but depends on the fact that in the persuasion problem, the speaker's preferences are common knowledge.

implies for example that in a binary version of Shin (2003) described in Section 4.1, there exists an equilibrium in the game without commitment in which in states in A the speaker presents all of his information and which is at least as good, from the listener's perspective as the sanitization equilibrium described by Shin (2003), in which the speaker withholds all unfavorable information in all states. This is certainly not a consequence of any version of the revelation principle offered in the literature on mechanism design with evidence.

7.3 Relation to Glazer and Rubinstein's (2004, 2006) Proofs of the Credibility Result

The Glazer and Rubinstein (2006) proof of the credibility result involves a metaphor in which a planner must decide which bridges to open given that he would like some people to cross them and others not, and attempts to maximize the "net flow" of good types minus bad types. This metaphor was quite insightful. Although the classical notion of a flow differs somewhat, it turns out that persuasion problem is formally related to the classical maximum flow problem. One of the main contributions of the current paper is to formalize this relationship. Glazer and Rubinstein (2004) studied a related model in which the speaker initially sends a cheap talk message to the listener and then the listener can partially verify the message which is sent. The credible implementation in that model is qualitatively similar to the canonical form which I establish for the credible implementation of the Glazer and Rubinstein (2006) model: in A, the speaker will tell the truth, and in R, the speaker will tell a partial lie, hoping that the part of the evidence checked by the listener will turn out to be true. In the current model "telling the truth", "telling lies", "telling as much of the truth as possible subject to time constraints" refers to the presentation of hard evidence rather than cheap talk. This observation helps to unify the two models.

7.4 Normality, Symmetry, Comparative Statics, and Multiple Actions

Many of the qualitative properties of the persuasion problems in Section 4.3 which depend on normality extend to the model with multiple actions. For example, with multiple actions there still always exists a symmetric optimal rule under normality. Moreover, one can show that if the interests of the speaker and the listener become more aligned in the sense that the slope of the listener's utility function increases everywhere, so that it is beneficial to take a higher action, it becomes even more beneficial, and if it is costly, it becomes less costly, then optimal persuasion rules become less difficult. This is somewhat analogous to Theorem 8 for the binary case.

Appendix A: Credibility and the Minimax Theorem

Here I present an alternative short and direct proof of the credibility result using the minimax theorem. Fix an optimal persuasion rule f^* in a binary persuasion problem $\mathcal{P}$, and a deterministic speaker best reply ζ_A^* on A. Consider the persuasion problem $\mathcal{P}^* := (X, M, \sigma_{\zeta_A^*}, A, p)$, and let Γ^* be the corresponding game without commitment. Let $U(f,\zeta)$ be the error probability given strategy profile (f,ζ) in Γ^*.[26] So the listener chooses f to minimize $U(f,\zeta)$. Assuming that the speaker's objective is to choose ζ to maximize $U(f,\zeta)$ does not alter the speaker's best response correspondence in Γ^* since the speaker is only free to make a nontrivial choice in R. Part (i) of Theorem 11[27] implies that f^* is a minimax strategy for the listener in Γ^*, and hence there is an equilibrium (f^*,ζ^*) in Γ^*. Since ζ_A^* agrees with ζ^* on A, (f^*,ζ^*) is an equilibrium in the game without commitment corresponding to $\mathcal{P}$.

Appendix B: Characterization of Normality

In this section, I provide several characterizations of normality. All lattice-theoretic definitions and notation are standard and can be found in Davey and Priestley (2002) and Topkis (1998). Define $\mathcal{F} := \{\alpha(f) : f \in F\}$ and $\mathcal{F}^* := \{\alpha(f) : f \text{ is an optimal persuasion rule.}\}$. $\mathcal{F}$ only depends on the message structure, and not on (A,p), whereas $\mathcal{F}^*$ depends on both. The persuasion problem may be viewed as the problem of choosing $\beta \in \mathcal{F}$ to minimize $\sum_{x\in A}(1-\beta_x)p_x + \sum_{x\in R}\beta_x p_x$, or equivalently to maximize $\sum_{x\in A}\beta_x p_x - \sum_{x\in R}\beta_x p_x$, where β_x is the x-component of β. $\mathcal{F}^*$ is the set of solutions to this problem. The lattices $[0,1]^X$ and $\mathcal{F}$ are both equipped with the componentwise order $\leq$ according to which $\beta \leq \beta' \Leftrightarrow \forall x \in X, \beta_x \leq \beta'_x$. For any quasi-order (i.e., reflexive and transitive relation) on X define the polytope $P_{\preccurlyeq} := \{\beta \in [0,1]^X : \forall x,y \in X, x \preccurlyeq y \Rightarrow \beta_x \leq \beta_y\}$.

Theorem 18 *The following are equivalent: 1. (X,M,σ) is normal. 2. $\mathcal{F}$ is convex. 3. For all (A,p), $\mathcal{F}^*$ is convex. 4. There exists a quasi-order $\preccurlyeq$ on X such that $\mathcal{F} = P_{\preccurlyeq}$. 5. For all (A,p), $\mathcal{F}^*$ (also $\mathcal{F}$) is a (subcomplete) sublattice of $[0,1]^X$.*
6. For all (A,p), there exist most difficult and least difficult optimal persuasion rules. 7. For all (A,p), the persuasion problem is the maximization of a modular (also quasi-supermodular) function on $\mathcal{F}$.

Moreover, (i) $\mathcal{F}$ (but not $\mathcal{F}^*$) is always a (complete) lattice, (ii) whenever most difficult or least difficult optimal rules exist, they are deterministic, (iii) for all quasi-orders $\preccurlyeq$, there

[26] Notice that since F already allows randomization, allowing the speaker to randomize over persuasion rules would not change the strategy set.

[27] Observe that part (i) of Theorem 11 is a simple argument that does not depend on Theorem 4, so we are not already implicitly assuming the credibility result here.

exists normal (X, M, σ) such that $\mathcal{F} = P_{\preccurlyeq}$ and (iv) whenever (X, M, σ) is not normal, there exist a collection $(\preccurlyeq_i)_{i\in I}$ of quasi-orders such that $\mathcal{F} = \bigcup_{i\in I} P_{\preccurlyeq_i}$.

Proof. 1 $\Rightarrow$ 4: Define $\preccurlyeq$ by $x \preccurlyeq y \Leftrightarrow \sigma(x) \subseteq \sigma(y)$. The argument is then straightforward. 4 $\Rightarrow$ 2, 2 $\Rightarrow$ 3, and 5 $\Rightarrow$ 6 are immediate.

3 $\Rightarrow$ 1: Assume normality fails. Then there must be a state x at which the speaker does not have a maximal message. Define $\overline{\sigma}(x) := \{m \in \sigma(x) : \forall m' \in \sigma(x), \sigma^{-1}(m') \not\subset \sigma^{-1}(m)\}$, where "$\subset$" denotes *proper* subset. Choose an arbitrary vector γ in $\mathbb{R}^Y_{++}$ where $Y := \bigcup\{\sigma^{-1}(m) : m \in \overline{\sigma}(x)\} \setminus x$. For $m \in \overline{\sigma}(x)$, define $\gamma(m) := \sum_{y\in\sigma^{-1}(m)\setminus x} \gamma_y$. Enumerate the elements of $\overline{\sigma}(x)$ so that $\gamma(m_1) \leq \gamma(m_2) \leq \cdots \leq \gamma(m_n)$, where $n = |\overline{\sigma}(x)|$. Since there is no maximal message at x, $n \geq 2$, so there exists a smallest index i with $\sigma^{-1}(m_1) \neq \sigma^{-1}(m_i)$. Since $m_1, m_i \in \overline{\sigma}(x)$, $\sigma^{-1}(m_i) \setminus \sigma^{-1}(m_1)$ and $\sigma^{-1}(m_1) \setminus \sigma^{-1}(m_i)$ are nonempty. So, noting that whenever $1 \leq j < i$, $\gamma(m_1) = \gamma(m_j)$, it is possible to choose γ–consistent with everything above–so that $\gamma(m_1) = \gamma(m_i)$. Next extend γ to $\mathbb{R}$ so that for all $z \notin Y$, (including $z = x$), $\gamma_z > \beta(m_1)$. Next define p so that for all $y \in X, p_y := \gamma_y / \sum_{z\in X} \gamma_z$. Let $A = \{x\}$. For any $j = 1, \ldots, n$, let f_j be the persuasion rule such that $f_j(m) = 1$ if $m = m_j$, and $f_j(m) = 0$ otherwise. Then by construction f_1 and f_i are optimal persuasion rules. Choose $\lambda \in (0, 1)$ and define $\delta = \lambda\alpha(f_1) + (1 - \lambda)\alpha(f_i)$. Observe that $\delta_y = 1$ if and only if $y \in \sigma^{-1}(m_1) \cap \sigma^{-1}(m_i)$. Also $x \in \sigma^{-1}(m_1) \cap \sigma^{-1}(m_i)$. So if there exists f such that $\delta = \alpha(f)$, then there must exist $m \in \sigma(x)$ such that $\sigma^{-1}(m) \subseteq \sigma^{-1}(m_1) \cap \sigma^{-1}(m_i)$, but this is contradicts the facts that (i) $m_1, m_i \in \overline{\sigma}(x)$ and (ii) $\sigma^{-1}(m_1) \neq \sigma^{-1}(m_i)$. So $\mathcal{F}^*$ is not convex.

4 $\Rightarrow$ (5 and 7): It is straightforward to show that any set of the form $P_{\preccurlyeq}$ is a subcomplete sublattice of $[0, 1]^X$. The listener would like to choose $\beta \in \mathcal{F}$ to maximize $U(\beta) := \sum_{x\in A} \beta_x p_x - \sum_{x\in R} \beta_x p_x$. Since U is modular on $[0, 1]^X$ and $\mathcal{F}$ is a sublattice of $[0, 1]^X$, U is modular (hence, also quasi-supermodular) on $\mathcal{F}$. Since U is continuous, it now follows from Corollary 2.7.1 of Topkis (1998) that $\mathcal{F}^*$ is a subcomplete sublattice of $[0, 1]^X$.

(6 $\Rightarrow$ 1) and (7 $\Rightarrow$ 1): Assume normality fails. Then construction in the proof that 3 $\Rightarrow$ 1 also provides an example with a finite number of optimizers in which 6 fails. (i) (below) shows that $\mathcal{F}$ is a lattice. If the listener's objective were quasi-supermodular on this lattice, then the set of optimizers would be a lattice, having a greatest element, implying now that 7 also fails.

(i): Let $\emptyset \neq \mathcal{E} \subseteq \mathcal{F}$ and $\gamma := \bigvee_{[0,1]^X} \mathcal{E}$. For each $\beta \in \mathcal{E}$, there exists $f^\beta \in F$ such that $\alpha(f^\beta) = \beta$. For all x, there exists $m^\beta \in \sigma(x)$ such that $f^\beta(m^\beta) = \beta_x$, and for all $m \in \sigma(x), f(m) \leq \beta_x$. Define $f(m) := \sup\{f^\beta(m) : \beta \in \mathcal{E}\}$. For each $x \in X, \sup\{\max_{m\in\sigma(x)} f^\beta(m) : \beta \in \mathcal{E}\} = \gamma_x$. Finiteness of $\sigma(x)$ implies that there exists $m \in \sigma(x)$ such that $f(m) = \sup\{f^\beta(m) : \beta \in \mathcal{E}\} = \gamma_x$. So $\alpha(f, x) = \gamma_x$. So $\gamma \in \mathcal{F}$. So $\gamma = \bigvee_{\mathcal{F}} \mathcal{E}$. So $\mathcal{F}$ is a complete join-semilattice, and $\mathcal{F}$ has a least element $(0, \ldots, 0)$ because this can be achieved by the

persuasion rule that assigns 0 to all messages. It follows that $\mathcal{F}$ is a complete lattice. In fact for any $\emptyset \neq \mathcal{E} \subseteq \mathcal{F}$, $\bigwedge_{\mathcal{F}} \mathcal{E} = \bigvee_{[0,1]^X} \{\beta \in \mathcal{F} : \forall \beta' \in \mathcal{E}, \beta \leq \beta'\}$.
(ii): By (i) and $1 \Rightarrow 4$, $\mathcal{F} = \bigcup_{i \in I} P_{\preccurlyeq_i}$ for some $(\preccurlyeq_i)_{i \in I}$. Any most difficult optimal persuasion rule f must be such that $\alpha(f)$ belongs to some polytope $P_{\preccurlyeq_i}$. Since the objective is linear, it follows $\alpha(f)$ belongs to some face Q of $P_{\preccurlyeq_i}$ such that for every $\beta \in Q$, there exists an optimal persuasion rule f' with $\alpha(f') = \beta$. Since f is a most difficult optimal rule, $\alpha(f)$ must be an extreme point of Q and hence of $P_{\preccurlyeq_i}$. One can verify that the extreme points of $P_{\preccurlyeq_i}$ are precisely the points β satisfying $\beta_x = 1$ if $x \in U$ and 0 otherwise for $U \subseteq X$ satisfying: if $x \in U$ and $x \preccurlyeq_i y$, then $y \in U$. So the extreme points of $P_{\preccurlyeq_i}$ are integral, and therefore correspond to deterministic persuasion rules. A similar argument applies to least difficult optimal rules.
(iii): Let $\preccurlyeq$ be a quasi-order on X. Then set $M = X$, and it is possible to verify that $\sigma(x) := \{y \in X : y \preccurlyeq x\}$ is normal with $m_x = x$ and leads to $\mathcal{F} = P_{\preccurlyeq}$.
(iv): Define $O := \{\preccurlyeq : \ \preccurlyeq \text{ is a quasi-order and } P_{\preccurlyeq} \subseteq \mathcal{F}\}$. Then $\bigcup_{\preccurlyeq \in O} P_{\preccurlyeq} \subseteq \mathcal{F}$. Next, choose $\beta \in \mathcal{F}$, and define $x \preccurlyeq y \Leftrightarrow \beta_x \leq \beta_y$. To complete the proof, one argues that $P_{\preccurlyeq} \subseteq \mathcal{F}$. $\square$

Appendix C: Proofs

Proof of Theorem 4

Theorem 3 and Section 3.5 imply that $V^\varphi \setminus \{s\}$ solves the maximal closure problem corresponding to the persuasion problem. So to show that the persuasion rule f in (8) is optimal, it is sufficient to show (i) if $x \in A \cap V^\varphi$, x has a message that is accepted by f, and (ii) if $x \in R \setminus V^\varphi$, then x does not have a message accepted by f. (8) implies (i). Next assume for contradiction that there exists $x \in R \setminus V^\varphi$ and $y \in A \cap V^\varphi$ such that $m_y \in \sigma(x)$. It follows that $(y, x) \in E$, and this edge has infinite capacity, so $x \in V^\varphi$, a contradiction. This establishes (ii). By (6), (5), the fact the φ and V^φ are respectively a maximum flow and a minimum cut and the the max-flow min-cut theorem: $\text{value}(\varphi) = \sum_{x \in A \setminus V^\varphi} p_x + \sum_{x \in R \cap V^\varphi} p_x$. Since, as just established, $V^\varphi \setminus \{s\} = \{x : \alpha(f, x) = 1\}$, value$(\varphi)$ is the error probability.

To complete the proof, I show that the strategy ζ in (9) and f are mutual best responses. First, I argue that ζ is a best response to f. (That ζ indeed defines a probability distribution over messages at each state follows from the flow conservation constraints). If $x \in A \cap V^\varphi$, the speaker is attaining the highest possible utility. The previous paragraph established that in $x \in R \setminus V^\varphi$, the speaker will be rejected no matter what he does. It also follows from this that if there exists $x \in A \setminus V^\varphi$ who has a message accepted by f, then by Theorem 2, in the corresponding maximal closure problem there is a closed set C containing x and $A \cap V^\varphi$ but disjoint from $R \setminus V^\varphi$ which attains a higher value for that problem than $V^\varphi \setminus \{s\}$, a contradiction. Finally, consider $x \in R \cap V^\varphi$. Since φ is a maximum flow, $t \notin V^\varphi$, so

$\varphi(x,t) = p_x > 0$. From the flow conservation constraints, it follows that there exists $y \in A$ with $(y,x) \in E$ such that $\varphi(y,x) > 0$. So by (9), at x the speaker only plays messages of the form m_y with $y \in A$ with positive probability, and for any such y, $\varphi(y,x) > 0$, so $y \in V^\varphi$, implying that $f(m_y) = 1$. This establishes that ζ is a best response to f. Now, I argue that f is a best response to ζ. First consider $m = m_x$ for some $x \in A \cap V^\varphi$. For any $y \in R$, $m_x \in \sigma(y) \Leftrightarrow (x,y) \in E$, so if $m_x \in \sigma(y)$, $c(x,y) = \infty$ implies $y \in V^\varphi$, and the optimality of φ implies $t \notin V^\varphi$, so $\varphi(y,t) = p_y > 0$. So

$$p_x \geq \varphi(s,x) = \sum_{y \in R : m_x \in \sigma(y)} \varphi(x,y) = \sum_{y \in R : m_x \in \sigma(y)} \frac{\varphi(x,y)}{\varphi(y,t)} p_y = \sum_{y \in R : m_x \in \sigma(y)} \zeta(m_x, y) p_y,$$

where the inequality follows from the capacity constraints, and the first equality from the flow conservation constraints. This implies that the probability that the state is in A conditional on m being sent is at least $1/2$. So conditional on seeing m, the listener would prefer to accept the message, which is what f does. To complete the proof, it is sufficient to show that conditional on any message $m = m_x$ for some $x \in A \setminus V^\varphi$, the probability that the state is in A conditional on m being sent is at most $1/2$. Observe that if $x \in A \setminus V^\varphi$, then $p_x = \varphi(s,x)$. So:

$$p_x = \varphi(s,x) = \sum_{y \in R : m_x \in \sigma(y)} \varphi(x,y) \leq \sum_{y \in R : m_x \in \sigma(y), \varphi(y,t) \neq 0} \frac{\varphi(x,y)}{\varphi(y,t)} p_y = \sum_{y \in R : m_x \in \sigma(y)} \zeta(y, m_x) p_y,$$

where the second equality follows from the flow conservation constraints and the inequality follows the capacity constraints $\varphi(y,t) \leq p_t$ and the fact that $\varphi(y,t) = 0 \Rightarrow \varphi(z,y) = 0$ for all $(z,y) \in E$. Observe that the last term in the equation above potentially understates the probability that m is sent in states in R because it does not include the probability that m is sent in a state $y \in R$ with $\varphi(y,t) = 0$. It follows from the above inequality that the probability that the state is in A conditional on m being sent is at most $1/2$, which is what we wanted to show. $\square$

Proof of Theorem 6

Let Π be the set of all symmetries. Let f be any optimal persuasion rule. Then for every $(\pi, \xi) \in \Pi$, $f \circ \xi$ is an optimal rule. Define $f^*(m) := \max\{f \circ \xi(m) : (\pi, \xi) \in \Pi\}$. Then for all x, $\alpha(f^*, x) = \max\{\alpha(f \circ \xi) : (\pi, \xi) \in \Pi\}$. It now follows from the implication $1 \Rightarrow 5$ in Theorem 18 that f^* is also an optimal rule. As a notational matter, let $(\pi, \xi) \circ (\pi', \xi') := (\pi \circ \pi', \xi \circ \xi')$, where $\circ$ denotes the composition of functions. For for any $(\pi', \xi') \in \Pi$ and any $m \in M$: $f^* \circ \xi'(m) = \max\{f \circ \xi(\xi'(m)) : (\pi, \xi) \in \Pi\} = \max\{f \circ \xi''(m) : (\pi'', \xi'') \in \Pi \circ (\pi', \xi')\} = \max\{f \circ \xi''(m) : (\pi'', \xi'') \in \Pi\} = f^*(m)$. The third equality follows from the fact that Π is a group under $\circ$. So f^* is symmetric. $\square$

Proof of Theorem 7

Assume f is a symmetric optimal rule which does not reject some message m: $f(m) > 0$. By optimality, there exists x^* with $\sum_i x_i^* =: k \geq \ell$ and $J \subseteq \{1, \ldots, n\}$, $|J| \leq h$ with $f(x_J^*) > 0$. In other words, optimality implies that if f does not reject all messages, there must be some message m that f accepts with positive probability which is available at some state in A. For every q with $h \leq q < k$, there exists $y^q \in X$ with $\sum_i y_i^q = q$ and $y_J^q = x_J^*$. By symmetry, f accepts some message with positive probability from every state with $h \leq \sum_i x_i \leq k$. It follows that the optimal rule f accepts every message of the form $\{(1,j) : j \in J\}$ with $|J| = h$ with positive probability, and by symmetry it must assign all these messages the same probability, in fact 1 (since otherwise it would be optimal to reject all these messages as well). Note that any optimal rule must reject any message which reveals that any component is 0 or shows less than h components, because in this way it is possible to reject every type who has at most h 1's without reducing the probability of acceptance conditional on A. But now consider the rule f' which agrees with f except on message $\{(1,j) : 1 \leq j \leq h\}$, which it rejects. Such a rule would reject on an additional state in R, without rejecting on any additional states in A, and hence would do better, contradiction. □

Proof of Theorem 8

(i): This follows from Theorem 3.7.4 of Topkis (1998), Theorem 3, implication $1 \Rightarrow 6$ of Theorem 18 and part (ii) of that theorem.
(ii): Let f be optimal given (A, p), and f' be suboptimal given (A, p) and less difficult than f. Optimality of f given (A, p) implies: (*) $\sum_{x \in A}[\alpha(f, x) - \alpha(f', x)]p_x - \sum_{x \in R}[\alpha(f, x) - \alpha(f', x)]p_x > 0$. If (A, p) and (A', p') are related by 1 or 2, then the fact that $\alpha(f', x) \geq \alpha(f, x)$ for all x imply that (*) would still hold if A were replaced by A', R by $R' = X \setminus A'$, and p_x by p'_x, implying the result.
(iii): Assume normality fails. Then construct (A, p) and optimal persuasion rules f_1 and f_i as in the the proof that statement 3 of Theorem 18 implies statement 1 of Theorem 18. Rename (A, p) as $(\tilde{A}, \tilde{p})$, which frees up the expression (A, p). By construction, there exists $x \in \tilde{A}$ such that $\tilde{A} = \{x\}$. Now choose some small ϵ and define $p_y := \tilde{p}_y + \epsilon$ for all y such that $\alpha(f_1, y) = 0$, $p_x := \tilde{p}_x - h\epsilon$ where $h = |\{y \in X : \alpha(f_1, y) = 0\}|$, and $p_y := \tilde{p}_y$ otherwise. Similarly, define $p'_y := \tilde{p}_y - \epsilon$ for all y such that $\alpha(f_1, y) = 0$, $p'_x := \tilde{p}_x + h\epsilon$, and $p'_y := \tilde{p}_y$ otherwise. Since $\alpha(f_1, y) = 0 \Rightarrow y \in R$, $p \leq^* \tilde{p} \leq^* p'$. Setting $A = A' = \tilde{A}$, the unique optimal rule given (A, p) is f_1, and the set of optimal rules given (A', p') is a subset of $\{f_i, f_{i+1}, \ldots, f_n\}$ (see the construction in Theorem 18). f_1 is neither more nor less difficult than any rules in $\{f_i, \ldots, f_n\}$. □

Proof of Lemma 2

Suppose that $f \in \mathcal{R}(\zeta)$ is such that for some $m^*, \sum_{j\in J} u(j)f(m^*,j) \notin \{u(j) : a \in J\}$. (i.e., f violates (12)). Let $Q := \{m \in M : \sum_{j\in J} u(j)f(m,j) = \sum_{j\in J} u(j)f(m^*,j)\}$. For each $m \in Q$, let $\underline{j}^m$ (resp., $\overline{j}^m$) be the speaker's least (resp., most) preferred action j such that $f(m,j) > 0$. The fact that every $m \in Q$ gives the speaker an expected utility outside of $\{u(j) : j \in J\}$ implies that $\underline{j}^m \neq \overline{j}^m$. Now define two persuasion rules $\underline{f}$ and $\overline{f}$ such that for all $m \in Q$:

$$\underline{f}_{m,\underline{j}^m} = f_{m,\underline{j}^m} + \epsilon^m, \quad \underline{f}_{m,\overline{j}^m} = f_{m,\overline{j}^m} - \epsilon^m, \quad \overline{f}_{m,\overline{j}^m} = f_{m,\overline{j}^m} + \epsilon^m, \quad \overline{f}_{m,\underline{j}^m} = f_{m,\underline{j}^m} - \epsilon^m$$

where for all $m \in Q$, ϵ^m solves: $(u(\overline{j}^m) - u(\underline{j}^m))\epsilon^m = \epsilon$ for some $\epsilon > 0$, and I have used the notation $f_{m,j} := f(m,j)$. In all other cases, $\overline{f}$ and $\underline{f}$ coincide with f. If ϵ is chosen small enough then (i) $\overline{f}$ and $\underline{f}$ are in fact persuasion rules (i.e., assign a probability distribution over actions to each message) and, (ii) the ranking of messages (including ties) according to the speaker's expected utility to sending them is the same under $f, \underline{f}$, and $\overline{f}$. It follows that $\underline{f}$ and $\overline{f}$ are in $\mathcal{R}(\zeta)$. On the other hand $f = \frac{1}{2}\underline{f} + \frac{1}{2}\overline{f}$. So f is not an extreme point of $\mathcal{R}(\zeta)$. So any extreme point of $\mathcal{R}(\zeta)$ satisfies (12).

Next, assume that $f \in \mathcal{R}(\zeta)$ violates (13). Then there exist m^* and distinct $j', j'', j''' \in J$ such that for all $j \in I := \{j', j'', j'''\}, f(m^*, j) > 0$. Define $e := \sum_{j\in I} u(j)h_j$, where $h_j := f(m,j)/(\sum_{i\in I} f(m^*,i))$. Notice that since the numbers, $u(j'), u(j'')$, and $u(j''')$ are all distinct, the set $K := \{r \in \mathbb{R}^I : e = \sum_{j\in I} u(j)r_j, \sum_{j\in I} r_j = 1\}$ is a one-dimensional affine space. Moreover, there must be two distinct pairs of elements in the set $\{u(j'), u(j''), u(j''')\}$ such that e is a (possibly degenerate for one pair) weighted average of the pair. So assume wlog that there exist $\alpha, \beta \in [0,1)$ such that $e = \alpha u(j') + (1-\alpha)u(j'')$ and $e = \beta u(j') + (1-\beta)u(j''')$. In other words, $(\alpha, 1-\alpha, 0), (\beta, 0, 1-\beta) \in K$. Moreover $(\alpha, 1-\alpha, 0), (\beta, 0, 1-\beta)$ are affinely independent. So since $(h_{j'}, h_{j''}, h_{j'''}) \in K$ and $h_j > 0$ for all $j \in I$, there exists $\gamma \in \mathbb{R}$ such that $\gamma(\alpha, 1-\alpha, 0) + (1-\gamma)(\beta, 0, 1-\beta) = (h_{j'}, h_{j''}, h_{j'''})$. Since $(h_{j'}, h_{j''} h_{j'''}) \geq 0$, it follows that $\gamma \in (0,1)$. So define persuasion rules f' and f'' so that:

$$\begin{array}{lll} f'_{m^*,j'} = \alpha \sum_{j\in I} f_{m^*,j}, & f'_{m^*,j''} = (1-\alpha) \sum_{j\in I} f_{m^*,j}, & f'_{m^*,j'''} = 0 \\ f''_{m^*,j''} = \beta \sum_{j\in I} f_{m^*,j}, & f''_{m^*,j''} = 0, & f''_{m^*,j'''} = (1-\beta) \sum_{j\in I} f_{m^*,j} \end{array}$$

In all other cases, f' and f'' coincide with f. Notice that the speaker's expected utility to any message is the same under f' and f'' as under f. It then follows from the fact that $f \in \mathcal{R}(\zeta)$ that both f' and f'' are in $\mathcal{R}(\zeta)$. Moreover $f = \gamma f' + (1-\gamma)f''$. So f is not an extreme point of $\mathcal{R}(\zeta)$. So any extreme point of $\mathcal{R}(\zeta)$ satisfies (13). $\square$

Proof of Theorem 16

Consider a persuasion problem $\mathcal{P} = (X, M, \sigma, J, v, u, p)$ with 3 actions $J = \{j-1, j, j+1\}$. Let $\underline{\mathcal{P}}$ (resp. $\overline{\mathcal{P}}$) be the same as $\mathcal{P}$ except that $J = \{j, j+1\}$ (resp. $J = \{j-1, j\}$). Let $\tilde{f}^j$ be the deterministic persuasion rule such that $\tilde{f}^j(m) = j$ for all $m \in M$. Next define two maximal closure problems $\underline{\mathcal{W}} = (X, E, \overline{w})$ and $\overline{\mathcal{W}} = (X, E, \underline{w})$, where $E = \{(x, y) \in X \times X : \sigma(x) \subseteq \sigma(y)\}$, $\underline{w}_x = \frac{v(j+1)-v(j)}{r(j+1)-r(j)} p_x$, $\overline{w}_x = \frac{v(j,x)-v(j-1,x)}{r(j)-r(j-1)} p_x$.

Lemma 5 *If $\mathcal{P}$ is normal and satisfies Assumption 2 and $\tilde{f}^j$ is optimal in $\underline{\mathcal{P}}$ and $\overline{\mathcal{P}}$, then $\emptyset$ is optimal in $\underline{\mathcal{W}}$ and X is optimal in $\overline{\mathcal{W}}$.*

Proof. A proof similar to that of Theorem 2 shows that $\overline{\mathcal{P}}$ corresponds $\overline{\mathcal{W}}$, $\underline{\mathcal{P}}$ corresponds to $\underline{\mathcal{W}}$, and $\tilde{f}^j$ corresponds to X in $\overline{\mathcal{W}}$ and $\emptyset$ in $\underline{\mathcal{W}}$. $\square$

Lemma 6 *If $\mathcal{P}$ is normal and satisfies Assumption 2, and $\tilde{f}^j$ is optimal in $\underline{\mathcal{P}}$ and $\overline{\mathcal{P}}$, then $\tilde{f}^j$ is credible.*

Proof. By Assumption 2, for all x, $\underline{w}_x \leq \overline{w}_x$. It follows from Lemma 3 and Lemma 5 that there exists w with $\underline{w}_x \leq w_x \leq \overline{w}_x$ for all $x \in X$ and such that both $\emptyset$ and X are optimal $\mathcal{W} := (X, E, w)$. Let $X^+ := \{x \in X : w_x \geq 0\}, X^- = X \setminus X^+$. Consider a network N such that for every $x \in X^+$ there is an edge (s, x) with capacity w_x, for every $y \in X^-$, there is an edge (y, t) with capacity $-w_y$ and for every $x \in X^+, y \in X^-$, such that there is an x-y path in (X, E), there is an edge (x, y) in N with infinite capacity. Considerations as in Section 3.5 imply that N corresponds to $\mathcal{W}$, that optimality of $\emptyset$ and X in $\mathcal{W}$ imply that $\{s\}$ and $\{s\} \cup X$ respectively are minimum cuts in N, which in turn implies via the max-flow min-cut theorem that for any maximum flow φ in N and any $x \in X^+$, and $y \in X^-$, $\varphi(s, x) = w_x$ and $\varphi(y, t) = -w_y > 0$. Define a speaker strategy ζ on X

$$\zeta(x, m) = \begin{cases} 1, & \text{if } x \in X^+,\ m = m_x; \\ \sum_{z \in X^+, m = m_y} \frac{\varphi(z,x)}{\varphi(x,t)}, & \text{if } x \in X^+,\ m = m_y \text{ for } y \in X^+ \text{ with } \sigma(y) \subseteq \sigma(x); \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

The flow conservation constraints imply that ζ indeed defines a probability distribution over messages at every state. It is immediate that ζ is a best reply to $\tilde{f}^j$. For all $x \in X_+$, let $K_x := \{y \in X_+ : m_y = m_x\}$. Then choose k_x with $x \in X^j$ and observe that:

$$\begin{aligned} &\sum_{y \in X} \frac{v(j,x) - v(j-1,x)}{r(j) - r(j-1)} p_x \zeta(y, m_x) \geq \sum_{y \in X} w_y \zeta(y, m_x) \\ &= \sum_{y \in K_x} [w_y + \sum_{z \in X^-: \sigma(y) \subseteq \sigma(z)} w_z \frac{\varphi(y,z)}{\varphi(z,t)}] = \sum_{y \in K_x} [\varphi(s,y) - \sum_{z \in X^-: \sigma(y) \subseteq \sigma(z)} \varphi(y,z)] = 0, \end{aligned}$$

where the inequality follows from the fact that $\frac{v(j,x)-v(j-1,x)}{r(j)-r(j-1)}p_x = \overline{w}_x \geq w_x$, the second to last equality follows from the fact that for $x \in X^+$, $w_x = \varphi(s,x)$ and for $z \in X^-$, $w_z = -\varphi(z,t)$, and the last equality follows from the flow conservation constraints. This implies that the speaker would not like to assign $j-1$ to message m_x. A similar argument using $\underline{w}_x \leq w_x$ implies that $\sum_{y\in X} \frac{v(j+1,x)-v(j,x)}{r(j+1)-r(j)}p_x\zeta(y,m_x) \leq 0$, so the speaker would not like to assign $j+1$ to m_x. $\square$

Lemma 7 *If $\mathcal{P}$ satisfies Assumption 2, and $\tilde{f}^j$ is deterministically optimal in $\mathcal{P}$, then $\tilde{f}^j$ is credible.*

Proof. First observe that deterministic optimality of $\tilde{f}^j$ in $\mathcal{P}$ implies optimality of $\tilde{f}^j$ in $\underline{\mathcal{P}}$ and $\overline{\mathcal{P}}$ since deterministic optimality implies optimality in the binary case. Define $\underline{X}^+ := \{x \in X : \underline{w}_x \geq 0\}$, $\underline{X}^- := X \setminus \underline{X}^+, \overline{X}^+ := \{x \in X : \overline{w}_x \geq 0\}, \overline{X}^- := X \setminus \overline{X}^+$. For each $x \in \underline{X}^+$ choose $k_x \in \sigma(x)$ and for each $x \in \underline{X}^-$ construct a new message h_x. Define $\sigma^*(x) = \{k_x\}$ if $x \in \underline{X}^+, \sigma^*(x) = \sigma(x) \cup \{h_x\}$ if $x \in \underline{X}^+$. Let $\underline{\mathcal{P}}^*$ be like $\underline{\mathcal{P}}$ except that σ is replaced by σ^*. An argument similar to that in Theorem 11 implies that $\tilde{f}^j$ is optimal in $\underline{\mathcal{P}}^*$. Moreover $\underline{\mathcal{P}}^*$ is normal, so construct the corresponding network N^* and find a maximum flow φ^*. Optimality of $\tilde{f}^j$ in $\underline{\mathcal{P}}^*$ implies that $\{s\}$ is a minimum cut in N^* and via the max-flow min-cut theorem, for every maximum flow φ^* in N^* and $x \in \underline{X}^+$, $\varphi^*(s,x) = \underline{w}_x$. Define $Y = \overline{X}^+ \setminus \underline{X}^+$. For each $y \in Y$ any x such that $(x,y) \in E^*$, the set of edges in N^*, with $\varphi^*(x,y) > 0$, create a new element y_x and let $Z := \{y_x : y \in Y, (x,y) \in E^*, \varphi^*(x,y) > 0\}$. Define $\widehat{X} := (X \setminus Y) \cup Z$. Now create a new network N^{**} with vertices $\widehat{X} \cup \{s,t\}$. Let E^{**} be the edge set in N^{**}, and suppose that any edge in E^* both of whose vertices are contained in $(X \setminus Y) \cup \{s,t\}$ is also contained in E^{**} with the same capacity as in N^*. Moreover for each $y_x \in Z$, there is an edge (x, y_x) in E^{**} with infinite capacity, and an edge (y_x, t) with capacity $\underline{w}_x \frac{\varphi^*(x,y)}{\varphi^*(y,t)}$. Now construct a flow φ^{**} in N^{**} which agrees with φ^* on $E^* \cap E^{**}$. For $(x, y_x) \in E^{**}$ and $(y_x, t) \in E^{**}$, define $\varphi^{**}(x, y_x) :=: \varphi^{**}(y_x, t) := \varphi^*(x,y)$. φ^{**}–like φ^* in N^*–induces minimum cut $\{s\}$ (as the vertex set reachable from the source) in N^{**}. Next replace X by $\widehat{X}$ in $\underline{\mathcal{P}}^*$, keep probabilities, utilities, and messages unchanged on $\widehat{X} \cap X$, but extend σ^*–which we now call σ^{**}–so that σ^* on Z so that $\sigma^*(y_x) = \{k_x\}$, and assume that for all i $v(i, y_x) = v(i, y)$ and the probability of y_x is $\frac{\varphi^*(x,y)}{\varphi^*(y,t)}p_y$, naming this altered problem $\underline{\mathcal{P}}^{**}$. N^{**} is the network corresponding to $\underline{\mathcal{P}}^{**}$ and since $\{s\}$ is a minimum cut in N^{**}, $\tilde{f}^j$ is optimal in $\underline{\mathcal{P}}^{**}$. Let $\overline{\mathcal{P}}^{**}$ be the same as $\overline{\mathcal{P}}$ except that $\widehat{X}$ replaces X and σ^{**} replaces σ and utilities and probabilities are altered as above. Now consider a persuasion problem $\overline{\mathcal{P}}^*$ which is the same as $\overline{\mathcal{P}}$ except that σ is replaced by σ^* with which agrees with σ on $\widehat{X} \cap X$ but for $y_x \in Z$, $\sigma(y) = \sigma(y_x)$ and utilities and probabilities are as above. Then $\overline{\mathcal{P}}^*$ is essentially identical to $\overline{\mathcal{P}}$, so that $\tilde{f}^j$ is optimal in $\overline{\mathcal{P}}^*$. Now using arguments similar to those in Theorem 11, one can show that $\tilde{f}^j$ is also optimal in $\overline{\mathcal{P}}^{**}$. Then combining $\underline{\mathcal{P}}^{**}$ and $\overline{\mathcal{P}}^{**}$ into a single persuasion problem with three action $\mathcal{P}^{**}$, and observing that this problem is normal, it follow from Lemma 6 that $\tilde{f}^j$ is credible in $\mathcal{P}^{**}$, and noting that messages h_x will never be used in the credible implementation constructed in that lemma, this will also amount to a credible implementation in $\mathcal{P}$. $\square$

Now consider an arbitrary discrete action multiple action persuasion problem $\mathcal{P}$ and a deterministically optimal persuasion rule $\tilde{f}$. Then for every action $j = 2, \ldots, n-1$ consider the persuasion problem $\mathcal{P}^j$, where the actions are $J = \{j-1, j, j+1\}$, the state space is restricted to $X^j = \{x \in X : \max_{m \in \sigma(x)} \tilde{f}(m) = x\}$, the messages are $M^j := \{m \in M : \tilde{f}(m) = j\}$ and $\sigma^j(x) = \sigma(x) \cap M^j$, utilities are unchanged and the probabilities are renormalized so as to sum to 1. Then Lemma 7 applied separately to each $\mathcal{P}^j$ yields a credible implementation in $\mathcal{P}$. Lemma 7 only says that after seeing a message with $\tilde{f}(m) = j$, the listener would not prefer $j-1$ or $j+1$ to j, but it then follows from Assumption 2 that the listener would not prefer any other action i to j. Note also that the cases $j = 1$ and $j = n$ must be treated separately but those cases are easier to deal with than the others, because there is only one adjacent action, which essentially reduces the situation to that of a binary persuasion problem.

Finally to extend the results to the continuous case, start with a deterministically optimal persuasion rule $\tilde{f}$ and any $j \in (0,1)$ with $X^j \neq \emptyset$. Then define a three action persuasion problem $\mathcal{P}^j$ with actions $\{j-1, j, j+1\}$ and listener utility function $\hat{v}$ with $\hat{v}(j,x) = v(j,x)$ for all $x \in X^j$, but suppose that $\hat{v}(j+1,x)$ and $\hat{v}(j-1,x)$ solve $\lim_{i \uparrow j} \frac{v(j,x)-v(i,x)}{r(j)-r(i)} = \frac{\hat{v}(j,x)-\hat{v}(j-1,x)}{r(j)-r(j-1)}$ and $\lim_{i \downarrow j} \frac{v(i,x)-v(j,x)}{r(i)-r(j)} = \frac{\hat{v}(j+1,x)-\hat{v}(j,x)}{r(j+1)-r(j)}$. Using $\mathcal{P}^j$, the rest of the argument is similar to the discrete case. $\square$

Proof of Lemma 3

Proof. For each $z \in Z$, let z^+ and z^- be two distinct elements. Define $Z^+ := \{z^+ : z \in Z\}$, $Z^- := \{z^- : z \in Z\}$ and $Z' := Z^+ \cup Z^-$. For all $z^+ \in Z^+$ define $\overline{w}'_{z^+} := \max\{0, \overline{w}_z\}$ and $\underline{w}'_{z^+} = \max\{0, \underline{w}_z\}$, and for all $z^- \in Z^-$ define $\overline{w}'_{z^-} := \min\{0, \overline{w}_z\}$ and $\underline{w}'_{z^-} = \min\{0, \underline{w}_z\}$. For each $U \subseteq Z$, define $U' := \{z^+ \in Z^+ : z \in U\} \cup \{z^- \in Z^- : z \in U\}$. Then for all $U \subseteq Z$, $\sum_{z \in U} \overline{w}_z = \sum_{z' \in U'} \overline{w}'_{z'}$ and $\sum_{z \in U} \overline{w}_z = \sum_{z' \in U'} \overline{w}'_{z'}$. Moreover, for all $z' \in Z'$, $\underline{w}'_{z'} \leq \overline{w}'_{z'}$. Finally define $\mathcal{C}' \subseteq 2^{Z'}$ by $\mathcal{C}' := \{U' : U \in \mathcal{C}\}$, and observe that for all $U \subseteq Z$, $(Z \setminus U)' = Z' \setminus U'$, and that $\mathcal{C}'$ is closed under union and intersection.

Define $E' = \{(x,y) \in Z' \times Z' : \forall U \in \mathcal{C}', x \in U \Rightarrow y \in U\}$. It is easy to show that $\mathcal{C}'$ is the set of closed sets in the graph (Z', E'). Let $\overline{N} = (V, E, \overline{c}, s, t)$ and $\underline{N} = (V, E, \underline{c}, s, t)$ be the networks corresponding to the maximal closure problems $(Z', E', \underline{w}')$ and $(Z', E', \overline{w}')$ respectively. In particular $V = Z' \cup \{s,t\}$ and

$$\begin{aligned} E \quad = \quad & \underbrace{\{(s,x) : x \in Z^+\}}_{E_1} \cup \underbrace{\{(x,y) : x \in Z^+, y \in Z^-, \text{ there is an } x\text{-}y \text{ path in } (Z', E')\}}_{E_2} \\ & \cup \underbrace{\{(y,t) : y \in Z^-\}}_{E_3} \end{aligned}$$

Observe that this is a slightly different formulation of the edges in the network corresponding to a maximal closure graph than the one given in Section 3.5, but it is also valid. Moreover $\overline{c}(s,x) = \overline{w}'_x$ and $\underline{c}(s,x) = \underline{w}'_x$ for all $(s,x) \in E_1$, $\overline{c}(y,t) = -\overline{w}'_y$ and $\underline{c}(y,t) = -\underline{w}'_y$ for all $(y,t) \in E_3$, and $\overline{c}(e) = \underline{c}(e) = \infty$ for all $e \in E_2$. Next define the network $N = (V, E, c, s, t)$ where for all $e \in E$, $c(e) = \min\{\overline{c}(e), \underline{c}(e)\}$. Let φ^0 be a maximum flow in N.

Lemma 8 *There exist maximum flows $\underline{\varphi}$ and $\overline{\varphi}$ in $\underline{N}$ and $\overline{N}$ respectively such that (i) for all $e \in E_1 \cup E_3$, $\varphi^0(e) \leq \min\{\underline{\varphi}(e), \overline{\varphi}(e)\}$, (ii) for any $(x,y) \in E$ such that $\{x,y\} \cap (V^{\varphi^0} \setminus \{s,t\}) \neq \emptyset$, $\overline{\varphi}(x,y) = \varphi^0(x,y)$, and (iii) for any $(x,y) \in E$ such that $\{x,y\} \cap (V \setminus (V^{\varphi^0} \cup \{s,t\})) \neq \emptyset$, $\underline{\varphi}(x,y) = \varphi^0(x,y)$.*

Proof. The following is a consequence of the definition of V^{φ^0} and the fact that for all $e \in E_2$, $c(e) = \infty$:

Fact 1 *For all $x \in V^{\varphi^0} \cap Z^+$ and $y \in (V \setminus V^{\varphi^0}) \cap Z^-$, $(x,y) \notin E$, and for all $y \in V^{\varphi^0} \cap Z^-$ and $x \in (V \setminus V^{\varphi^0}) \cap Z^+$ with $(x,y) \in E$, $\varphi^0(x,y) = 0$.*

Observe that φ^0 is a feasible flow in $\overline{N}$. So one can arrive at a maximum flow in $\overline{N}$ by applying the Ford-Fulkerson algorithm starting with φ^0. I argue that at each stage of the algorithm, the flow φ satisfies:

1. For all $e \in E_1 \cup E_3$, $\varphi^0(e) \leq \varphi(e)$.
2. For any $(x,y) \in E$ such that $\{x,y\} \cap (V^{\varphi^0} \setminus \{s,t\}) \neq \emptyset$, $\varphi(x,y) = \varphi^0(x,y)$.

These two properties are immediate initially when $\varphi = \varphi^0$. Now suppose that both properties hold during some stage of the algorithm when the current flow is φ. Now find an s-t path P (without cycles) in $\overline{N}^{\varphi}$, the residual graph induced by φ in $\overline{N}$. P is of the form $(s, x_1, y_1, x_2, y_2, \ldots, x_h, y_h, t)$ where each x_i belongs to Z^+ and each y_i belongs to Z^-. By the definition of the residual graph, it must be the case that $\varphi(y_h, t) < \overline{c}(y_h, t)$. Note that for all $y \in V^{\varphi^0}$, with $(y,t) \in E, \varphi^0(y,t) = c(y,t) = \overline{c}(y,t)$. It then follows from the fact that φ satisfies 1 that $y_h \notin V^{\varphi^0}$. Next assume that $y_i \notin V^{\varphi^0}$. Observe that $(x_i, y_i) \in E$. But since $c(x_i, y_i) = \infty$, it follows that $x_i \notin V^{\varphi^0}$. If $i \neq 1$, then $(x_i, y_{i-1}) \in E$ and $\varphi(x_i, y_{i-1}) > 0$. If $y_{i-1} \in V^{\varphi^0}$, then by Fact 1, $\varphi^0(x_i, y_{i-1}) = 0$, but then since φ satisfies 2, $\varphi(x_i, y_{i-1}) = 0$, a contradiction. This establishes that every vertex in P other than s and t, does not belong to V^{φ^0}. But this implies that when additional flow is sent through P, the resulting flow will still satisfy 2. Moreover, since the only edges in P which belong to $E_1 \cup E_3$ are (s, x_1) and (y_h, t), and the flow along these edges is increased, the resulting flow will also satisfy 1. It follows that $\varphi(e) \leq \overline{\varphi}(e)$ and $\overline{\varphi}$ satisfies (ii), where $\overline{\varphi}$ is the maximum flow in $\overline{N}$ derived through the Ford-Fulkerson algorithm. An analogous argument, which constructs a maximum flow $\underline{\varphi}$ from φ^0 in $\underline{N}$ completes the proof. $\square$ Next define the flow φ' by:

$$\varphi'(e) := \begin{cases} \overline{\varphi}(e), & \text{if } e = (x,y) \text{ with } \{x,y\} \cap (V \setminus (V^{\varphi^0} \cup \{s,t\})) \neq \emptyset \ ; \\ \underline{\varphi}(e), & \text{otherwise.} \end{cases}$$

For $x \in (V \setminus V^{\varphi^0}) \cap Z^+$, we have $\varphi'(s,x) = \overline{\varphi}(s,x) = \sum_{y \in Z^-:(x,y)\in E} \overline{\varphi}(x,y) = \sum_{y\in Z^-:(x,y)\in E} \varphi'(x,y)$, where the first and third inequalities follow from the definition of φ' and the second inequality follows from the fact that $\overline{\varphi}$ satisfies the flow conservation constraints. For $x \in V^{\varphi^0} \cap Z^+$, $\varphi'(s,x) = \underline{\varphi}(s,x) = \sum_{y\in Z^-:(x,y)\in E} \underline{\varphi}(x,y) = \sum_{y \in Z^-:(x,y)\in E} \varphi'(x,y)$, where the first equality follows from the definition of φ', the second from the fact that $\underline{\varphi}$ satisfies the flow conservation constraints, and the third from the definition of φ' and parts (ii) and (iii) of Lemma 8, which imply that if $(x,y) \in E$ and $y \in (V \setminus V^{\varphi^0}) \cap Z^-$, then $\varphi'(x,y) = \overline{\varphi}(x,y) = \varphi^0(x,y) = \underline{\varphi}(x,y)$. Similar arguments apply for $y \in (V \setminus V^{\varphi^0}) \cap Z^-$ and $y \in V^{\varphi^0} \cap Z^-$, implying that φ' satisfies all of the flow conservation constraints. Next define the network $N' = (V, E, c', s, t)$ with $c'(e) = \varphi'(e)$ for all $e \in E_1 \cup E_3$ and $c'(e) = \infty$ for $e \in E_2$. Then, by construction φ' is a maximum flow in N'. Let (Z', E', w') be the maximal closure problem corresponding to N'. Observe that Z' and E' are as defined above, and $w'_x = c'(s,x)$ for all $x \in Z^+$ and $w'_y = -c'(y,t)$ for all $y \in Z^-$. Since φ' is a feasible flow in $\overline{N}$, $\sum_{x\in Z^+} c'(s,x) = \sum_{x\in Z^+} \varphi'(s,x) = \sum_{y \in Z^-} \varphi(y,t) = \sum_{y\in Z'} c'(y,t)$, where the first and third equalities hold by construction and the second follows from the fact that the flow out of the source equals the flow into the sink for any feasible flow. It follows from the previous equation and the max-flow min-cut theorem that both $\{s\}$ and $\{s\} \cup Z'$ are minimum cuts in N'. The relationship between the maximal closure problem and the minimum cut problem on the corresponding network as explained in Section 3.5 implies that every minimum cut in N' corresponds to a maximum weight closed set in (Z', E', w') (up to a set of zero weight vertices).[28] It follows $\emptyset$ and Z' are both maximum weight closed sets in (Z', E', w'), which implies that for all $U' \in \mathcal{C}'$, $\sum_{x\in U'} w'_x \leq 0$ and $\sum_{x \in Z' \setminus U} w'_x \geq 0$. Next define $(w_z)_{z\in Z}$ by $w_z := w'_{z^+} + w'_{z^-}$, where Z is as above. Then observe that for any $U \in \mathcal{C}$, the corresponding set U' in $\mathcal{C}'$ is such that $z \in U \Leftrightarrow \{z^+, z^-\} \in U'$. It follows that $\sum_{z\in U} w_z = \sum_{z'\in U'} w'_{z'}$ and $\sum_{z \in Z\setminus U} w_z = \sum_{z'\in Z' \setminus U'} w'_{z'}$. So for all $U \in \mathcal{C}$, $\sum_{z\in U} w_z \leq 0$ and $\sum_{z\in Z\setminus U} w_z \geq 0$.

Lemma 9 *For all $e \in E_1$, $\underline{c}(e) = \underline{\varphi}(e)$ and for all $e \in E_3$, $\overline{c}(e) = \overline{\varphi}(e)$.*

Proof. It follows from the fact that $\sum_{x\in U'} \underline{w}_x \leq 0$ for all $U' \in \mathcal{C}'$ that $\emptyset$ is a maximum weight closed set in $(Z', E', \underline{w}')$, which implies that $\{s\}$ is a minimum cut in $\underline{N}$, and by the max-flow min-cut theorem, it then follows that $\sum_{e\in E_1} \underline{c}(e) = \sum_{e \in E_1} \overline{\varphi}(e)$ and since for all $e \in E_1$, $\overline{\varphi}(e) \leq \overline{c}(e)$, it follows that $\overline{\varphi}(e) = \overline{c}(e)$. A similar argument shows that for all $e \in E_3$, $\overline{\varphi}(e) = \overline{c}(e)$. $\square$

Lemma 10 *For all $e \in E_1$, $\underline{c}(e) \leq c'(e) \leq \overline{c}(e)$ and for all $e \in E_3$, $\overline{c}(e) \leq c'(e) \leq \underline{c}(e)$.*

Proof. First observe that for all $e \in E$, $\varphi'(e) \leq \max\{\overline{\varphi}(e), \underline{\varphi}(e)\} \leq \max\{\overline{c}(e), \underline{c}(e)\}$. For $e \in E_1$, $\max\{\overline{c}(e), \underline{c}(e)\} = \overline{c}(e)$ so $\varphi'(e) \leq \overline{c}(e)$. For $x \in V^{\varphi^0} \cap Z^+$, $\underline{c}(s,x) = \underline{\varphi}(s,x) = \varphi'(s,x) = c'(s,x)$, where the first equality follows from Lemma 9, the second follows from the definition of φ' and the third follows from the definition of c'. For $x \in (V \setminus V^{\varphi^0}) \cap Z^-$,

[28] The qualification concerning zero weight vertices has to do with the fact that the network is constructed in a slightly different way in this proof from the way it is constructed in Section 3.5, as mentioned above.

$\underline{c}(s,x) = \underline{\varphi}(s,x) = \varphi^0(s,x) \leq \overline{\varphi}(s,x) = \varphi'(s,x) = c'(s,x)$, where the first equality follows from Lemma 9, the second from part (iii) of Lemma 8, the inequality from part (i) of Lemma 8, the third equality from the definition of φ', and the fourth from the definition of c'. A similar argument establishes the inequalities for edges $e \in E_3$. □

It is an immediate consequence of Lemma 10 that for all $x \in Z'$, $\underline{w}'_x \leq w'_x \leq \overline{w}'_x$. But this implies that for all $z \in Z$, $\underline{w}_z = \underline{w}'_{z+} + \underline{w}'_{z-} \leq w_z = w'_{z+} + w'_{z-} \leq \overline{w}_z = \overline{w}'_{z+} + \overline{w}'_{z-}$, completing the proof. □

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