Basis for a Subspace

Typically a subspace contains an infinite number of vectors. Consequently, there are issues with the tractability of such subspaces. To resolve this caveat, some problems involving a subspace, are handled best by working with a small finite set of vectors that span the subspace. The smaller the set, the better. It can be shown that the smallest possible spanning set must be linearly independent.

A **basis** for a subspace Φ of $\Re^n$ is a linearly independent set in Φ that spans Φ.

Example

Find a basis for the column space and the null space

of a matrix $A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -2 & 3 & 1 \end{bmatrix}$.

To find the basis for $Col(A)$, I need to first locate the pivot columns.

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & -2 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$
$$\quad \mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4$$

The two pivot columns correspond to vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Therefore $Col(A) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\}$

To find the basis for $Nul(A)$, I need to write the solution of $A\mathbf{x} = \mathbf{0}$ in parametric vector form:

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 & 0 \\ 0 & -2 & 3 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -2 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

$x_1 = 3x_3 + 2x_4 = 3\alpha + 2\beta$
$x_2 = \frac{3}{2}x_3 + \frac{1}{2}x_4 = \frac{3}{2}\alpha + \frac{1}{2}\beta$
$x_3 = \alpha$
$x_4 = \beta$

where x_3 and x_4 are free variables.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3\alpha + 2\beta \\ \frac{3}{2}\alpha + \frac{1}{2}\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 3 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Thus, $\left\{ \begin{bmatrix} 3 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$ spans $Nul\ A$.